Glasgow Math. J. **41** (1999) 33–41.

RELATIONS BETWEEN RICCI CURVATURE AND SHAPE OPERATOR FOR SUBMANIFOLDS WITH ARBITRARY CODIMENSIONS

BANG-YEN CHEN

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824-1027, USA
e-mail: bychen@math.msu.edu

(Received 15 February, 1997)

Abstract. First we define the notion of k-Ricci curvature of a Riemannian n-manifold. Then we establish sharp relations between the k-Ricci curvature and the shape operator and also between the k-Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. Several applications of such relationships are also presented.

1. Introduction. According to the well-known Nash's immersion theorem, every Riemannian n-manifold admits an isometric immersion into the Euclidean space $E^{n(n+1)(3n+11)/2}$. In general, there exist enormously many isometric immersions from a Riemannian manifold into Euclidean spaces if no restriction on the codimension were made. For a submanifold of a Riemannian manifold there associate several extrinsic invariants beside its intrinsic invariants. Among extrinsic invariants, the shape operator and the squared mean curvature are the most important ones. Among the main intrinsic invariants, sectional, Ricci and scalar curvatures are the well-known ones.

One of the most fundamental problems in submanifold theory is the following.

PROBLEM 1. *Establish simple relationship between the main extrinsic invariants and the main intrinsic invariants of a submanifold.*

Several famous results in differential geometry, such as isoperimetric inequality, Chern-Lashof's inequality, and Gauss-Bonnet's theorem among others, can be regarded as results in this respect. For some recent progress in this direction, see for instances [**2–8**].

In this paper we consider isometric immersions of a Riemannian manifold into Riemannian space forms with arbitrary codimension. In Section 2 we extend the well-known notion of Ricci curvature to k-Ricci curvature for a Riemannian manifold. In Section 3 we obtain a solution to Problem 1 by establishing a sharp relationship between the k-Ricci curvatures and the shape operator for submanifolds in Riemannian space forms with arbitrary codimension. In Section 4 we obtain another solution to Problem 1 by establishing a sharp relationship between the k-Ricci curvatures and the squared mean curvature also for submanifolds in Riemannian space forms with arbitrary codimension. Results obtained in this paper can be regarded as generalizations of some results obtained in [**4**].

2. Preliminaries. Let M^n be an n-dimensional submanifold of a Riemannian space form $R^m(c)$ of constant sectional curvature c. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M^n and $R^m(c)$, respectively. Then the Gauss and Weingarten formulas of M^n in $R^m(c)$ are given respectively by

https://doi.org/10.1017/S0017089599970271 Published online by Cambridge University Press

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for vector fields X, Y tangent to M^n and ξ normal to M^n, where h denotes the second fundamental form, D the normal connection, and A the shape operator of the submanifold. The second fundamental form and the shape operator of M^n in $R^m(c)$ are related by

$$\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle, \tag{2.3}$$

The mean curvature vector H of the submanifold M^n is defined by $H = \frac{1}{n}$ trace h

Denote by R the Riemann curvature tensor of M^n. Then the equation of Gauss is given by

$$\begin{aligned} R(X, Y; Z, W) = (\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle)c \\ +\langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \end{aligned} \tag{2.4}$$

for vectors X, Y, Z, W tangent to M^n.

For a Riemannian n-manifold M^n, denote by $K(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_pM^n$, $p \in M^n$. Suppose L^k is a k-plane section of T_pM^n and X a unit vector in L^k. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$. Define the Ricci curvature Ric_{L^k} of L^k at X by

$$Ric_{L^k}(X) = K_{12} + \ldots + K_{1k}, \tag{2.5}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. We simply called such a curvature a *k-Ricci curvature*. The scalar curvature τ of the k-plane section L^k is defined by

$$\tau(L^k) = \sum_{1 \le i < j \le k} K_{ij}. \tag{2.6}$$

Let V^l be an l-plane section in a tangent space T_pM^n of a Riemannian n-manifold M^n. Then V^l is said to be *k-Einsteinian* if the k-Ricci curvatures of all k-plane sections in V^l are equal. In particular, if V^l is the whole tangent space T_pM^n at p, then M^n is said to be *k-Einsteinian* at p.

An l-plane section V^l is said to have constant sectional curvature if it is 2-Einsteinian; in this case, sectional curvatures of all 2-plane secions in V^l are equal. It follows from (2.5) that an l-plane section is 2-Einsteinian if and only if it is k-Einsteinian for some $k < l$.

