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A REMARK ON CONVOLUTION WITH MEASURES SUPPORTED ON CURVES

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ABSTRACT. Let λ be a certain measure supported on a curve in $\mathbf{R}^3$. We prove that if the curve has nonzero curvature and torsion, then $\lambda * L^{3/2} \subset L^2$.

1. Let $I \subset \mathbf{R}^1$ be a closed interval and let $\mathbf{C}$ be a curve in $\mathbf{R}^3$ defined by

$$\mathbf{C}: t \in I \longrightarrow \big(t, \phi_1(t), \phi_2(t)\big) \in \mathbf{R}^3,$$

where ϕ_1 and ϕ_2 are real-valued functions. Let λ be the measure on $\mathbf{R}^3$ defined by

$$\langle \lambda, g \rangle = \int_I g\big(t, \phi_1(t), \phi_2(t)\big)\, dt.$$

Clearly, λ is a singular measure and is supported on the curve $\mathbf{C}$. The following theorem was proved in [1].

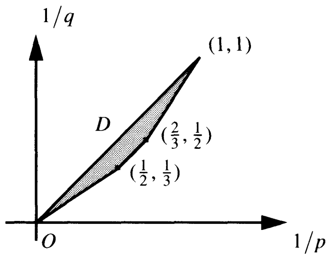
THEOREM 0 (OBERLIN, [1]). *Let $\phi(t) = \big(\phi_1(t), \phi_2(t)\big)$ and $\phi^{(j)}$ be the j-th derivative of ϕ. Suppose that ϕ satisfies the following conditions:*

(i) Given any $t_1, t_2 \in I$, the vectors $\phi^{(2)}(t_1)$ and $\phi^{(3)}(t_2)$ span $\mathbf{R}^2$.

(ii) ϕ_1 and ϕ_2 are both polynomial functions or both trigonometric functions on I.

Then $\lambda * L^{3/2} \subset L^2$.

The L^p spaces that appeared in the theorem are spaces of functions on $\mathbf{R}^3$. By duality and interpolation (with the trivial $L^1 \longrightarrow L^1$ and $L^\infty \longrightarrow L^\infty$ estimates), one is able to get $\lambda * L^p \subset L^q$, whenever $(1/p, 1/q)$ lies inside or on the boundary of the following region D:



The measure λ is said to be *L^p-improving*. For more on the background of this problem, see [1].

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Condition (i) in Theorem 0 seems to be a natural geometric condition. It is a little bit stronger than saying that the curve **C** has nonzero curvature and torsion. The condition that the curve **C** has nonzero curvature and torsion at every point means precisely that $\{\phi^{(2)}(t), \phi^{(3)}(t)\}$ spans $\mathbf{R}^2$, for every $t \in I$.

However, condition (ii) is not a natural condition, because, for one thing, it is not invariant under diffeomorphism. It also makes the scope of application of the theorem rather limited.

The purpose of this note is to prove the result of Theorem 0 without imposing condition (ii). Namely, we have the following:

THEOREM 1. *Suppose $\phi_1, \phi_2 \in C^3(I)$. Let λ and ϕ be given as above. Suppose that for every $t \in I$, $\phi^{(2)}(t)$ and $\phi^{(3)}(t)$ span $\mathbf{R}^2$. Then $\lambda * L^{3/2} \subset L^2$.*

The proof of Theorem 1 will be based on Oberlin's method and some simple observations.

2. Let $S^1 \subset \mathbf{R}^2$ be the unit circle, γ be an arc on S^1 (*i.e.* γ is a connected subset of S^1), and $l(\gamma)$ be the length of γ. We use $-\gamma$ to denote the set $\{-z \mid z \in \gamma\}$. The set $\Gamma = \{rz \mid r \geq 0, z \in \gamma\}$ is called a *sector*. For $z, w \in S^1$, let $d(z,w)$ be the S^1 distance between z and w (there are two arcs on S^1 which have z and w as their endpoints. $d(z,w)$ is the length of the shorter arc). For $U, V \subset S^1$, let $d(U,V) = \inf\{d(z,w) \mid z \in U, w \in V\}$.

