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A LOWER BOUND FOR THE NUMBER OF ZEROS OF A MEROMORPHIC FUNCTION AND ITS SECOND DERIVATIVE

by J. K. LANGLEY

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We prove that for a function $f(z)$ transcendental and meromorphic in the plane and not of the form $\exp(az+b)$, we have either $N(r,1/ff'')\neq o(T(r,f'/f))$ or $\overline{\lim}_{r\to\infty} \dfrac{\log N(r,1/ff'')}{\log\log r}\geqq 2$.

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1. Introduction

The following theorem was proved in [12], confirming a conjecture of Hayman [8].

Theorem A. *Suppose that f is meromorphic in the plane such that f and F have only finitely many zeros, where $F=f''+a_1f'+a_0f$ and the a_j are rational functions with $a_j(z)=O(|z|^{j-2})$ as $z\to\infty$. Then f'/f is rational. In particular, if f and f'' have no zeros, we have $f(z)=\exp(az+b)$ or $f(z)=(az+b)^{-n}$ with a and b constants and n a positive integer.*

The proof of Theorem A in [12] begins by using a device of Frank [5,6,7]. If f_1,f_2 are linearly independent solutions of the associated homogeneous equation $w''+a_1w'+a_0w=0$, then we define g by $g^2=f/F$, and the functions $w_j=f_j'g-f_j(f'/f)g$ solve an equation $w''+a_1w'+bw=0$ in which b is meromorphic with only finitely many poles. The paper [15] uses this method to determine all functions f meromorphic in the plane for which f and F have only finitely many zeros, with the a_j any rational functions. In the particular case $F=f''$, an alternative, but related, approach, used by Mues in [16], is to write $H=z-f/f'$ so that H has only finitely many multiple points and its Schwarzian derivative $\{H,z\}=H^{(3)}/H'-\frac{3}{2}(H''/H')^2$ [10,11] has only finitely many poles. Using the modified auxiliary function $G=z-hf/f'$, with h a constant, Bergweiler proved the following in [2].

Theorem B. *Suppose that a is a constant such that $a\neq 1$ and $1/(a-1)$ is not a positive integer, and suppose that f is transcendental and meromorphic of finite order in the plane such that $L=ff''-af'^2$ has only finitely many zeros. Then*

$$f(z)=\exp(Az+B), \quad A,B\in\mathbb{C}. \tag{1.1}$$

It is necessary to assume in Theorem B that a is not of the form $(n+1)/n$, with n a positive integer, because of the example $f(z)=g(z)^{-n}$, with g any entire function such that g'' has only finitely many zeros. The case $a=1$ must also be excluded, because of examples such as $f(z)=\cos z, f(z)=(1+e^z)^n (n\leqq 1), f(z)=e^{g(z)}, g''(z)\neq 0$ (see also [17]). Bergweiler's proof in [2] does not use the Schwarzian derivative, but proceeds by applying the main result of [3] to show that the inverse function of G has only finitely many singularities, this leading to an estimate for $G'(z)$ at fixpoints of G.

In the present paper, we remove the order restriction in Theorem B, and strengthen the conclusion that if (1.1) does not hold then L must have infinitely many zeros, thus also improving Theorem A in the most important case $F=f''$. We state the following result, part (i) of which was proved by Frank and Hellerstein in [6], with part (ii) appearing in [14].

Theorem C. *Suppose that f is transcendental meromorphic in the plane and $N(r,1/ff^{(k)})=o(T(r,f'/f))$ for some $k\geqq 2$.* (i) *If $k\geqq 3$ then* (1.1) *holds.* (ii) *The same conclusion holds if $k=2$ and f has finite lower order.*

It seems possible that Theorem C holds without the order restriction in part (ii). The proof in [14] uses the auxiliary function $H=z-f/f'$ and the following combination of results of Shea [18] and Eremenko [4]. If H is transcendental and meromorphic of finite lower order in the plane, and the counting function

$$N_1(r,H)=N(r,1/H')+N(r,H)-\bar{N}(r,H)$$

of the multiple points of H satisfies $N_1(r,H)=o(T(r,H))$, and if H is normalised so that $\delta(\infty,H)=0$, then the order ρ of H is such that 2ρ is an integer not less than two, and there are slowly varying functions $L_j(r)$ such that

$$-\log|H'(re^{i\theta})|=\pi r^{\rho}L_1(r)(|\cos(\rho(\theta-L_2(r)))|+o(1)) \tag{1.2}$$

outside an exceptional set. With the hypotheses of Theorem D, part (ii), the assumption that H is transcendental and (1.2) lead to an upper estimate for $m(r,1/f)$, which in turn contradicts the fact that $N(r,1/f)$ is small. This approach does not seem to work for the problem considered in Theorem B, and (1.2) is not available when H has infinite order. We prove here the following theorem.

