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A NOTE ON STRESS CONCENTRATION AROUND AN ELLIPTIC HOLE IN MICROPOLAR ELASTICITY

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Abstract

Within the scope of Eringen's linearised micropolar theory, this note outlines a solution for the stress concentration around an elliptic hole in an infinite plate under axial tension.

1. Introduction

In classical linear elasticity theory, the solutions for the stress concentration around circular and elliptic holes are well known [2]. For elastic materials with microstructure, the fundamental system of field equations includes a couple-stress tensor, in addition to the force-stress tensor. Eringen [1] has formulated the tensor equations for elastic materials with microstructure. Using an approach based on stress functions, the problem of stress concentration around a circular hole has been solved [4]. Here, instead of using stress functions we use a method based on the Helmholtz representation of the displacement vector $\vec{u}$, and the micro-rotation vector $\vec{\phi}$, as sums of two fields, one with a scalar and the other a vector potential.

2. Outline of method

Following Eringen [1], the basic equations for a micro-polar elastic solid in the absence of inertia forces are

$$(\lambda + 2\mu + \kappa)\nabla\nabla . \vec{u} - (\mu + \kappa)\nabla \times \nabla \times \vec{u} + \kappa \nabla \times \vec{\phi} = 0,$$

$$(\alpha + \beta + \gamma)\nabla\nabla . \vec{\phi} + \gamma \nabla \times \nabla \times \vec{\phi} + \kappa \nabla \times \vec{u} - 2\kappa\vec{\phi} = 0. \tag{2.1}$$

The constants λ, μ, κ, α, β and γ denote the material constants of the micro-polar elastic solid. Using

$$\vec{u} = \nabla u_0 + \nabla \vec{U},$$

$$\vec{\phi} = \nabla \phi_0 + \nabla \times \vec{\Phi}, \tag{2.2}$$

subject to conditions

$$\nabla . \vec{U} = 0, \qquad \nabla . \vec{\Phi} = 0,$$
$$\phi_0 = 0, \qquad \vec{\phi} = \phi \vec{e}_3,$$
$$l^2 = \frac{\gamma(\kappa+\mu)}{\kappa(\kappa+2\mu)}, \tag{2.3}$$

the equations (2.1) reduce to

$$\nabla^4 u_0 = 0,$$
$$\nabla^4 \phi - \frac{1}{l^2}\nabla^2 \phi = 0,$$
$$\nabla^6 U - \frac{1}{l^2}\nabla^4 U = 0. \tag{2.4}$$

We consider now an infinite plate in a state of uniform tension S (along the x-direction) disturbed by an elliptical hole of semiaxes a and b parallel to the x- and y-axes, which is free from stress. As the boundary of the hole is an ellipse, it is advantageous to use elliptic coordinates (ξ, η) given by $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\cos\eta$.

As the hole is stress free, and the plate is subjected to uniform tension S at infinity, we have as boundary conditions

$$t_\xi^\xi = t_\eta^\xi = 0 \text{ on the hole } \xi = \xi_0, \qquad m_z^\xi = 0 \text{ on } \xi = \xi_0,$$
$$t_\xi^\xi = \frac{S}{2}(1+\cos 2\eta), \qquad t_\eta^\xi = -\frac{S}{2}\sin 2\eta \text{ at } \xi = \infty, \tag{2.5}$$

where m_z^ξ is the couple-stress and t_1^k is the stress-tensor.