A Riemannian manifold is called *k-Einsteinian* if it is k-Einsteinian at every point. Obviously, a Riemannian n-manifold is an Einstein space if it is n-Einsteinian. On the other hand, a Riemannian n-manifold is a Riemannian space form if it is k-Einsteinian for some $k < n$.

3. k-Ricci curvature and shape operator. The main purpose of this section is to obtain a solution to Problem 1 by establishing a sharp relationship between the k-Ricci curvature and the shape operator for a submanifold in a Riemannian space

https://doi.org/10.1017/S0017089599970271 Published online by Cambridge University Press

form with arbitrary codimension. In order to do so, for each integer k, $2 \leq k \leq n$, we introduce a Riemannian invariant, denoted by θ_k, on a Riemannian n-manifold M_n defined by

$$\theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{L^k, X} Ric_{L^k}(X), \qquad p \in M^n, \tag{3.1}$$

where L^k runs over all k-plane sections in T_pM and X runs over all unit vectors in L^k. Recall that for a submanifold M^n in a Rieniannian manifold, the *relative null space* of M^n at a point $p \in M^n$ is defined by

$$N_p = \{X \in T_pM^n : h(X, Y) = 0 \text{ for all } Y \in T_pM^n\}.$$

THEOREM 1. *Let $x{:}M^n \to R^m(c)$ be an isometric immersion of a Riemannian n-manifold M^n into a Riemannian space form $R^m(c)$ of constant sectional curvature c. Then for any integer k, $2 \leq k \leq n$ and any point $p \in M^n$, we have*

1. *If $\theta_k(p) \neq c$, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}(\theta_k(p) - c)I \text{ at } p, \tag{3.2}$$

where I denotes the identity map of T_pM^n.

2. *If $\theta_k(p) = c$, then $A_H \geq 0$ at p.*
3. *A unit vector $X \in T_pM$ satisfies $A_HX = \frac{n-1}{n}(\theta_k(p) - c)X$ if and only if $\theta_k(p) = c$ and X lies in the relative null space at p.*
4. *$A_H \equiv \frac{n-1}{n}(\theta_k - c)I$ at p if and only if p is a totally geodesic point, i.e., the second fundamental form vanishes identically at p.*

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM^n. Denote by $L_{i_1 \ldots i_k}$ the k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. It follows from (2.5) and (2.6) that

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} Ric_{L_{i_1 \ldots i_k}}(e_i), \tag{3.3}$$

$$\tau(p) = \frac{(k-2)!(n-k)!}{(n-2)!} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \tau(L_{i_1 \ldots i_k}). \tag{3.4}$$

Combining (3.1), (3.3) and (3.4) we find

$$\tau(p) \geq \frac{n(n-1)}{2} \theta_k(p). \tag{3.5}$$

On the other hand, Lemma 1 of [**4**] yields

$$H^2(p) \geq \frac{2}{n(n-1)} \tau(p) - c. \tag{3.6}$$

https://doi.org/10.1017/S0017089599970271 Published online by Cambridge University Press

From (3.5) and (3.6), we obtain $H^2(p) \geq \theta_k(p) - c$. This shows that $H(p) = 0$ may occur only when $\theta_k(p) \leq c$. Consequently, if $H(p) = 0$, statements (1) and (2) hold automatically. Therefore, without loss of generality, we may assume $H(p) \neq 0$. Choose an orthonormal basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ at p such that e_{n+1} is in the direction of the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator A_H. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix}, \tag{3.7}$$

$$A_r = (h^r_{ij}), \quad \sum_{i=1}^{n} h^r_{ii} = 0, \qquad r = n+2, \ldots, m. \tag{3.8}$$

From the equation of Gauss we get

$$a_i a_j = K_{ij} - c + \sum_{r=n+2}^{m} (h^r_{ij})^2 - \sum_{r=n+2}^{m} h^r_{ii} h^r_{jj}, \quad 1 \leq i \neq j \leq n, \tag{3.9}$$

From (3.9) we obtain

$$\begin{aligned} a_1(a_{i_2} + \cdots + a_{i_k}) &= Ric_{L_{1i_2\cdots i_k}}(e_1) - (k-1)c \\ &+ \sum_{r=n+2}^{m} \sum_{j=2}^{k} (h^r_{1i_j})^2 - \sum_{r=n+2}^{m} \sum_{j=2}^{k} h^r_{11} h^r_{i_j i_j}, \qquad 1 < i_2 < \cdots < i_k, \end{aligned} \tag{3.10}$$

which yields

$$\begin{aligned} a_1(a_2 + \cdots + a_n) = \frac{1}{\binom{n-2}{k-2}} & \sum_{2 \leq i_2 < \cdots < i_k \leq n} Ric_{L_{1i_2\ldots i_k}}(e_1) \\ & -(n-1)c + \sum_{r=n+2}^{m} \sum_{j=1}^{n} (h^r_{1j})^2. \end{aligned} \tag{3.11}$$