Theorem 1. *Suppose that f is transcendental and meromorphic in the plane, and that $F=ff''-af'^2$, $a\in\mathbb{C}$.*

(i) *If $a\neq 1$ and $1/(a-1)$ is not a positive integer and*

$$N(r)=o(T(r,f'/f)) \quad \textit{and} \quad \overline{\lim_{r\to\infty}}\frac{\log N(r)}{\log\log r}<2 \tag{1.3}$$

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holds with $N(r) = N_2(r) = N(r, 1/F)$, *then* (1.1) *holds.*

(ii) *Suppose that* $a \neq 1$ *and* $1/(a-1)$ *is not an integer, or that* $a = 0$. *If* (1.3) *holds with* $N(r) = N_3(r)$, *where* $N_3(r)$ *counts the zeros of* F *which are not multiple zeros of* f, *then* (1.1) *holds.*

In particular, if $N(r)$ counts the simple zeros of f and the zeros of f'' which are not multiple zeros of f, and (1.3) holds, then we have (1.1). Theorem 1 will be proved using the following auxiliary result.

Theorem 2. *Suppose that* H *is transcendental and meromorphic in the plane of order* $\rho(H)$ *such that*

$$N_1(r, H) = o(T(r, H)), \overline{\lim_{r \to \infty}} \frac{\log N_1(r, H)}{\log \log r} < 2, \tag{1.4}$$

and suppose further that q *is a constant with* $q < \rho(H)$. *Then* H *has fixpoints* z *with* $|z|$ *arbitrarily large and* $|H'(z)| > |z|^q$.

We make some remarks about Theorem 2. First, we are free to assume that $\rho(H)$ is positive, by the results of Shea and Eremenko already cited. Further, the first inequality of (1.4) implies that H has infinitely many fixpoints (as noted in [2]) and, if H has finite lower order, implies additionally that (1.2) holds. However, these fixpoints may and almost certainly must lie in regions where the cosine term in (1.2) is small.

Of course, if f'/f is not of small growth then Theorem 1 gives only a weak estimate for the number of zeros of f and f''. We indicate in Section 5 where the second inequality of (1.4) seems to be necessary for our method, and in Section 8 we outline the proof of a stronger result which holds when f'/f has very large growth.

2. Preliminaries

We begin by summarizing some facts from the Wiman–Valiron theory [9]. If $\gamma > 1/2$ is a constant and $g(z) = \sum_{k=0}^{\infty} a_k z^k$ is a transcendental entire function with maximum modulus $M(r,g)$ on $|z| = r > 1$, the central index $\nu(r,g)$ is the largest n such that $|a_n| r^n = \mu(r,g) = \max_m |a_m| r^m$, and there is a set of *normal* r, the complement of this set having finite logarithmic measure, such that if r is normal and $|z_0| = r$ with $|g(z_0)| > \frac{1}{2} M(r,g)$, we have

$$(\log M(r,g))/2 \log r \leqq \nu(r,g) \leqq (\log M(r,g))^{3/2},$$

$$M(r,g') \sim M(r,g)\nu(r,g)/r, \text{ while } g(z) \sim (z/z_0)^{\nu(r,g)} g(z_0),$$

and

$$g^{(j)}(z)/g(z) \sim (\nu(r,g)/z)^j \quad \text{for} \quad j = 1, 2 \quad \text{and} \quad |\log(z/z_0)| \leqq \nu(r,g)^{-\gamma}.$$

We require in addition representations in annuli for entire functions of very small growth. Suppose that δ and ε are positive constants, and that $f_1, f_2, \ldots, f_{n-1}$ and $f_n = f$ are transcendental entire functions each satisfying $T(r, f_j) = O(\log r)^{2-\delta}$. Then if r_0 is large the number of zeros of the f_j in $r_0 \leqq |z| \leqq (r_0)^{1+\varepsilon}$ is $O(\log r_0)^{1-\delta}$, and we can find r_1, r_2 with

$$\log r_0 \leqq \log r_1 \leqq \log r_1 + (\log r_0)^{2\delta/3} = \log r_2 \leqq (1+\varepsilon)\log r_0$$

and such that none of the f_j have zeros in

$$\log s_1 = \log r_1 - (\log r_0)^{2\delta/3} \leqq \log |z| \leqq \log r_2 + (\log r_0)^{2\delta/3} = \log s_2.$$

If $f(z) = \prod_{j=1}^{\infty} (1 - z/a_j)$, assuming that $f(0) = 1$, then for $r_1 \leqq |z| \leqq r_2$ we can write

$$f(z) = z^{n(s_1, 1/f)} \Pi_{|a_j| \leqq s_1}(-1/a_j)(1 - a_j/z) \Pi_{|a_j| \geqq s_2}(1 - z/a_j),$$

and we have (see [**1, 13**] for details)

$$|\Pi_{|a_j| \leqq s_1}(1 - a_j/z) - 1| \leqq \exp(n(s_1, 1/f)s_1/r_1) - 1 \leqq \exp(-(\log r_0)^{\delta/2}),$$

$$|\Pi_{|a_j| \geqq s_2}(1 - z/a_j) - 1| \leqq \exp\left(r_2 \int_{s_2}^{\infty} (1/t)dn(t, 1/f)\right) - 1$$

$$\leqq O((r_2/s_2)\log s_2) \leqq \exp(-(\log r_0)^{\delta/2}).$$

Lemma 1. *There is a positive constant c_1 with the following property. Suppose that $c > 0$ and $L > (1+c)s > 1 + c$ and $F(z)$ is analytic in the closed rectangular region Ω given by $|\mathrm{Re}(z)| \leqq 3\pi L$, $Ls \leqq \mathrm{Im}(z) \leqq LS \leqq 2Ls$, with $|F(z)| \leqq cL^{-2}$ there. Then the equation*

$$w''(z) + (1 - F(z))w(z) = 0 \tag{2.1}$$

has solutions $U(z)$, $V(z)$ in Ω such that, with $|\varepsilon_j(z)| \leqq c_1 cs/L$,

$$U(z) = e^{-iz}(1 + \varepsilon_1(z)), \quad U'(z) = -ie^{-iz}(1 + \varepsilon_2(z)),$$
$$V(z) = e^{iz}(1 + \varepsilon_3(z)), \quad V'(z) = ie^{iz}(1 + \varepsilon_4(z)). \tag{2.2}$$

