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Chaotic group actions

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Chaotic dynamical systems have been the subject of intensive research in recent years. In this thesis we develop the theory of chaotic group actions. Our research is motivated by the fact that although in its most general setting a dynamical system is defined as an action of a group on some space, most of the work in chaotic dynamics has been concerned with the iteration of single maps (discrete dynamical systems); in other words, with group action of the additive group $\mathbb{Z}$ or the semigroup $\mathbb{Z}^+$.

An increasingly popular defintion of chaos is due to Devaney (see [**5**]). A map f is said to be chaotic on a metric space X if it satisfies the following conditions:

(1) f has *sensitive dependence on initial conditions*; that is, there exists $\delta > 0$ such that for any $x \in X$ and any open neighbourhood U of x, there exists $y \in U$ and $n \geqslant 0$ such that $d(f^n(x), f^n(y)) > \delta$,

(2) f is *topologically transitive*; that is, for any pair of non-empty open disjoint sets U and V there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$,

(3) f has a *dense set of periodic points*; that is the set $\{x \in X : f^n(x) = x$ for some $n \in \mathbb{N}\}$ is dense in X.

These conditions reflect respectively "unpredictability, indecomposability, and an element of regularity" [**5**] of a chaotic system. It has been more recently shown that topological transitivity and a dense set of periodic orbits imply sensitive dependence in any metric space [**1**].

As a natural generalization of Devaney's definition of chaotic maps, we introduce in this thesis the notion of chaotic action of arbitrary groups: an action is *chaotic* if it is topologically transitive and the set of points with finite orbits is dense. This notion is not merely an artificial generalization of Devaney's definition, since there exist chaotic actions of a group G for which the restriction to every one generator subgroup is not chaotic (Exampe 3.4, Chapter III).

There are some questions which immediately come to mind when one familiarizes oneself with the definition of chaotic group action: which groups can have interesting

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chaotic dynamics on some Hausdorff space; which spaces admit chaotic actions of some group? We give answers to both questions. In answering the first question we show that a group G possesses a faithful chaotic action on some Hausdorff space if and only if G is residually finite (Theorem 1.9, Chapter I). In fact, the density of finite orbits of a chaotic action, the condition in the definition of chaos which reflects "an element of regularity" of a chaotic system, is just a disguised form of residual finiteness. This result is especially interesting since residually finite groups are a well studied class of groups; they often occur in geometrical problems and have interesting group-theoretical properties. We remark in passing that the above result also provides an elementary and unified method to prove residual finiteness of groups in many cases.

We give an answer to the second question by showing that every compact triangulable n-manifold of dimension greater that 1 admits a faithful chaotic action of every countably generated free group (Theorem 3.1, Chapter III). It is known that every smooth compact manifold of dimension greater that 1 admits a topologically transitive smooth map (in fact, a Bernoulli diffeomorphism) [**2**]. However, it is not clear if there is a chaotic Bernoulli diffeomorphism on every manifold. In fact, there are known examples of Bernoulli diffeomorphisms which do not have any periodic points at all. Our result amounts to a particular construction of a chaotic action of a countably generated free group for which the rational points have periodic orbits. Previously the result was apparently unknown even in the special case of $\mathbb{Z}$-actions.

A large part of the thesis deals with constructions of chaotic actions. We provide constructions of chaotic actions of products (direct and wreath) of groups on product spaces (Chapters I and IV). We study chaotic actions of a group and its quotient groups on quotient spaces (Chapter II). We lift a chatoic action of a group on some space X to a chaotic action of an extension of the group on a covering space of X (Chapter II). These constructions prove to be useful since they allow one to build up examples of various chaotic group actions on a wide range of domains. This part of the thesis does not have an analogue in the theory of chaotic maps for the obvious reasons.

To obtain stronger version of the definition of a chaotic group action one can substitute the second condition in the definition by a topological property which is stronger than transitivity. We introduce the following properties: strong mixing, weak mixing, strong transitivity and total transitivity (Chapter IV). Our definitions are generalizations of corresponding mixing properties of maps and flows. We give a detailed study of the relations between these properties. It turns out that although Abelian group actions enjoy the same system of implications which hold for $\mathbb{Z}$-actions (as one might expect), for general group actions this is not the case.

In topological dynamics there are two slightly different approaches to studying group actions. One approach is to study an action of a group G by homeomorphisms on a topological space M; that is, the action defined by the homomorphsm $\phi : G \to Homeo(M)$. Another approach is to study an action of a transformation group G on M; that is the action of a topological group G defined by the homomorphism $\phi : G \to Homeo(M)$ with the additional assumptions that the map $f : G \times M \to M$ is continuous. Clearly, a group G acting on a space M by homeomorphisms can be made into a transformation group by equipping it with the discrete topology. In our study we adopt the first, more general, approach. So, we do not usually assume any topological hypothesis on the groups. However we make one exception at the end of Chapter IV, to study a natural Hausdorff topology associated with residually finite groups, the *profinite* topology. We show that when a group G is given the profinite topology and the action of G is chaotic on a complete metric space M then the function $f : G \times M \to M$ is discontinuous; that is, G is not a transformation group (Theorem 4.28, Chapter IV).

The thesis is supplemented by the Appendix, where we give a short review on Devaney's definition of a chaotic map, as well as other interesting definitions, and where we briefly discuss their potential for generalization.

Some of the results of the thesis have appeared in [**3**] and [**4**].

References

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