Let $J \subset \mathbf{R}^1$ be a closed interval, $\psi_1, \psi_2 \in C^2(J)$, and $\psi(t) = \big(\psi_1(t), \psi_2(t)\big)$. Define the measure μ on $\mathbf{R}^2$ by

$$\langle \mu, f\rangle = \int_J f\big(\psi(t)\big)\, dt.$$

The key lemma that was used to prove Theorem 0 in [1] is the following:

LEMMA 2. *Suppose Γ_1 and Γ_2 are two sectors and $\Gamma_2 \cap (\Gamma_1 \cup -\Gamma_1) = \{0\}$. Suppose $\delta > 0$, $|J| \leq \delta^{-1}$ and*

(a) for every $t \in J$, $\psi'(t) \in \Gamma_1$, $\psi^{(2)}(t) \in \Gamma_2$, $|\psi'(t)| \geq \delta$, $|\psi^{(2)}(t)| \geq \delta$;

(b) for any $x \in \mathbf{R}^2$, J splits into disjoint subintervals $J_1, \ldots, J_K$ with $K \leq \delta^{-1}$ such that the scalar product $x \cdot \psi^{(2)}(t)$ is of constant sign on each t-interval J_n.

Then, there is a positive constant $C(\Gamma_1, \Gamma_2, \delta)$ such that

$$\|\mu * f\|_{L^3(\mathbf{R}^2)} \leq C(\Gamma_1, \Gamma_2, \delta)\|f\|_{L^{3/2}(\mathbf{R}^2)}.$$

The presence of condition (b) in Lemma 2 is the reason why condition (ii) in Theorem 0 is needed, as pointed out in [1]. Our strategy is to replace Lemma 2 by the following:

LEMMA 3. *Let γ_1, γ_2 be two arcs on S^1, and Γ_1, Γ_2 be the corresponding sectors. Assume that $l(\gamma_1) \leq \pi/4$, $l(\gamma_2) \leq \pi/4$ and $d(\gamma_1 \cup -\gamma_1, \gamma_2 \cup -\gamma_2) = d_0 > 0$. Suppose for every $t \in J$, $\psi'(t) \in \Gamma_1$, $\psi^{(2)}(t) \in \Gamma_2$, and $|\psi'(t)| \geq \delta$, $\delta \leq |\psi^{(2)}(t)| \leq M$, for some* *positive δ and M. We also assume that $|J| \leq M$. Then there is a constant $C(d_0, \delta, M)$ such that*

$$\|\mu * f\|_{L^3(\mathbf{R}^2)} \leq C(d_0, \delta, M)\|f\|_{L^{3/2}(\mathbf{R}^2)}. \tag{2.1}$$

PROOF. Let $\eta \in S^1$ and $d(\eta, \gamma_1 \cup -\gamma_1) \geq \pi/4$. For $z \in C$, we introduce the analytic family of operators $\{T_z\}$ by setting

$$T_z f(x) = \frac{1}{\Gamma(z/2)} \int_J \int_{-\infty}^{\infty} f\Big(x - \big(\psi(t) + s\eta\big)\Big)|s|^{-1+z}\, ds\, dt,$$

where $x \in \mathbf{R}^2, f \in C_0^\infty(\mathbf{R}^2)$. We note that

$$T_0 f = \mu * f.$$

To prove (2.1), it suffices to prove that

$$\|T_{1+iy}\|_{L^1 \to L^\infty} \leq C(d_0, \delta, M)/\Gamma\Big(\frac{1}{2} + \frac{iy}{2}\Big) \tag{2.2}$$

$$\|T_{-\frac{1}{2}+iy}\|_{L^2 \to L^2} \leq C(d_0, \delta, M)/\Gamma\Big(\frac{3}{4} - \frac{iy}{2}\Big) \tag{2.3}$$

and then apply Stein's interpolation theorem ([2]) for analytic family of operators.

The proof of (2.2) is exactly the same as the argument in the proof of Lemma 2 in [1]. It follows from the observation that the Jacobian of the map $(s, t) \longrightarrow \psi(t) + s\eta$ is always $\geq \delta/\sqrt{2}$.