Proof. This is similar to Lemma 1 of [**12**], but with a different region Ω. We choose a solution v of the equation $v'' + 2iv' - Fv = 0$ such that $v(X) = 1, v'(X) = 0$, where $X = iLS$. Differentiating twice shows that

$$v(z) = 1 - (1/2i)\int_X^z (e^{2i(t-z)} - 1)F(t)v(t)\,dt, \; v'(z) = \int_X^z e^{2i(t-z)}F(t)v(t)\,dt. \tag{2.3}$$

We take as path of integration the straight line segment from X to $\mathrm{Re}(z) + iLS$ followed by that from $\mathrm{Re}(z) + iLS$ to z. If ds denotes arc-length on this path, (2.3) gives

$$|v(z)-1| \leqq \int_X^z |F(t)v(t)|\,ds.$$

We set

$$W(\zeta)=\log\left(1+\int_X^\zeta |F(t)v(t)|\,ds\right),$$

with ζ lying on the above path. Then $dW/ds \leqq |F(\zeta)|$, so that using $d_1, d_2, \ldots$ to denote positive constants not depending on c, we obtain $W(z) \leqq \int_X^z |F(t)|\,ds \leqq d_1 csL^{-1}$. Thus

$$|v(z)-1| \leqq \exp(d_1 csL^{-1})-1 \leqq d_2 csL^{-1} \leqq d_3, |v'(z)| \leqq \int_X^z |F(t)|\,d_3\,ds \leqq d_4 csL^{-1}.$$

Now we set $V(z)=e^{iz}v(z)$ so that V solves (2.1) and (2.2) follows at once. To form U, choose a solution u of $u''-2iu'-Fu=0$, such that $u(Y)=1$, $u'(Y)=0$, where $Y=iLs$. The integral equation for u is $u-1=(1/2i)\int_Y^z(e^{-2i(t-z)}-1)F(t)u(t)\,dt$ and we choose a path of integration on which $\mathrm{Im}(t-z) \leqq 0$. Finally we set $U=u\,e^{-iz}$.

3. Proof of Theorem 2, first part

Suppose that H satisfies the hypotheses of the theorem, with

$$N_1(r,H)=o(T(r,H)), \quad N_1(r,H)=O(\log r)^{2-\delta}, \tag{3.1}$$

δ being a positive constant. Then [10, 11] we can write, locally,

$$H=w_1/w_2, \quad W(w_1,w_2)=1, \tag{3.2}$$

where the w_j are solutions of the equation

$$w''+b(z)w=0, \tag{3.3}$$

with $b(z)=\frac{1}{2}\{H,z\}$. The function b is meromorphic in the plane, while the solutions w_j admit analytic continuation along any path avoiding poles of b. For convenience later on we choose constants ϕ, ψ such that $H_1(z)=(H(z)-\phi)/(H(z)-\psi)$ has only simple zeros and poles and such that $\bar{N}(r,H_1) \sim T(r,H)$. Further, we write $H_1(z)=\sigma(z)/\tau(z)$ with σ and τ entire functions with no common zeros, and we choose a zero u_0 of $\tau(z)$. This gives $H_1'(z)=W(\tau,\sigma)\tau^{-2}=h\tau^{-2}$, with $N(r,1/h)=O(\log r)^{2-\delta}$, and we can assume that $T(r,h)=O(\log r)^{2-\delta}$, because otherwise we can write $h(z)=h^*(z)e^{-2\lambda(z)}$ with $\lambda(z)$ entire and $T(r,h^*)=O(\log r)^{2-\delta}$, and we need only replace σ and τ by σe^λ and τe^λ.

Returning to the function H, we have

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$$H' = -1/V_2, \quad (1/H)' = 1/V_1, \quad V_j = (w_j)^2, \tag{3.4}$$

and so the V_j are meromorphic in the plane and

$$N(r, V_1) + N(r, V_2) + N(r, b) = O(N_1(r, H)) = O(\log r)^{2-\delta}. \tag{3.5}$$

We can write

$$V_j = g_j/h_j, \quad b = a/c, \quad T(r, h_j) + T(r, c) = O(\log r)^{2-\delta}, \tag{3.6}$$

with g_j, h_j, a, c entire, with in each case the numerator and denominator having no common zeros. We now divide the proof into certain cases.

Case A. Suppose that $T(r,a) \neq O(\log r)^{2-\delta/2}$.

