SOME REMARKS ON TRANSFORMATIONS IN METRIC SPACES

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1. Introduction. Recently A. Haimovici [1] has proved a general fixed point theorem of transformations in metric spaces from which he obtained existence theorems for certain types of ordinary and partial differential equations. However, both the result and the proof are given for a rather special case. One of the purposes of this present note is to put his result on a more concrete basis and give a stronger characterization of the kind of transformations used in [1]. (Theorem 3).

In proving the existence of solutions to general ordinary and partial differential equations, the method of successive approximations is commonly used. (Examples for ordinary differential equations are numerous. For partial differential equations, see for example [2].) It hence seems desirable to consider transformations for which every sequence of iterates converges, to a certain unique fixed point. In certain interesting cases, these transformations are in fact contractions. (We refer to [3] in case of ordinary differential equations and to [4] for the corresponding case of partial differential equations.) It is interesting to note that these transformations form various subspaces of the space of all bounded continuous transformations in the given metric space. We present similar characterizations for these transformations as in the case of the fixed point transformations introduced in [1]. Finally, along the same line we present an open problem relating this work to the converse of the contraction mapping principle.

2. The space of bounded continuous transformations. Let X be a complete metric space with metric denoted by ρ, and T a transformation of X into itself. We call T a bounded transformation if $T(X)$ is bounded. Denote by X^*

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the set of all bounded continuous transformations of X into itself, and define a metric d on X^* by:

$$d(A, B) = \sup_{x \in X} \rho(Ax, Bx) \qquad (*)$$

for any pair of elements $A, B \in X^*$. Clearly d is finite and satisfies the following:

(i) $d(A, B) = 0$ if and only if $A = B$,

(ii) $d(A, B) = d(B, A)$,

(iii) $d(A, B) \leq d(A, C) + d(C, B)$,

for arbitrary $A, B, C \in X^*$. In fact, we have the following statement:

THEOREM 1. X^* is a complete metric space with metric defined by (1).

Proof. Let $\{A_n\}$ be a Cauchy sequence in X^*. From the definition of d, we conclude that for each $x \in X$, $\{A_n x\}$ is again a Cauchy sequence in X and hence has a limit, say $\bar{x}$. Define the transformation A by $\bar{x} = Ax$ for each $x \in X$, and observe that for any $x, y \in X$, we have:

$$\rho(Ax, Ay) \leq \rho(Ax, A_n x) + \rho(A_n x, A_n y) + \rho(A_n y, Ay) .$$

Since each A_n is bounded and continuous and $\{A_n x\}$ tends to Ax for each $x \in X$, we conclude that A is bounded and continuous. This completes the proof.

3. Fixed point transformations. A transformation T defined on X into itself is said to have a fixed point $x_o \in X$ if and only if $Tx_o = x_o$. Denote by Φ the set of all bounded continuous transformations which have at least one fixed point. In terms of our terminologies, the main result in [1] may be stated as follows:

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THEOREM 2. $\bar{\Phi}$ is closed in X^*.

Proof. Let $\{A_n\}$ be a Cauchy sequence in $\bar{\Phi}$, and denote its limit in X^* by A. For each n, let x_n be some fixed point of A_n, i.e., $A_n x_n = x_n$. Since we always have $\rho(x_n, x_m) \leq d(A_n, A_m)$, it follows that $\{x_n\}$ is also a Cauchy sequence in X. Denote by $\bar{x}$ the limit of $\{x_n\}$ and observe that

$$\rho(A\bar{x}, \bar{x}) \leq \rho(A\bar{x}, Ax_n) + \rho(Ax_n, A_n x_n) + \rho(A_n x_n, \bar{x})$$

$$\leq \rho(A\bar{x}, Ax_n) + d(A, A_n) + \rho(x_n, \bar{x}) .$$

Since every term on the right tends to zero as n tends to infinity, $A \in \bar{\Phi}$ and the proof is complete.

THEOREM 3. $A \in \bar{\Phi}$ if and only if:

(i) There exists a sequence $A_n \in X^*$ such that $d(A_n, A) \to 0$ as $n \to \infty$; and

(ii) There exists a compact sequence $\{x_n\}$ in X such that $\rho(A_n x_n, x_n) < \frac{1}{n}$. (Recall a compact sequence is one which contains a convergent subsequence.)[1)]

Proof. The conditions are clearly necessary. To show sufficiency, let $\{x_{n_i}\}$ be the convergent subsequence contained in $\{x_n\}$ and $\bar{x}$ denote its limit. We claim that $A\bar{x} = \bar{x}$. To see this, we write down the following estimate:

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1) The necessary and sufficient conditions given here are motivated from the work of J. V. Wehausen [5].

