Bull. Aust. Math. Soc. **84** (2011), 79–89
doi:10.1017/S0004972710002078

TETRAVALENT ARC-TRANSITIVE GRAPHS WITH UNBOUNDED VERTEX-STABILIZERS

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(Received 13 October 2010)

Abstract

It has long been known that there exist finite connected tetravalent arc-transitive graphs with arbitrarily large vertex-stabilizers. However, beside a well-known family of exceptional graphs, related to the lexicographic product of a cycle with an edgeless graph on two vertices, only a few such infinite families of graphs are known. In this paper, we present two more families of tetravalent arc-transitive graphs with large vertex-stabilizers, each significant for its own reason.

2010 *Mathematics subject classification*: primary 20B25.
Keywords and phrases: valency four, arc-transitive.

1. Introduction

A celebrated theorem of Tutte [7, 8] states that, for a finite connected cubic arc-transitive graph, a vertex-stabilizer has order at most 48. As is well known and will be shown below, Tutte's result has no immediate generalization to graphs of valency four. However, it is still an interesting question whether there exists a relatively tame function f such that, for every connected tetravalent arc-transitive graph with n vertices, a vertex-stabilizer has order at most $f(n)$.

Here is a standard example showing that f must grow at least exponentially with the number of vertices: for $r \geq 3$, let W_r denote the lexicographic product $\mathrm{C}_r[\bar{\mathrm{K}}_2]$ of a cycle of length r with an edgeless graph on two vertices. The graph W_r has vertex set $\mathbb{Z}_r \times \mathbb{Z}_2$ with (v, i) adjacent to $(v \pm 1, j)$ for $v \in \mathbb{Z}_r$ and $i, j \in \mathbb{Z}_2$. This graph admits an arc-transitive action of the group $G \cong \mathrm{C}_2 \mathrm{~wr~} \mathrm{D}_r$, with the base group $\mathrm{C}_2^r \leq G$ preserving each *fibre* $V_j = \{j\} \times \mathbb{Z}_2 \subseteq \mathbb{Z}_r \times \mathbb{Z}_2$ setwise and D_r acting naturally on the set of fibres $\{V_j\}_j$. The vertex-stabilizer G_v is then isomorphic to the group $\mathrm{C}_2^{r-1} \rtimes \mathrm{C}_2$. In particular, the order of G_v is 2^r and grows exponentially with the number of vertices $2r$ of the graph W_r. Further examples of families of tetravalent

The second author is supported by UWA as part of the Australian Council Federation Fellowship Project FF0776186.

arc-transitive graphs exhibiting an exponential growth of $|G_v|$ were found by Praeger and Xu in [5]. The graphs $\mathrm{C}(r, s)$, constituting these families, will be described in Section 3.

On the other hand, it has recently been shown by the authors of this paper that a completely different picture emerges once the exceptional graphs $\mathrm{C}(r, s)$ are excluded [4, Corollary 3]. Namely, if Γ is a connected tetravalent G-arc-transitive graph not isomorphic to a graph $\mathrm{C}(r, s)$, then either $|G_v| \le 2^4 3^6$ or

$$|\mathrm{V}\Gamma| \ge 2|G_v| \log_2(|G_v|/2). \tag{1.1}$$

Note that (1.1) implies that $|G_v|$ is bounded above by a sublinear function of $|\mathrm{V}\Gamma|$. The first aim of this paper is to construct a family of connected G-arc-transitive graphs, not isomorphic to $\mathrm{C}(r, s)$, attaining the bound given in (1.1). This will done in Section 4 (see Definition 4.1). Both the graphs $\mathrm{C}(r, s)$ and the graphs presented in Section 4 have soluble groups of automorphisms.

The only family of connected tetravalent G-arc-transitive graphs with arbitrarily large vertex-stabilizers and with G nonsoluble that was previously known to us is the family constructed by Conder and Walker in [2]. For a member Γ of this family, $G \cong \mathrm{Sym}(n)$ (for some n) and $|\mathrm{V}\Gamma| \ge (|G_v| - 1)!$. In particular, $|G_v|$ grows more slowly than any logarithmic function of $|\mathrm{V}\Gamma|$.