In this case we easily obtain arbitraily large s_0 with

$$\log M(s_0, a) > 2(\log s_0)^{2-\delta/2}, \quad \nu(s_0, a) > (\log s_0)^{1-\delta/2}. \tag{3.7}$$

By the discussion in Section 2, we can find s_1, s_2 satisfying

$$s_0 \leqq s_1 < s_2 \leqq (s_0)^2, \quad \log(s_2/s_1) \geqq (\log s_0)^{\delta/2} \tag{3.8}$$

and such that we have, for some non-zero constant γ_0,

$$c(z) = \gamma_0 z^\nu(1 + o(1)), \quad c'(z)/c(z) = (\nu/z)(1 + o(1)), \quad c''(z)/c(z) = (\nu/z)^2(1 + o(1)),$$
$$\nu = n(s_1, 1/c) = O(\log s_0)^{1-\delta}, \quad s_1 \leqq |z| \leqq s_2, \tag{3.9}$$

with similar representations for h_1 and h_2 in the same annulus.

We now choose r, normal for the functions a, g_1 and g_2, and z_0 and constants γ_1, γ, such that $\frac{1}{4}s_2 \leqq r \leqq \frac{1}{2}s_2$, $|a(z_0)| = M(r, a)$, and $\frac{1}{2} < \gamma_1 < \gamma < \frac{2}{3}$. We then have, provided s_0 was chosen large enough,

$$a(z) = a(z_0)(z/z_0)^{\nu(r,a)}(1 + o(1)), \quad a^{(j)}(z)/a(z) = (\nu(r,a)/z)^j(1 + o(1)), \tag{3.10}$$

for $j = 1, 2$ and $z = z_0 e^\tau, |\tau| < \nu(r,a)^{-\gamma_1}$. Setting $N = \nu(r,a) - \nu = \nu(r,a)(1 + o(1))$, using (3.7), we now have, by (3.9),

$$b(z) = b(z_0)(z/z_0)^N(1 + o(1)), \quad \text{for } z = z_0 e^\tau, |\tau| \leqq N^{-\gamma}, \tag{3.11}$$

and we note that, with $M(r, b) = \max\{|b(z)| : |z| = r\}$,

$$|b(z_0)| \geqq (1 - o(1))M(r, b), \quad (\log s_0)^{1-\delta} < N \leqq (\log|b(z_0)|)^2. \tag{3.12}$$

We now make the same local change of variables as in [**12**, Section 7], and [**15**,

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Section 4]. We set $D(s)=\{z:|\log|z/z_0||\leqq sN^{-\gamma},\ |\arg(z/z_0)|\leqq sN^{-\gamma}\}$. We write $z_1=z_0\exp(-N^{-\gamma})$ and on $D(1)$ we define branches of $b(z)^{1/2}$ and $z^{N/2}$. Defining Z exactly as in [**12**, Section 7], we obtain, in $D(1/2)$,

$$Z=b(z_1)^{1/2}z_1 2/(N+2)+\int_{z_1}^{z} b(t)^{1/2}\,dt=(1+o(1))b(z_0)^{1/2}2z^{(N+2)/2}z_0^{-N/2}(N+2)^{-1}$$
$$=(1+o(1))b(z)^{1/2}2z/(N+2). \tag{3.13}$$

As in [**12**] we conclude that the function Z has in $D(1/4)$ simple islands over the closed region D^* given by

$$|\log|Z/Z_0||\leqq N^{1/3},\quad |\arg Z|\leqq\pi/4,\quad Z_0=|b(z_0)^{1/2}2z_0/(N+2)|, \tag{3.14}$$

and we choose such a pre-image D^{**}. By (3.12), $Z_0\exp(-N^{1/3})$ and the minimum modulus of $b(z)$ on D^{**} are large. Indeed, we have, using (3.7),

$$\log r+\log N=o(\log Z_0),\quad r=|z_0|. \tag{3.15}$$

As in [**12**, Section 8] we make the transformation $W(Z)=b(z)^{1/4}w(z)$, where w solves (3.3), z lies in D^{**} and Z in D^*. The equation (3.3) transforms to

$$d^2W/dZ^2+(1-F_0(Z))W=0,\quad F_0(Z)=b''(z)/4b(z)^2-5b'(z)^2/16b(z)^3, \tag{3.16}$$

in which, using (3.7), (3.9), (3.10) and (3.13), $|F_0(Z)|\leqq c_4|Z|^{-2}$ in D^*. By Lemma 1 of [**12**] there exist solutions U, V of (3.16) in D^* such that

$$U(Z)=(1+o(1))e^{-iZ},\quad V(Z)=(1+o(1))e^{iZ},\quad W(U,V)=2i+o(1). \tag{3.17}$$

We can write, in D^{**}, for some constants A,B,C,D,

$$u(z)=b(z)^{-1/4}U(Z),\quad v(z)=b(z)^{-1/4}V(Z),\quad w_1=Au+Bv,\quad w_2=Cu+Dv. \tag{3.18}$$

We estimate A, B, C and D, using a method different to that of [**12**]. Since

$$-b=w_j''/w_j=\tfrac{1}{2}(g_j''/g_j-h_j''/h_j)-\tfrac{1}{4}(g_j'/g_j)^2+\tfrac{3}{4}(h_j'/h_j)^2-\tfrac{1}{2}(g_j'/g_j)(h_j'/h_j), \tag{3.19}$$

we obtain, denoting absolute constants by c_j and using (3.9), $v(r,g_j)\leqq c_1 rM(r,b)^{1/2}$ and so

$$\log M(r,g_j)+\log M(r,g_j')\leqq\log M_0=c_2 r(\log r)M(r,b)^{1/2}. \tag{3.20}$$