In elliptic coordinates (ξ, η) the general solutions of the equations (2.4) are given by

$$u_0 = \phi_0 + \sum_{n=1}^{\infty} \phi_{2n}\cos 2n\eta,$$
$$\phi = \sum_{n=1}^{\infty} [A_{2n}e^{-2n\xi}\sin 2n\eta + B_{2n}\,\mathrm{Gek}_{2n}(\xi, -q_1)\mathrm{se}_{2n}(\eta, -q_1)],$$
$$U = \sum_{n=1}^{\infty} [\psi_{2n}\sin 2n\eta + C_{2n}\,\mathrm{Gek}_{2n}(\xi, -q_1)\mathrm{se}_{2n}(\eta, -q_1)], \tag{2.6}$$

where $\mathrm{se}_{2n}(\eta, -q_1)$ is the periodic Mathieu function of order $2n$ and $\mathrm{Gek}_{2n}(\xi, -q_1)$ is a modified Mathieu function involving K-Bessel functions [5]. Further, $q_1 = c^2/4l^2$ and A_{2n}, B_{2n}, and C_{2n} are constants which are determined from the boundary conditions. Also

$$\phi_0 = a_0 e^{-2\xi} + c_0\xi,$$

$$\phi_{2n} = a_{2n-2}e^{-(2n-2)\xi} + a_{2n}e^{-(2n+2)\xi} + a'_{2n}e^{-2n\xi} \qquad (n \geqq 1),$$

$$\psi_0 = b_0 e^{-2\xi} + c'_0\xi,$$

$$\psi_{2n} = b_{2n-2}e^{-(2n-2)\xi} + b_{2n}e^{-(2n+2)\xi} + b'_{2n}e^{-2n\xi} \qquad (n \geqq 1).$$

The unknown coefficients a_{2n}, b_{2n}, a'_{2n}, b'_{2n}, b_0, c_0, etc. are determined from the boundary condition (2.5), by solving an infinite set of linear equations.* The stress components in elliptic coordinates are given by

$$\begin{aligned}
\frac{c^2}{2}(\cosh 2\xi - \cos 2\eta)^2 t^{\xi}_{\xi} = {} & \lambda(\cosh 2\xi - \cos 2\eta)\left(\frac{\partial^2 u_0}{\partial \xi^2} + \frac{\partial^2 u_0}{\partial \eta^2}\right) \\
& + (2\mu + \kappa)\left[(\cosh 2\xi - \cos 2\eta)\left(\frac{\partial^2 u_0}{\partial \xi^2} + \frac{\partial^2 U}{\partial \xi \partial \eta}\right)\right. \\
& \left. - \sinh 2\xi\left(\frac{\partial u_0}{\partial \xi} + \frac{\partial U}{\partial \eta}\right) + \sin 2\eta\left(\frac{\partial u_0}{\partial \eta} - \frac{\partial U}{\partial \xi}\right)\right], \\
\frac{c^2}{2}(\cosh 2\xi - \cos 2\eta)^2 t^{\eta}_{\eta} = {} & \lambda(\cosh 2\xi - \cos 2\eta)\left(\frac{\partial^2 u_0}{\partial \xi^2} + \frac{\partial^2 u_0}{\partial \eta^2}\right) \\
& + (2\mu + \kappa)\left[(\cosh 2\xi - \cos 2\eta)\left(\frac{\partial^2 u_0}{\partial \eta^2} - \frac{\partial^2 U}{\partial \xi \partial \eta}\right)\right. \\
& \left. + \sinh 2\xi\left(\frac{\partial u_0}{\partial \xi} + \frac{\partial U}{\partial \eta}\right) - \sin 2\eta\left(\frac{\partial u_0}{\partial \eta} - \frac{\partial U}{\partial \xi}\right)\right], \\
\frac{c^2}{2}(\cosh 2\xi - \cos 2\eta)^2 t^{\xi}_{\eta} = {} & \mu(\cosh 2\xi - \cos 2\eta)\left(2\frac{\partial^2 u_0}{\partial \xi \partial \eta} - \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2}\right) \\
& - (2\mu + \kappa)\sin 2\eta\left(\frac{\partial u_0}{\partial \xi} + \frac{\partial U}{\partial \eta}\right) \\
& - (2\mu + \kappa)\sinh 2\xi\left(\frac{\partial u_0}{\partial \eta} - \frac{\partial U}{\partial \xi}\right) \\
& - \kappa\frac{c^2}{2}(\cosh 2\xi - \cos 2\eta)^2\phi. \qquad (2.7)
\end{aligned}$$

Let T' denote the solution (2.6) with the boundary conditions:

(i) uniformly pressurised hole on $\xi = \xi_0$,

(ii) $t^{\xi}_{\xi} = t^{\xi}_{\eta} = m^{\xi}_{z} = 0$ at $\xi = \infty$.