The second family of graphs that we construct in this paper (see Definition 5.1) consists of tetravalent G-arc-transitive graphs Δ_n with $G \cong \mathrm{Sym}(4n)$. The vertex-stabilizer G_v in this family has order 2^{2n}, and hence $|\mathrm{V}\Delta_n| = (4n)!/2^{2n}$. Using Stirling's formula, one can see that, asymptotically, $|G_v|$ grows more slowly than $|\mathrm{V}\Gamma|^c$ for any $c > 0$, but more rapidly than any logarithmic function of $|\mathrm{V}\Gamma|$. This is much slower than the growth from (1.1) but still considerably faster than the growth exhibited by the family in [2].

2. Preliminaries

The following standard notation and terminology will be used throughout the paper. All the graphs will be finite, simple and connected. Let Γ be a graph and let $G \le \mathrm{Aut}(\Gamma)$. We say that Γ is G-arc-transitive provided that G acts transitively on the set of arcs of Γ. In this case, the permutation group $G_v^{\Gamma(v)}$ induced by the action of the stabilizer G_v of a vertex $v \in \mathrm{V}\Gamma$ on the neighbourhood $\Gamma(v)$ is transitive. A pair (Γ, G) is called *locally*-D_4 if Γ is a connected tetravalent G-arc-transitive graph with $G_v^{\Gamma(v)}$ isomorphic to the dihedral group D_4 in its natural action on four points. For a group G, a subgroup H and an element $a \in G \setminus H$, the *coset graph* $\mathrm{Cos}(G, H, a)$ is the graph with vertex set the set of right cosets $G/H = \{Hg : g \in G\}$ and edge set $\{\{Hg, Hag\} : g \in G\}$. It was proved by Sabidussi [6] that every G-vertex-transitive graph is isomorphic to some coset graph of G. More precisely, we have the following well-known result.

https://doi.org/10.1017/S0004972710002078 Published online by Cambridge University Press

THEOREM 2.1. *Let Γ be a connected G-arc-transitive graph, let G_v be the stabilizer of the vertex $v \in \mathrm{V}\Gamma$ and let $a \in G$ be an automorphism with $v^a \in \Gamma(v)$. Then $\Gamma \cong \mathrm{Cos}(G, G_v, a)$.*

Conversely, let H be a core-free subgroup of a finite group G and let $a \in G$ be such that $G = \langle H, a\rangle$ and $a^{-1} \in HaH$. Then the graph $\Gamma = \mathrm{Cos}(G, H, a)$ is connected and G-arc-transitive. The valency of Γ is $|HaH|/|H|$ and the neighbourhood $\Gamma(H)$ of the vertex $H \in \mathrm{V}\Gamma$ is the set $\Gamma(H) = \{Hah : h \in H\}$.

Given a connected graph Γ, a subgroup $G \le \mathrm{Aut}(\Gamma)$ and a normal subgroup $N \trianglelefteq G$, the *normal quotient graph* Γ/N is the graph with vertex set the orbit space $\mathrm{V}\Gamma/N = \{v^N : v \in \mathrm{V}\Gamma\}$ and with two orbits u^N and v^N adjacent in Γ/N whenever there exist a pair of vertices $u', v' \in \mathrm{V}\Gamma$ with $u' \in u^N$ and $v' \in v^N$. Note that there exists a natural (but possibly not faithful) action of G/N on Γ/N. Moreover, if G is transitive on the vertices (respectively, arcs) of Γ, then G/N is transitive on vertices (respectively, arcs) of Γ/N.

A special case of normal quotients arises when the quotient projection $\pi : \Gamma \to \Gamma/N$, $v \mapsto v^N$, is locally bijective (that is, π maps the neighbourhood of an arbitrary vertex $v \in \mathrm{V}\Gamma$ bijectively onto the neighbourhood of $\pi(v)$ in Γ/N). It is well known that, in this case, G/N acts faithfully on $\mathrm{V}(\Gamma/N)$, and that the vertex-stabilizers G_v and $(G/N)_{\pi(v)}$ are isomorphic and induce permutation isomorphic local groups $G_v^{\Gamma(v)}$ and $(G/N)_{\pi(v)}^{\Gamma/N(\pi(v))}$. When the quotient projection $\pi : \Gamma \to \Gamma/N$ is locally bijective, we will say that the pair (Γ, G) is an *N-cover* of the pair $(\Gamma/N, G/N)$. If, in addition, N is contained in the centre of G, then we say that the pair (Γ, G) is a *central N-cover* of the pair $(\Gamma/N, G/N)$.