By taking the Fourier transform of $T_z f$, we get

$$\widehat{T_z f}(\xi) = \frac{2^z \pi^{\frac{1}{2}}}{\Gamma(\frac{1-z}{2})}|\xi \cdot \eta|^{-z}\Big(\int_J e^{i\xi \cdot \psi(t)}\, dt\Big)\hat{f}(\xi) \tag{2.4}$$

To prove (2.3), it suffices to show that

$$\Big|\int_J e^{i\xi \cdot \psi(t)}\, dt\Big| \leq C(d_0, \delta, M)|\xi|^{-\frac{1}{2}} \tag{2.5}$$

for all $\xi \in \mathbf{R}^2$. It is easy to see that (2.5) holds if $|\xi| \leq 1$. So we assume that $|\xi| \geq 1$.

Let $\gamma_1^\perp = e^{i\frac{\pi}{2}}\gamma_1 \cup e^{-i\frac{\pi}{2}}\gamma_1$, $\gamma_2^\perp = e^{i\frac{\pi}{2}}\gamma_2 \cup e^{-i\frac{\pi}{2}}\gamma_2$, $A = \{z \in S^1 \mid d(z, \gamma_1^\perp) \geq d_0/2\}$, and $B = \{z \in S^1 \mid d(z, \gamma_2^\perp) \geq d_0/2\}$. Clearly we have $S^1 = A \cup B$.

Given $\xi \in \mathbf{R}^2$, $|\xi| \geq 1$, if $\xi/|\xi| \in B$, then

$$\big(\xi \cdot \psi(t)\big)'' = |\xi|\Big(\frac{\xi}{|\xi|} \cdot \psi^{(2)}(t)\Big) \geq \delta|\xi| \sin(d_0/2),$$

for all $t \in J$. By Van der Corput's lemma ([3]), we get

$$\Big|\int_J e^{i\xi \cdot \psi(t)}\, dt\Big| \leq C\Big(\delta \sin\Big(\frac{d_0}{2}\Big)\Big)^{-\frac{1}{2}}|\xi|^{-\frac{1}{2}}. \tag{2.6}$$

If $\xi/|\xi| \in A$, we have $\big(\xi \cdot \psi(t)\big)' \geq \delta|\xi| \sin(d_0/2)$. Using integration by parts, we get

$$\begin{aligned}\Big|\int_J e^{i\xi\cdot\psi(t)}\,dt\Big| &= \Big|\int_J \frac{1}{\big(\xi\cdot\psi'(t)\big)}\frac{d}{dt}(e^{i\xi\cdot\psi(t)})\,dt\Big| \\ &\leq 2|\xi|^{-1}\Big(\delta\sin\Big(\frac{d_0}{2}\Big)\Big)^{-1} + M|J|\,|\xi|^{-1}\Big(\delta\sin\Big(\frac{d_0}{2}\Big)\Big)^{-2} \\ &\leq C(d_0,\delta,M)|\xi|^{-\frac{1}{2}}.\end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we see that (2.5) holds. The proof of Lemma 3 is complete. ∎

3. We now prove Theorem 1. Throughout the proof of Theorem 1, we use Oberlin's ideas. Our emphasis is placed on showing how Oberlin's arguments work when one uses Lemma 3 instead of Lemma 2.

PROOF OF THEOREM 1. Since $\phi^{(2)}(t)$ and $\phi^{(3)}(t)$ span $\mathbf{R}^2$ for every $t \in I$, we can find a constant $\delta_0 > 0$ such that

$$|\phi^{(2)}(t)| \geq \delta_0, \quad |\phi^{(3)}(t)| \geq \delta_0, \text{ for } t \in I. \tag{3.1}$$

Let

$$h_1 = \inf\Big\{\Big|\frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|} - \frac{\phi^{(3)}(t)}{|\phi^{(3)}(t)|}\Big| \;\Big|\; t \in I\Big\}, \tag{3.2}$$

$$h_2 = \inf\Big\{\Big|\frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|} + \frac{\phi^{(3)}(t)}{|\phi^{(3)}(t)|}\Big| \;\Big|\; t \in I\Big\}. \tag{3.3}$$

By the same assumption, we have $h_1 > 0$, $h_2 > 0$. Set $h = \min\{h_1, h_2, \pi/4\}$. We also assume that $|\phi^{(3)}(t)| \leq M$, for $t \in I$.