Let $\bar{T}$ denote the solution (2.6) with the undisturbed uniform field of uni-axial tension (i.e. as if there is no hole) $t^{x}_{x} = S$.

* Further details are available on request to the author.

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If T denotes the required solution of the problem defined by (2.4) and (2.5) then we have

$$T = T' + \bar{T}.$$

The solution $\bar{T}$ is given by

$$t_\xi^\xi = S \sinh^2\xi \frac{1+\cos 2\eta}{\cosh 2\xi - \cos 2\eta},$$

$$t_\eta^\xi = -\frac{S}{2}\sinh 2\xi \frac{\sin 2\eta}{\cosh 2\xi - \cos 2\eta},$$

$$t_\eta^\eta = S \cosh^2\xi \frac{1-\cos 2\eta}{\cosh 2\xi - \cos 2\eta},$$

$$m_z^\xi = 0. \tag{2.8}$$

We are only interested in the stress concentration factor $t_\eta^\eta|_{\eta=\pi/2}$ on the hole $\xi = \xi_0$, and this reduces to

$$t_\eta^\eta\Big|_{\substack{\xi=\xi_0\\ \eta=\pi/2}} = S \cosh^2\xi \frac{1-\cos 2\eta}{\cosh 2\xi - \cos 2\eta}\Big|_{\substack{\xi=\xi_0\\ \eta=\pi/2}} = S. \tag{2.9}$$

Similarly, using the solution T', the stress concentration factor reduces to

$$t_\eta^\eta\Big|_{\substack{\xi=\xi_0\\ \eta=\pi/2}} = \frac{2Sr_2/r_1}{1+M_0}, \tag{2.10}$$

where

$$M_0 = \frac{\left(\frac{r_2}{r_1}\right)^2 - 1 + 32d(1-\nu')\frac{b'^2}{r_1^2}}{\frac{2}{r_1}\left(1+\frac{r_2}{r_1}\right)},$$

$$d = 1 + \frac{2\mathrm{Gek}_{2n}(\xi_0, -q_1)}{\mathrm{Gek}'_{2n}(\xi_0, -q_1)}, \qquad r_1 = c\cosh\xi_0, \qquad r_2 = c\sinh\xi_0,$$

$$b'^2 = \frac{\gamma}{2(2\mu+\kappa)}, \qquad \nu' = \frac{\lambda}{2\lambda+2\mu+\kappa},$$

and the prime in Gek'_{2n} denotes differentiation.

Using the complete solution T, the stress concentration factor is given by

$$t_\eta^\eta\Big|_{\substack{\xi=\xi_0\\ \eta=\pi/2}} = S\left(1 + \frac{2r_2/r_1}{1+M_0}\right). \tag{2.11}$$

Transition to circular case

As the ellipse of semi-axes r_1 and r_2 tends to a circle with radius 'a', $\xi_0 \to \infty$, $c \to 0$ such that $r_1, r_2 \to a$ we get from (2.11)

$$t^\eta_\eta \bigg|_{\substack{\eta=\pi/2 \\ r_1, r_2 = a}} = \frac{3+M_0}{1+M_0} S,$$

which becomes $3S$ in the classical case where $M_0 = 0$.

Conclusion

It is evident that the stress concentration factor proves to be smaller than in the classical theory (as $M_0 > 0$), depending on the elastic constants of the microstructure of the media. These results are important for solids composed of dumbell macro-molecules, such as fibrous and coarse grain structure materials.

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