Therefore we have, for $s_1\leqq|z|\leqq r$, using (3.7),

$$\log(|w_j(z)|+|w_j'(z)|)\leqq\log M_0+O(\log r)^{2-\delta}\leqq\log M_1=c_3 r(\log r)M(r,b)^{1/2}. \tag{3.21}$$

To estimate A,B,C,D we note that, in D^{**},

$$w_1 = Au + Bv, \quad w_1' = Au' + Bv', \quad W(u,v) = 2i + o(1). \tag{3.22}$$

We choose Z^* in D^* with $|Z^*| \leqq Z_0/2$ and $|U(Z^*)| \leqq 2$, $|U'(Z^*)| \leqq 2$. Let z^* be the preimage of Z^* in D^{**}, so that $|z^*| \leqq r$ and $|u(z^*)| \leqq 2$ and $|u'(z^*)| \leqq 2M(r,b)^{1/2}$. Now (3.22) gives $B = W(u,w_1)W(u,v)^{-1}$ so that

$$|B| \leqq 2M_1 M(r,b)^{1/2} \leqq \exp(c_4(\log r) r M(r,b)^{1/2}),$$

and the same estimate holds for A, C and D. Further,

$$1 = W(w_1, w_2) = (AD - BC)W(u,v) = (AD - BC)(2i + o(1)). \tag{3.23}$$

Thus, in any pair $\{A,B\}$, $\{A,C\}$, $\{B,D\}$, $\{C,D\}$, each term has modulus at most

$$M = \exp(c_5 N Z_0 \log r), \quad \log M = o(\exp(N^{1/4})Z_0), \tag{3.24}$$

while at least one term has modulus at least $1/M$.

4. Completion of the proof of Theorem 2, in Case A

We choose a small positive constant η, and we write

$$g(z) = w_2 z - w_1 = F_1 u + F_2 v, \quad F_1(z) = Cz - A, \quad F_2(z) = Dz - B,$$
$$G(Z) = b(z)^{1/4} g(z), \quad P_j(Z) = F_j(z). \tag{4.1}$$

Now (3.23) implies that either $|AD| \geqq 1/6$ or $|BC| \geqq 1/6$. In this section we use D_j to denote positive constants not depending on η. It is easy to see that we have

$$|1 - Cz/A| \geqq (D_1(N+2))^{-1}, |1 - B/Dz| \geqq (D_1(N+2))^{-1},$$
$$|F_j'(z)/F_j(z)| \leqq D_1(N+2)/|z|, \tag{4.2}$$

if $AD \neq 0$ and, if $CD \neq 0$,

$$|1 - Dz/B| \geqq (D_1(N+2))^{-1}, |1 - A/Cz| \geqq (D_1(N+2))^{-1},$$
$$|F_j'(z)/F_j(z)| \leqq D_1(N+2)/|z|, \tag{4.3}$$

outside at most four discs each of radius at most $D_1 r(N+2)^{-1}$. Further, these discs have images in the Z-plane which each lie in an annulus of form $R_1 \leqq |Z| \leqq R_2$ with $\log(R_2/R_1) \leqq D_2$. This easily gives us $d \geqq 1/2$ such that

$$|P_j'(Z)/P_j(Z)| \leqq D_3/|Z|, dN^{1/3} \leqq \log|Z/Z_0| \leqq (d+1/12)N^{1/3}, |\arg Z| \leqq \pi/4, \tag{4.4}$$

and such that we have (4.2) and/or (4.3) in the pre-image D' in D^{**} of the region in (4.4). Now suppose that $z \in D'$ is a zero of g. Then we have $H(z)=z$, since the w_j have no common zeros, and $\mu = v/u = (A-Cz)/(Dz-B)$ so that

$$-1/H' = (w_2)^2 = uv\mu^{-1}(C+D\mu)^2 = uv(Dz-B)^{-1}(A-Cz)^{-1}(AD-BC)^2.$$

If $|AD| \geqq 1/6$ this is, using (3.12), (3.13), (3.15), (3.23) and (4.2),

$$\begin{aligned}(-1/4+o(1))b(z)^{-1/2}(ADz(1-B/Dz)(1-Cz/A))^{-1} &= O(|b(z)|^{-1/2}|z|^{-1}(N+2)^2)\\ &= O((N+2)|Z|^{-1}) = O(Z_0^{-1}) = o(|z|^{-q}).\end{aligned}$$

We argue similarly if $|BC| \geqq 1/6$. Thus the conclusion of Theorem 2 holds in Case A, provided that we can show that D' contains a zero of g.