Using (3.1) and (3.11) we find

$$a_1(a_2 + \cdots + a_n) \geq (n-1)(\theta_k(p) - c), \tag{3.12}$$

with the equality holding if and only if

$$Ric_L(e_1) = 0 \text{ and } h^r_{1j} = 0, \qquad r = n+2, \ldots, m; \qquad j = 2, \ldots, n, \tag{3.13}$$

for any k-plane section L which contains e_1.

Inequality (3.12) implies

$$a_1(a_1 + \cdots + a_n) \geq (n-1)(\theta_k(p) - c) + a_1^2 \geq (n-1)(\theta_k(p) - c). \tag{3.14}$$

Similar inequalities hold when 1 were replaced by $j \in \{2, \ldots, n\}$. Hence, we have

$$a_j(a_1 + \cdots + a_n) \geq (n-1)(\theta_k(p) - c) + a_j^2, \qquad j = 1, \ldots, n, \tag{3.15}$$

which yields

$$A_H \geq \frac{n-1}{n}(\theta_k(p) - c)I. \tag{3.16}$$

Now, suppose that the equality case of (3.16) is achieved for some unit vector $X \in T_pM^n$. Then at least one of the n eigenvalues of A_H is equal to $(n-1)(\theta_k(p)-c)/n$. Without loss of generality, we may assume a_1 is such an eigenvalue. Thus, we have

$$a_1(a_1 + \cdots + a_n) = (n-1)(\theta_k(p) - c). \tag{3.17}$$

On the other hand, from (3.14) and (3.17) we obtain $a_1 = 0$ and $\theta_k(p) = c$. Moreover, in this case we also know from (3.13) that e_1 must lie in the relative null space N_p. Hence, statements (1) and (2) follow. Moreover, this also implies one part of statement (3). The remaining part of statement (3) is obvious.

Now, if $A_H = \frac{n-1}{n}(\theta_k - c)I$ identically at a point p, then every tangent vector of M^n at p lies in the relative null space N_p at p, according to statement (3). Therefore, p is a totally geodesic point. Conversely, if p is a totally geodesic point, then $\theta_k(p) = c$ and $A_H = 0$ which imply $A_H = \frac{n-1}{n}(\theta_k - c)I$ identically at a point p. Thus we have statement (4). □

REMARK 1. Clearly the estimate of A_H given in statement (2) of Theorem 1 is sharp.

Consider a hyper-ellipsoid in E^{n+1} defined by

$$ax_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1, \tag{3.18}$$

where $0 < a < 1$. The principal curvatures $a_1, \ldots, a_n$ of the hyper-ellipsold are given by (cf. [**9**])

$$a_1 = \frac{a}{(1 + a(a-1)x_1^2)^{\frac{3}{2}}}, \qquad a_2 = \cdots = a_n = \frac{1}{(1 + a(a-1)x_1^2)^{\frac{1}{2}}}. \tag{3.19}$$

Therefore, for any k, $2 \leq k \leq n$, the k-Ricci curvatures at a point p satisfies

$$Ric_{L^k}(X) \geq (k-1)\theta_k(p) := \frac{(k-1)a}{(1 + a(a-1)x_1^2)^2} > 0 \tag{3.20}$$

for any k-plane section L^k and any unit vector X in L^k and, moreover, the eigenvalues $\kappa_1, \ldots, \kappa_n$ of the shape operator A_H are given by

$$\kappa_1 = \cdots = \kappa_{n-1} = \frac{a + (n-1)(1 + a(a-1)x_1^2)}{n(1 + a(a-1)x_1^2)^2}, \qquad \kappa_n = \frac{a(a + (n-1)(1 + a(a-1)x_1^2))}{n(1 + a(a-1)x_1^2)^3}. \tag{3.21}$$

From (3.20) and (3.21) it follows that $A_H > \left(\frac{n-1}{n}\right)\theta_k(p) I_n$ and

$$\kappa_1 - \frac{n-1}{n}\theta_k(p) = \frac{a^2}{n(1 + a(a-1)x_1^2)^3} \to 0$$

as $a \to 0$.