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$$\rho(A\bar{x},\bar{x}) \leq \rho(A\bar{x}, Ax_{n_i}) + \rho(Ax_{n_i}, Ax_{n_i}x_{n_i}) + \rho(A_{x_{n_i}}, x_{n_i}) + \rho(x_{n_i}, \bar{x})$$

and observe that each of the terms on the right tends to zero as i tends to infinity. Hence $A \in \Phi$.

4. Transformations with property (S, x_f).[2] A transformation T is said to have property (S, x_f) if for each $x \in X$, $\rho(T^n x, x_f) \to 0$ as $n \to \infty$ for a unique fixed element $x_f \in X$. Denote by Σ_{x_f} the set of all bounded continuous transformations which have property (S, x_f); and Σ the set of T's having property (S, x_f) for some $x_f \in X$. It is evident that Σ_{x_f}'s separate Σ into disjoint subsets in the sense that $\Sigma_{x_f} = \Sigma_{x'_f}$ if and only if $x_f = x'_f$. A sequence of mappings $\{A_n\}$ in X^* is called a C-sequence if for every $\epsilon > 0$, there correspond two positive numbers $\delta(\epsilon)$ and $N(\epsilon)$ such that $\rho(x, y) < \delta(\epsilon)$ implies $\rho(A_n x, A_n y) < \epsilon$ for every $n \geq N(\epsilon)$. A subset $E \subset X^*$ is called C-closed in X^* if every Cauchy C-sequence converges to some point in E. From the definition of Σ_{x_f} it is clear that $\Sigma_{x_f} \subset \Phi$. Moreover we have the following description for Σ_{x_f}.

THEOREM 4. Σ_{x_f} is C-closed in X^*.

Proof. Let $\{A_n\}$ be a Cauchy C-sequence in Σ_{x_f}, and denote its limit in X^* by A. We now wish to estimate the following expression:

$$\rho(A^n x, x_f) \leq \rho(A^n x, A_m^n x) + \rho(A_m^n x, x_f)$$

for arbitrary $x \in X$. Let $\epsilon > 0$ be given. Since $\{A_m\}$ is a

2) For a discussion on C-sequences, see A. Edrei [6].

C-sequence, the sequence of n^{th} iterates $\{A_m^n\}$ again forms a C-sequence. Note that for each m, A_m^n converges to the constant mapping T defined by $Tx = x_f$ for all $x \in X$. Hence we may choose $m \geq M(\epsilon)$ such that $\rho(A^n x, A_m^n x) < \frac{\epsilon}{2}$ for all $n \geq N_1(\epsilon)$. Now $A_m^n \in \Sigma_{x_f}$ implies that there exists $N_2(\epsilon, m)$ such that $n \geq N_2(\epsilon, m)$ implies $\rho(A_m^n x, x_f) < \frac{\epsilon}{2}$. Choose $n \geq \mathrm{Max}(N_1, N_2)$, and obtain $\rho(A^n x, x_f) < \epsilon$. Since ϵ is arbitrary, this completes the proof.

THEOREM 5. $A \in \Sigma_{x_f}$ if and only if:

(i) there exists a C-sequence $A_n \in X^*$ such that $d(A_n, A) \to 0$ as $n \to \infty$;

(ii) there exists a sequence $\{x_n\}$ in X, $\rho(x_n, x_f) \to 0$ as $n \to \infty$ and $\rho(A_n^m x_n, x_n) < \epsilon_n^m \downarrow 0$ as $n \to \infty$ for each iterate A_n^m; and

(iii) there exists a convergent subsequence $\{A_{n_i}\} \subset \{A_n\}$ such that $\rho(A_{n_i}^m x, A_{n_i}^m x_{n_i}) < \frac{1}{m}$ for i sufficiently large.

Proof. The conditions are clearly necessary. To show sufficiency, we consider the following inequality:

$$\rho(A^m x, x_f) \leq \rho(A^m x, A_{n_i}^m x) + \rho(A_{n_i}^m x, A_{n_i}^m x_{n_i}) + \rho(A_{n_i}^m x_{n_i}, x_{n_i}) + \rho(x_{n_i}, x_f)$$