To prove the last assertion, we assume that D' contains no zero of g. In the region in (4.4) we have, using (3.15), (3.24), the last remark in Section 3 and (4.2) or (4.3)

$$|\log|P_j(Z)|| \leqq D_4(\log M + \log r + \log(N+2)) \leqq o(|Z|),$$

and $\log^+|G(Z)| = O(|Z|)$. Further, we have, provided s_0 was chosen large enough,

$$|\log|e^{iZ}|| \geqq D_5|Z||\arg Z| \geqq D_5\eta \exp(\tfrac{1}{2}N^{1/3})Z_0, \quad \text{for } \pi/4 \geqq |\arg Z| \geqq \eta,$$

so that $|G(Z)|>1$ on $\arg Z = \eta$, and we obtain, arguing as in [12], $|G'(Z)/G(Z)| \leqq D_6$ in the region $(d+1/48)N^{1/3} \leqq |\log|Z/Z_0|| \leqq (d+1/16)N^{1/3}$, $|\arg Z| \leqq \pi/16$. We apply the argument principle to G on the boundary of the region

$$m_1 = (d+1/48)N^{1/3} \leqq |\log|Z/Z_0|| \leqq m_2 = (d+1/16)N^{1/3}, |\arg Z| \leqq \eta. \tag{4.5}$$

On the straight line segments which form part of this boundary we have

$$G(Z) = (1+o(1))U(Z)P_1(Z), \arg Z = \eta, \text{ and } G(Z) = (1+o(1))V(Z)P_2(Z), \arg Z = -\eta.$$

The change in $\log G(Z)$ as Z describes once counter-clockwise the boundary of the region in (4.5) is then, using (4.4),

$$2i\cos(\eta)Z_0(e^{m_2}-e^{m_1})+O(\eta Z_0 e^{m_2})+O((m_2-m_1))\neq 0.$$

5. Proof of Theorem 2: Case B

Here we suppose that $T(r,a)=O(\log r)^{2-\delta/2}$. By (3.4) and (3.6), we must have $\rho(g_j)=\rho(H)$ for each j. We choose q' with $q<q'<\rho(H)$ and $\varepsilon>0$ with $(1+\varepsilon)q<q'$, and we can find arbitrarily large s_0 such that, for some j,

$$\nu(s_0,g_j)>(s_0)^{q'}. \tag{5.1}$$

Since a and c both have small growth we can find s_1, s_2 such that

$$s_0\leqq s_1\leqq(s_0)^{1+\varepsilon/2}, \log(s_2/s_1)=(\log s_0)^{\delta/5} \tag{5.2}$$

and such that in the annulus $s_1\leqq|z|\leqq s_2$ we have

$$b(z)=\alpha z^N(1+\varepsilon(z)), \varepsilon(z)=o(1), \varepsilon'(z)=o(1)/z, \varepsilon''(z)=o(1)/z^2, \tag{5.3}$$

with α a non-zero constant (depending on the annulus) and $N=n(s_1,1/a)-n(s_1,1/c)$, and

$$h_j^{(m)}(z)/h(z)=O(n(s_1,1/h_j)/|z|)^m \tag{5.4}$$

for $j=1,2, m=1,2$. We remark that at this point the second condition of (1.4) seems to be necessary for our method to work, because if the entire functions a and c are of roughly the same growth we have otherwise apparently no information on the local behaviour of a/c. We then have, by (3.19) and (5.1), if $r\in[s_1,s_2]$ is normal for both g_j, and z is a maximum modulus point of g_j with $|z|=r$,

$$\nu(r,g_j)^2=(-4+o(1))\alpha z^{N+2}, \tag{5.5}$$

and it follows that since $\nu(r,g_j)$ is increasing we must have $N\geqq -2$ in (5.3). Further, $|\alpha|^{1/2}r^{(N+2)/2}$ must be large, if r is normal for the g_j. We postpone the case $N=-2$ to Section 6 and assume for the remainder of this section that $N\geqq -1$. We write $\alpha=\beta^2$ and we choose a real θ_0 such that $\arg\beta+(N+2)\theta_0/2=0 \pmod{2\pi}$. We set $z_1=s_1e^{i\theta_0}$, and we make the change of variables

$$Z=2\beta(N+2)^{-1}(z_1)^{(N+2)/2}+\int_{z_1}^{z} b(t)^{1/2}\,dt=(2+o(1))\beta(N+2)^{-1}z^{(N+2)/2},$$

$$\text{for} \quad s_1\leqq|z|\leqq s_2, |\arg z-\theta_0|<\pi, \tag{5.6}$$

using (5.3). Now the function Z maps a sub-domain D^{**} of the annulus $4s_1 \leqq |z| \leqq s_2$ univalently onto the region

$$T_1 = 2|\beta|(N+2)^{-1}(8s_1)^{(N+2)/2} \leqq |Z| \leqq T_2 = 2|\beta|(N+2)^{-1}(s_2/2)^{(N+2)/2},$$
$$|\arg Z| \leqq \pi/4, \tag{5.7}$$

and T_1 must be large, by (5.2) and the remark following (5.5). The same transformation $W(Z) = b(z)^{1/4} w(z)$ as in Section 3 gives (3.16) again, in which $F_0(Z) = O(|Z|^{-2})$, by (5.3). Again, we obtain the solutions U, V as in (3.17) in the region (5.7), and we estimate the coefficients A, B, C, D. We choose r with $12s_1 \leqq r \leqq 16s_1$, normal for both the g_j, and we obtain (3.20) again. Using the remark following (5.5), this leads to (3.21), valid for $|z| = r$. We choose Z^* in the region (5.9) with $|U(Z^*)| \leqq 2$, $|U'(Z^*)| \leqq 2$, and with pre-image z^* in D^{**} satisfying $|z^*| = r$. We then have

$$|u(z^*)| \leqq 2|b(z^*)|^{-1/4} \leqq 4|\alpha|^{-1/4}|z^*|^{-(N+2)/4}|z^*|^{1/2} \leqq o(|z^*|^{1/2}),$$

using (5.5). Further, using (5.3),

$$|u'(z^*)| \leqq o(|z^*|^{1/2})(|b(z^*)|^{1/2} + |b'(z^*)/4b(z^*)|) \leqq o(|\beta| r^{(N+2)/2} + (N+2)).$$