We decompose the interval I into n disjoint subintervals: $I = \bigcup_{k=1}^n I_k$, such that, for $1 \leq k \leq n$, $t, t' \in I_k$, the following are satisfied:

$$\Big|\frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|} - \frac{\phi^{(2)}(t')}{|\phi^{(2)}(t')|}\Big| \leq \frac{h}{2}, \tag{3.4}$$

$$\Big|\frac{\phi^{(3)}(t)}{|\phi^{(3)}(t)|} - \frac{\phi^{(3)}(t')}{|\phi^{(3)}(t')|}\Big| \leq \frac{\pi}{8}, \tag{3.5}$$

$$|\phi^{(2)}(t) - \phi^{(2)}(t')| \leq \frac{\delta_0}{4}, \tag{3.6}$$

$$|\phi^{(3)}(t) - \phi^{(3)}(t')| \leq \frac{\delta_0}{4}. \tag{3.7}$$

Since all the functions involved are continuous, thus uniformly continuous on the closed interval I, this can be achieved with an n that depends only on ϕ_1, ϕ_2 and I. For $J \subset I$, let

$$\gamma_1(J) = \Big\{\frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|} \;\Big|\; t \in J\Big\}, \quad \gamma_2(J) = \Big\{\frac{\phi^{(3)}(t)}{|\phi^{(3)}(t)|} \;\Big|\; t \in J\Big\},$$

and $\Gamma_1(J)$, $\Gamma_2(J)$ be the corresponding sectors. Let λ_k be the measure defined by

$$\langle \lambda_k, f\rangle = \int_{I_k} f\big(t, \phi_1(t), \phi_2(t)\big)\, dt.$$

We have $\lambda = \sum_{k=1}^n \lambda_k$. To prove the theorem, it suffices to show that

$$\|\lambda_k * f\|_2 \le C\|f\|_{3/2}, \tag{3.8}$$

for $k = 1, \ldots, n$.

Fix k, $1 \le k \le n$. By (3.2), (3.3) and (3.4), we get

$$\left|\frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|} - \frac{\phi^{(3)}(t')}{|\phi^{(3)}(t')|}\right| \ge \left|\frac{\phi^{(2)}(t')}{|\phi^{(2)}(t')|} - \frac{\phi^{(3)}(t')}{|\phi^{(3)}(t')|}\right| - \left|\frac{\phi^{(2)}(t')}{|\phi^{(2)}(t')|} - \frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|}\right| \ge \frac{h}{2}, \tag{3.9}$$

$$\left|\frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|} + \frac{\phi^{(3)}(t')}{|\phi^{(3)}(t')|}\right| \ge \left|\frac{\phi^{(2)}(t')}{|\phi^{(2)}(t')|} + \frac{\phi^{(3)}(t')}{|\phi^{(3)}(t')|}\right| - \left|\frac{\phi^{(2)}(t')}{|\phi^{(2)}(t')|} - \frac{\phi^{(2)}(t)}{|\phi^{(2)}(t)|}\right| \ge \frac{h}{2}, \tag{3.10}$$

for $t, t' \in I_k$. By (3.4), (3.5), (3.9) and (3.10), we get

$$d\big(\gamma_1(I_k) \cup -\gamma_1(I_k), \gamma_2(I_k) \cup -\gamma_2(I_k)\big) \ge \frac{h}{2}, \tag{3.11}$$

$$l\big(\gamma_1(I_k)\big) \le \frac{\pi}{4}, \quad l\big(\gamma_2(I_k)\big) \le \frac{\pi}{4}. \tag{3.12}$$

Let $I_k = [a,b]$. For $0 \le u \le b-a$, let $J(u) = [a, b-u]$; for $a - b \le u < 0$, let $J(u) = [a-u, b]$. Let

$$\psi_1(t,u) = \frac{1}{u}\big(\phi_1(t+u) - \phi_1(t)\big), \quad \psi_2(t,u) = \frac{1}{u}\big(\phi_2(t+u) - \phi_2(t)\big),$$

and $\psi(t,u) = \big(\psi_1(t,u), \psi_2(t,u)\big)$. For fixed u, we define the measure σ_u on $\mathbf{R}^2$ by

$$\langle \sigma_u, g\rangle = \int_{J(u)} g\big(\psi(t,u)\big)\, dt.$$

We want to show that there is a constant C which is independent of u, $|u| \le b-a$, such that

$$\|\sigma_u * g\|_{L^3(\mathbf{R}^2)} \le C\|g\|_{L^{3/2}(\mathbf{R}^2)}. \tag{3.13}$$

To prove (3.13), we use Lemma 3. For fixed u, $|u| \le b-a$, we shall show that, if $t \in J(u)$, then

$$|\psi'(t,u)| \ge \frac{\delta_0}{2}, \quad |\psi^{(2)}(t,u)| \ge \frac{\delta_0}{2}; \tag{3.14}$$

$$\psi'(t,u) \in \Gamma_1(I_k), \quad \psi^{(2)}(t,u) \in \Gamma_2(I_k) \tag{3.15}$$

(all derivatives are in the t-variable).