for an arbitrary $x \in X$. Let $\epsilon > 0$ be given. By (iii), we may

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choose, for sufficiently large i (say $i \geq I_1$), a number $M(\epsilon)$ such that $m \geq M(\epsilon)$ implies $\rho(A^m_{n_i} x, A^m_{n_i} x_{n_i}) < \frac{\epsilon}{4}$. Since $\{A_n\}$ is a C-sequence, so is $\{A^m_n\}$ for each m. Hence we may choose $I_2(\epsilon, M(\epsilon))$ such that $i \geq I_2$ implies $\rho(A^m x, A^m_{n_i} x) < \frac{\epsilon}{4}$ for all $m \geq M(\epsilon)$. From (ii) it follows that there exists a number $I_3(\epsilon, M)$ such that $i \geq I_3$ implies $\rho(A^m_{n_i} x_{n_i}, x_{n_i}) < \frac{\epsilon}{4}$ for all $m \geq M(\epsilon)$. Finally, since $\rho(x_n, x_f) \to 0$ as $n \to \infty$, we can choose $i \geq I_4(\epsilon)$ such that $\rho(x_{n_i}, x_f) < \frac{\epsilon}{4}$ holds. Let $i \geq \mathrm{Max}(I_1, I_2, I_3, I_4)$ and $m \geq M(\epsilon)$. We conclude that $\rho(A^m x, x_f) < \epsilon$. Since ϵ is arbitrary and $M(\epsilon) \to \infty$ as $\epsilon \to 0$, this completes the proof.

5. <u>λ-contractive transformations.</u>[3)] A transformation T is called <u>λ-contractive</u> if there exists a constant λ, $0 \leq \lambda < 1$, such that for every pair of $x, y \in X$, $\rho(Tx, Ty) \leq \lambda \rho(x, y)$ (λ is independent of T). As a consequence of the contraction mapping principle (see for example [3]), every λ-contractive transformation T belongs to Σ_{x_f} for some element $x_f \in X$, which is the unique fixed point of T. Denote by Λ_{x_f} the set of all λ-contractive transformations having x_f as their fixed point, and Λ the set of all λ-contractive transformations in X^*. Similarly, Λ_{x_f}'s separate Λ into disjoint subsets in the sense that $\Lambda_{x_f} = \Lambda_{x'_f}$ if and only if $x_f = x'_f$. Also, if $\mathcal{M}$ denotes the set of all μ-contractive transformations in X^*, where $0 \leq \mu < 1$, and $\lambda \leq \mu$, then $\Lambda \subset \mathcal{M}$. Finally, it is again a routine matter to show the following statement.

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3) This terminology is adopted here after [7]. In fact this is called uniformly λ-contractive in [7].

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THEOREM 6. Λ is closed in X^*.

6. A problem on contractions. Denote the given complete metric space by the pair (X, ρ). For every equivalent metric $\tilde{\rho}$ we may associate as in section 2 the space $(X^*, \tilde{d})$, where $\check{d}$ is defined in terms of $\tilde{\rho}$ by (*). The sets $(\Phi, \check{d})$, $(\Sigma, \tilde{d})$ and $(\Lambda, \check{d})$ are defined similarly, and satisfy the inclusion statement: $\Lambda \subset \Sigma \subset \Phi \subset X^*$.

Along an entirely different approach, the author proposed the following problem in [7]: "What are the necessary and sufficient conditions on T (or on X, or both) such that $T \in (\Sigma, d) \cap (\Lambda, \tilde{d})$ for some $\tilde{d}$?" The statement is obviously false without imposing any condition at all. The following example shows that the statement is not true even if X is a compact metric space:[3)]

Let X be the circumference of the unit circle on the complex plane with the ordinary Euclidean metric, and T be the mapping which maps $z = e^{i\theta}$ into $Tz = e^{i\varphi}$ where $\varphi = (\frac{\theta}{2\pi})^2$ and θ is restricted to $0 \leq \theta < 2\pi$. It is clear that $T \in \Sigma_{z_o}$ where $z_o = (1, 0)$ and that $T \notin \Lambda_{z_o}$ for any λ $0 \leq \lambda < 1$. Assume there exists a metric $\tilde{\rho}$ equivalent to the Euclidean metric on X such that for every pair $x, y \in X$, $\tilde{\rho}(Tx, Ty) \leq \lambda\, \tilde{\rho}(x, y)$ for some λ, $0 \leq \lambda < 1$. Now consider the unique inverse of T defined by $T^{-1}z = e^{i\psi}$, $\psi = \sqrt{\frac{\theta}{2\pi}}$, and observe that for any $x, y \in X$, $x \neq y$ we have:

$$\rho(x, y) \leq \lambda^n\, \tilde{\rho}(T^{-n}x, T^{-n}y) \leq M\, \lambda^n$$

where M is the diameter of X under $\tilde{\rho}$. Letting n tend to infinity we arrive at the desired contradiction.

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4) Cf. L. Janos [8], [9].

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