This reasoning shows that the coefficients A, B, C, D all have modulus at most M, where

$$\log M = \log(N+2) + d_1|\beta|(16s_1)^{(N+2)/2}\log(16s_1), \tag{5.8}$$

denoting constants by d_j. We now write (4.1) again and we have (4.2) and/or (4.3) for all z in D^{**} outside at most four discs whose images each lie in an annulus in the Z-plane of logarithmic measure at most d_2. Since $\log(T_2/T_1) \geqq d_3(N+2)(\log s_0)^{\delta/5}$, we can choose S_1, S_2 such that

$$\log T_1 + d_4(N+2)(\log s_0)^{\delta/5} \leqq \log S_1 < \log S_2 = \log S_1 + d_4(N+2)(\log s_0)^{\delta/5} \leqq \log T_2, \tag{5.9}$$

and such that we have (4.2) and/or (4.3) in the pre-image in D^{**} of the region $S_1 \leqq |Z| \leqq S_2, |\arg Z| \leqq \pi/4$. We also have

$$(\log M)/S_1 \leqq o(1) + d_5 2^{(N+2)/2}(N+2)\log(16s_1)\exp(-d_4(N+2)(\log s_0)^{\delta/5}) = o(1).$$

Therefore we have

$$\log M = o(|Z|), |\log|P_j(Z)|| = o(|Z|), |P_j'(Z)/P_j(Z)| \leqq d_6/|Z|,$$
$$\text{for } S_1 \leqq |Z| \leqq S_2, |\arg Z| \leqq \pi/4. \tag{5.10}$$

We argue as in Section 4, assuming again that η is a small positive constant. If the region (5.10) contains a zero of $G(Z)$ then at the pre-image z in D^{**} we have

$$|1/H'(z)| \leqq d_7(N+2)^2|b(z)|^{-1/2}|z|^{-1} \leqq d_8(N+2)^2|\beta|^{-1}(2s_1)^{-(N+2)/2}(|z|/2s_1)^{-(N+2)/2}$$
$$\leqq d_9 \nu(s_0, g_j)^{-1}(N+2)^2 2^{-(N+2)/2} = O((s_0)^{-q'}) = o(|z|^{-q})$$

using (5.1), (5.2), (5.5) and (5.9). To show that such a fixpoint of H exists, if s_0 is large enough, we suppose that the region in (5.10) contains no zero of $G(Z)$. Then $G(Z)$ is large on $\arg Z = \eta$, by (5.10), and $|G'(Z)/G(Z)| \leqq d_{10}$ for $2S_1 \leqq |Z| \leqq S_2/2, |\arg Z| \leqq \pi/64$. Thus the change in $\log G(Z)$ as Z describes once counter-clockwise the boundary of the region $2S_1 \leqq |Z| \leqq S_2/2, |\arg Z| \leqq \eta$ is

$$2i((1/2)S_2 - 2S_1)\cos(\eta) + O(\eta S_2) + O(\log(S_2/S_1)) \neq 0.$$

6. The case where $N = -2$ in (5.3)

At this stage we recall the function H_1 defined following (3.3), to which we will apply an argument similar to that of **[18]**. By (5.5) and the discussion in Section 2, we can write $b(z)^{1/2} = iLz^{-1}(1+\omega(z))$, with L large and positive and $|\omega(z)| \leqq O(\exp(-(\log s_0)^{\delta/10}))$. As L is large, we can again use the transformation $w(z) = b(z)^{-1/4}W(Z)$, where

$$Z = iL\log s_1 + \int_{s_1}^{z} b(t)^{1/2}\,dt = iL\log z + o(L), s_1 \leqq |z| \leqq s_2, |\arg z| \leqq 4\pi,$$

mapping onto a region containing the rectangle

$$|\mathrm{Re}(Z)| \leqq 3\pi L, L\log(2s_1) \leqq \mathrm{Im}(Z) \leqq L\log(s_2/2). \tag{6.1}$$

The equation (3.3) transforms to (3.16) again, with this time $|F_0(Z)| \leqq C_1 L^{-2}$, denoting absolute constants by C_j, and we note that, by (5.1) and (5.5), we have $L > (s_1)^q$. Lemma 1 gives us solutions U, V of (3.21) in the region (6.1) satisfying $U(Z) \sim e^{-iZ}$, $V(Z) \sim e^{iZ}$. This gives us, on multiplying by a constant, solutions u, v of (3.3) satisfying

$$u(z) = z^{L+1/2}e^{o(L)}, \quad v(z) = z^{-L+1/2}e^{o(L)},$$

for $4s_1 \leqq |z| \leqq s_2/4, |\arg z| \leqq \pi$, and there are constants A, B, not both zero, such that $H_1'(z) = h(z)\tau(z)^{-2} = (Au(z) + Bv(z))^{-2}$. We assert the existence of R such that, for some non-zero constant γ_2,

$$H_1'(z) = \gamma_2 z^{-1 \pm 2L}e^{o(L)} \quad \text{for} \quad 4s_1 \leqq \tfrac{1}{4}R \leqq |z| \leqq 8R \leqq s_2/4, \tag{6.2}$$