This example shows that our estimate of A_H in statement (1) is also sharp.

One may apply Theorem 1 to obtain a lower bound of the eigenvalues of the shape operator A_H for all isometric immersions of a given Riemannian n-manifold with arbitrary codimension. For instance, Theorem 1 implies immediately the following.

COROLLARY 2. *Let $x : M^n \to E^m$ be any isometric immersion of an open portion of the unit n-sphere in a Euclidean m-space with arbitrary codimension. Then every eigenvalue of the shape operator A_H is greater than $\frac{n-1}{n}$.*

For an n-dimensional submanifold M^n in E^m let E^{n+1} be the linear subspace of dimension $n+1$ spanned by the tangent space at a point $p \in M$ and the mean curvature vector $H(p)$ at p. Geometrically, the shape operator A_{n+1} of M^n in E^m at p is the shape operator of the orthogonal projection of M^n into E^{n+1}. Moreover, it is known that if the shape operator of a hypersurface in E^{n+1} is definite at a point p, then it is strictly convex at p. For this reason a submanifold M^n in E^m is said to be *H-strictly convex* if the shape operator A_H is positive-definite at each point in M^n.

Theorem 1 implies immediately the following.

COROLLARY 3. *Let M^n be a submanifold of a Euclidean space with arbitrary codimension. If there is an integer k, $2 \le k \le n$, such that k-Ricci curvatures of M^n are positive, then M^n is H-strictly convex.*

2. Ricci curvature and squared mean curvature. In this section we give another solution to Problem 1 by establishing a sharp relationship between the k-Ricci curvature and the squared mean curvature.

THEOREM 4. *Let x: $M^n \to R^m(c)$ be an isometric immersion of a Riemannian n-manifold M^n into a Riemannian space form $R^m(c)$. Then*

1. *For each unit tangent vector $X \in T_pM^n$, we have*

$$H^2(p) \ge \frac{4}{n^2}\{Ric(X) - (n-1)c\}, \tag{4.1}$$

where $H^2 = \langle H, H\rangle$ is the squared mean curvature and $Ric(X)$ the Ricci curvature of M^n at X.

https://doi.org/10.1017/S0017089599970271 Published online by Cambridge University Press

2. *If $H(p)=0$, then a unit tangent vector X at p satisfies the equality case of (4.1) if and only if X lies in the relative null space N_p at p.*
3. *The equality case of (4.1) holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n=2$ and p is a totally umbilical point.*

Proof. Let $X \in T_pM^n$ be a unit tangent vector at p. We choose an orthonormal basis $e_1,\ldots,e_n,e_{n+1},\ldots,e_m$ such that $e_1,\ldots,e_n$ are tangent to M^n at p and $e_1=X$. Then from the equation of Gauss we have

$$n^2H^2 = 2\tau + \| h \|^2 - n(n-1)c, \tag{4.2}$$

where $\|h\|^2$ is the squared length of the second fundamental form. From (4.2) we find

$$\begin{aligned} n^2H^2 &= 2\tau + \sum_{r=n+1}^{m}\left\{(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2\sum_{i<j}(h_{ij}^r)^2\right\} \\ &\quad - 2\sum_{r=n+1}^{m}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - n(n-1)c \\ &= 2\tau + \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2\} \\ &\quad + 2\sum_{r=n+1}^{m}\sum_{i<j}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{m}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - n(n-1)c \\ &\ge 2\tau + \frac{n^2}{2}H^2 - 2\sum_{2\le i<j\le n} K_{ij} + 2\sum_{r=n+1}^{m}\sum_{j=2}^{n}(h_{1j}^r)^2 - 2(n-1)c, \end{aligned} \tag{4.3}$$

which implies

$$n^2H^2 \ge 4(Ric(e_1) - (n-1)c). \tag{4.4}$$

Since $e_1 = X$ can be chosen to be any arbitrary unit tangent vector at p, we obtain statement (1).

From (4.3) we know that the equality case of (4.4) holds if and only if

$$h_{12}^r = \cdots = h_{1n}^r = 0 \text{ and } h_{11}^r = h_{22}^r + \cdots + h_{nn}^r, \qquad r = n+1, \ldots, m. \tag{4.5}$$

If $H(p)=0$, (4.5) implies that $e_1 = X$ lies in the relative null space N_p at p. Conversely, if $e_1 = X$ lies in the relative null space, then (4.5) holds automatically, since $H(p)=0$. This proves statement (2).