For $t \in J(u)$, there are $\tau, \tau' \in I_k$, such that $\psi'(t,u) = \big(\phi_1''(\tau), \phi_2''(\tau')\big)$. Hence by (3.1) and (3.6),

$$|\psi'(t,u)| \ge |\phi^{(2)}(\tau)| - |\phi_2''(\tau') - \phi_2''(\tau)| \ge \frac{\delta_0}{2}.$$

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The second inequality in (3.14) follows similarly.

By the mean-value theorem, there is a $\tau'' \in I_k$, such that

$$\psi_1'(t,u)\phi_2''(\tau'') = \psi_2'(t,u)\phi_1''(\tau''). \tag{3.16}$$

Using (3.1), (3.6) and (3.16), we get $\psi'(t,u) \in \Gamma_1(I_k)$. Similar argument shows that $\psi^{(2)}(t,u) \in \Gamma_2(I_k)$. Also by our assumption, we have $|\psi^{(2)}(t,u)| \leq M$. By (3.11), (3.12), (3.14) and (3.15), we see that (3.13) holds uniformly in u, $|u| \leq b-a$.

The rest of the proof is the same as in [1]. We sketch it here, for the sake of completeness. Let $x = (x_1, x_2, x_3)$, $Tf(x) = \lambda_k * f(x)$. Then

$$T^*Tf(x) = \int_{|u|\leq b-a} \int_{J(u)} f\big(x-u, u\psi_1(t,u), u\psi_2(t,u)\big)\,dt\,du.$$

By (3.13), we get

(3.17)

$$\begin{aligned}
&\|T^*Tf\|_{L^3(\mathbf{R}^3)} \\
&\leq \Big\| \int_{|u|\leq b-a} \Big\| \int_{J(u)} f\Big(x_1-u, u\Big(\frac{x_2}{u}-\psi_1(t,u)\Big), u\Big(\frac{x_3}{u}-\psi_2(t,u)\Big)\Big)\,dt \Big\|_{L^3(dx_2\,dx_3)}\,du \Big\|_{L^3(dx_1)} \\
&\leq \Big\| \int_{|u|\leq b-a} |u|^{-\frac{2}{3}} \big\| f(x_1-u, x_2, x_3)\big\|_{L^{3/2}(dx_2\,dx_3)}\,du \Big\|_{L^3(dx_1)} \leq C\|f\|_{L^{3/2}(\mathbf{R}^3)}.
\end{aligned}$$

The last step uses the $L^p \longrightarrow L^q$ boundedness of the Riesz potential. By (3.17), we get

$$\|Tf\|^2_{L^2(\mathbf{R}^3)} \leq \|T^*Tf\|_{L^3(\mathbf{R}^3)}\|f\|_{L^{3/2}(\mathbf{R}^3)} \leq C\|f\|^2_{L^{3/2}(\mathbf{R}^3)}.$$

The proof is now complete. ∎

4. The following theorem follows immediately from Theorem 1.

THEOREM 4. *Let* $\mathbf{C}\colon t \in I \longrightarrow \big(\phi_1(t), \phi_2(t), \phi_3(t)\big)$ *be a* C^3 *curve on* $\mathbf{R}^3$, *i.e.* $\phi_i \in C^3(I)$, *for* $i = 1,2,3$. *Define the measure* λ *on* $\mathbf{R}^3$ *by*

$$\langle \lambda, f\rangle = \int_I f\big(\phi_1(t), \phi_2(t), \phi_3(t)\big)\,dt.$$

If the curve $\mathbf{C}$ *has nonzero curvature and torsion at every point, then* $\lambda * L^{3/2} \subset L^2$.

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