and such that $|h(z)| \geqq 1$ on $|z| = R$. If $AB = 0$, this is obvious. Otherwise, we choose R such that $|h(z)| \geqq 1$ on $|z| = R$ and $|\log R - (1/2L)\log|B/A|| > 2$ and (6.2) follows, as L is large. Since $n(r, 1/H_1') = O(\log r)^{1-\delta}$, (5.1), (5.5) and (6.2) give

$$2L \sim n(R/2, H'_1) \leqq C_2 N(R, 1/\tau) \leqq C_2 \log M(R,\tau) + C_3.$$

Therefore (6.2) gives $\log M(R,\tau) \sim \log m_0(R,\tau)$, with $m_0(R,\tau)$ denoting the minimum modulus. We choose a fixed z_2 with $|z_2| = R$. Integrating $h\tau^{-2}$ around $|z| = R$ we obtain $|\sigma(z)/\tau(z) - \sigma(z_2)/\tau(z_2)| \leqq RM(R,h)m_0(R,\tau)^{-2}$ and so

$$|\sigma(z)\tau(z_2) - \sigma(z_2)\tau(z)| \leqq RM(R,h)m_0(R,\tau)^{-2}M(R,\tau)|\tau(z_2)|$$

for $|z| \leqq R$. Applying this inequality with $z = u_0$, the zero of $\tau(z)$ fixed in Section 3, we obtain $m_0(R,\tau)^2 \leqq C_4 RM(R,h)M(R,\tau)$, a contradiction.

7. Proof of Theorem 1

We shall prove parts (i) and (ii) simultaneously, and we denote non-integrated counting functions, as usual, by $n(r)$. As in [2], we use the auxiliary function

$$H = z - hf/f', h = 1/(1-a), H' = h(ff'' - af'^2)/(f'^2).$$

Now multiple poles of H can only occur at zeros of f' of multiplicity $m \geqq 2$ which are not also zeros of f, and they contribute $m-1$ to $n_1(r,H)$ and the same to $n_2(r)$ and $n_3(r)$. Further, zeros of H' cannot occur at poles of f, since a is not of the form $(m+1)/m$ with m a positive integer, and with the hypotheses of part (ii) they cannot occur at multiple zeros of f either. Therefore, we have $N(r, 1/H') \leqq N(r)$ in both cases. However, at any fixpoint of H we clearly have $f = 0$ or $f = \infty$, and $H' = O(1)$.

Thus H must be a rational function, with no multiple points, and so a Möbius transformation, and we obtain $f'(z)/f(z) = h(Cz+D)/(Cz^2 + (D+S)z + T)$, with C, D, S, T constants, and since f is transcendental we see at once that C and $D+S$ must vanish.

8. A better estimate when f'/f has large growth

Suppose that H is transcendental meromorphic in the plane, such that $\log^+|H'(z)| \leqq O(\log|z|)$ at all fixpoints z of H with $|z|$ large, and suppose that $N_1(r,H) = O(r^m)$, with m a positive constant. It seems reasonable to believe that these hypotheses might imply that H has finite order. Our method does not appear to give this, but we can write (3.2), (3.3), (3.4) and (3.6) with $T(r,h_j) + T(r,c) = O(r^{2m})$. Suppose that $\rho(a)$ is large compared to m. Choosing $\gamma \in (1/2, 2/3)$, we can find arbitrarily large r, normal for a and the g_j, such that $(\log N)/(m \log r)$ is large, where $N = \nu(r,a)$, and such that, choosing z_0 with $|z_0| = r$ and $|a(z_0)| = M(r,a)$, we have (3.10), and

$$|c'(z)/c(z)| + |\log|c(z)|| \leqq r^{d_1 m} \quad \text{for} \quad ||z| - r| \leqq r^{-d_2 m},$$

with similar estimates for the h_j, and with d_j denoting constants not depending on m. This implies that (3.11) holds, and we have

$$\log|b(z_0|\geqq\log|a(z_0)|-O(r^{d_1m})\sim\log|a(z_0)|,\quad N\leqq(\log|b(z_0)|)^2.$$

Applying the same change of variables as in Section 3, we have, in particular, (3.15). The estimate (3.20) holds for the g_j, and we obtain (3.21), for $r-r^{-d_2m}\leqq|z|\leqq r$, and with $(\log r)^{2-\delta}$ replaced by r^{d_1m}. The coefficients A,B,C,D all have modulus at most

$$\exp(\exp(r^{d_3m})r|b(z_0)|^{1/2})\leqq M=\exp(\exp(N^{1/12})Z_0).$$

The argument of Section 4 goes through, since $\log M=o(|Z|)$ for Z satisfying (4.4), and we obtain fixpoints z of H with H' large, contradicting our initial assumption. Hence the order of a can be bounded in terms of m. Estimating the g_j using (3.19), we have the following.

Theorem 3. *Suppose that $0<m<\infty$ and f is meromorphic in the plane, and that $N(r)$ counts the zeros of ff'' which are not multiple zeros of f. If $N(r)=O(r^m)$, then $\log T(r,f'/f)=O(r^n)$, the positive constant n depending only on m.*

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University of Nottingham
Nottingham
NG7 2RD
E-mail: jkl@maths.nott.ac.uk.