Now, assume that the equality case of (4.1) holds identically for all unit tangent vectors at p. Then, for any $r=n+1,\ldots,m$, we have

$$h_{ij}^r = 0, \qquad i \ne j, \tag{4.6}$$

$$h_{11}^r + \cdots + h_{nn}^r - 2h_{ii}^r = 0, \qquad i = 1, \ldots, n. \tag{4.7}$$

Conditions (4.6) and (4.7) imply that either p is a totally geodesic point or $n=2$ and p is a totally umbilical point. The converse of this is trivial. Thus, we also have statement (3). □

Theorem 4 can be applied to obtain sharp estimates of the squared mean curvature for some submanifolds with arbitrary codimensions. For instance, Theorem 4 implies immediately the following.

COROLLARY 5. *Let $x: M^n \to E^m$ be an isometric immersion of a Riemannian n-manifold M^n in a Euclidean m-space with arbitrary codimension. Then*

$$H^2(p) \geq \left(\frac{4}{n^2}\right) \max_X Ric(X) \tag{4.8}$$

where X runs over all unit tangent vectors at p.

REMARK 2. There exist many examples of submanifolds in a Euclidean m-space which satisfy the equality case of (4.8) identically. Two simple examples are spherical hypercylinder $S^2(r) \times R$ and round hypercone in E^4.

REMARK 3. In general, given an integer k, $2 \leq k \leq n-1$, there *does not* exist a positive constant, say $C(n, k)$, such that

$$H^2(p) \geq C(n, k) \max_{L^k, X} Ric_{L^k}(X) \tag{4.9}$$

where L^k runs over all k-plane sections in T_pM^n and X runs over all unit tangent vectors in L^k. This fact can be seen from the following example:

Let $x: M^3 \to E^4$ be a minimal hypersurface whose shape operator is non-singular at some point $p \in M^3$. Then by the minimality there exist two principal directions at p, say e_1, e_2, such that their corresponding principal curvatures κ_1, κ_2 are of the same sign. This implies that the sectional curvature K_{12} at p is positive. Now, consider the minimal hypersurface in E^{n+1} which is given by the product of $x: M^3 \to E^4$ and the identity map $\iota: E^{n-3} \to E^{n-3}$. It is clear that, for any integer k, $2 \leq k \leq n-1$, the maximum value of the k-th Ricci curvatures of $M^n := M^3 \times E^{n-3}$ at a point (p,q), $q \in E^{n-3}$ is given by $K_{12} = \kappa_1 \kappa_2$ which is positive. Since $H=0$, this shows that there does not exist any positive constant $C(n,k)$ which satisfies (4.9).

On the contrary, by applying Theorem 4 we have the following relationship between the minimum value of the k-Ricci curvatures and the squared mean curvature for submanifolds with arbitrary codimensions.

COROLLARY 6. *Let $x: M^n \to R^m(c)$ be an isometric immersion of a Riemannian n-manifold M^n in a Riemannian space form $R^m(c)$ of constant sectional curvature c. Then, for any integer k, $2 \leq k \leq n$, we have*

$$H^2(p) \geq \frac{4(n-1)}{n^2} \left(\frac{\theta_k(p)}{k-1} - c\right), \tag{4.10}$$

where θ_k is the Riemannian invariant on M^n introduced in (3.1).

https://doi.org/10.1017/S0017089599970271 Published online by Cambridge University Press

The equality case of (4.1) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $k=n=2$ and p is a totally umbilical point.*

REFERENCES

1. B. Y. Chen, *Geometry of submanifolds* (Marcel Dekker, New York, 1973).

2. B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Archiv der Math*. **60** (1993), 568–578.

3. B. Y. Chen, A Riemannian invariant and its applications to submanifold theory, *Results Math*. **27** (1995), 17–26.

4. B. Y. Chen, Mean curvature and shape operator of isometric immersions in real-space-form, *Glasgow Math. J*. **38** (1996), 87–97.

5. B. Y. Chen, A report on submanifolds of finite type, *Soochow J. Math*. **22** (1996), 117–337.

6. P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, The normal curvature of totally real submanifolds of $S^6(1)$, *Glasgow Math. J*. **40** (1998), 199–204.

7. P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, A pointwise inequality in submanifold theory, *Arch. Math. (Brno)*, to appear.

8. P. F. Leung, On a relation between the topology and the intrinsic and extrinsic geometries of a compact submanifold, *Proc. Edinburgh Math. Soc*. **28** (1985), 305–311.