We will need the following lemma describing the relationship between covers and coset graphs.

LEMMA 2.2. *Let G be a group generated by a core-free subgroup H and an element a. Further, let $\Gamma = \mathrm{Cos}(G, H, a)$ and let N be a normal subgroup of G not containing a and intersecting the set H^aH trivially. Let $\bar{G} = G/N$, let $\bar{H} = HN/N$, and let $\bar{a} = Na \in G/N$. Then $\Gamma/N \cong \mathrm{Cos}(\bar{G}, \bar{H}, \bar{a})$ and (Γ, G) is an N-cover of $(\Gamma/N, \bar{G})$.*

PROOF. Let v denote the vertex of $\Gamma = \mathrm{Cos}(G, H, a)$ corresponding to the coset $H \in G/H$. Then $H = G_v$ and $H^a = G_{v^a}$. As Γ is G-arc-transitive, to show that the quotient projection $\pi : \Gamma \to \Gamma/N$ is locally bijective, it suffices to show that the N-orbit of v^a intersects the neighbourhood of v only in v^a, that is, $(v^a)^N \cap (v^a)^H = \{v^a\}$. Now, if $u \in (v^a)^N \cap (v^a)^H$, then $u = v^{az} = v^{ah}$ for some $z \in N$ and $h \in H$. Therefore $azh^{-1}a^{-1} \in G_v = H$, and thus $z \in H^aH$. Since $N \cap H^aH = 1$, this implies that $z = 1$ and that $u = v^a = v^{ah}$, and hence $(v^a)^N \cap (v^a)^H = \{v^a\}$. This shows that $\pi : \Gamma \to \Gamma/N$ is indeed a covering projection and hence (Γ, G) is an N-cover of $(\Gamma/N, G/N)$. It follows that G/N acts faithfully and arc-transitively on Γ/N. The stabilizer of the vertex v^N in G/N is the group $\bar{H} = HN/N \cong H/(H \cap N) \cong H$, and $\bar{a}$ maps the vertex $\pi(v)$ to the neighbour $\pi(v^a)$. By Theorem 2.1 we may thus conclude that $\Gamma/N \cong \mathrm{Cos}(\bar{G}, \bar{H}, \bar{a})$, as claimed. □

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3. The family of graphs with exponential growth of the vertex-stabilizer

In this section we describe the family of graphs $\mathrm{C}(r, s)$ mentioned in Section 1, which generalize the graphs W_r. We give a definition that is slightly different from but equivalent to the definition used in [5], where they were first introduced.

Let $\mathrm{C}(r, 1) = \mathrm{W}_r$. Let s be an integer satisfying $2 \leq s \leq r-2$ and let $\mathrm{C}(r, s)$ be the graph with vertices being the $(s-1)$-paths of $\mathrm{C}(r, 1)$ containing at most one vertex from each fibre V_j, $j \in \mathbb{Z}_r$, and with two such $(s-1)$-paths being adjacent in $\mathrm{C}(r, s)$ if and only if their intersection is an $(s-2)$-path in $\mathrm{C}(r, 1)$. The number of vertices of $\mathrm{C}(r, s)$ is clearly

$$|\mathrm{VC}(r, s)| = r2^s. \tag{3.1}$$

It is easy to see that the girth of $\mathrm{C}(r, s)$ is 4. Furthermore, $\mathrm{C}(r, s)$ is bipartite provided that r is even.

For $i \in \mathbb{Z}_r$, let x_i denote the automorphism of W_r which interchanges the two vertices in the fibre V_i and fixes all other vertices. Further, let a be the automorphism of W_r which maps each $(v, i) \in \mathrm{VW}_r$ to $(v+1, i)$, and let b be the automorphism acting on the vertices of W_r according to the rule $(v, i)^b = (-v, i)$ for every $v \in \mathbb{Z}_r$ and $i \in \mathbb{Z}_2$. Then the group

$$G_r = \langle x_0, \ldots, x_{r-1}\rangle \rtimes \langle a, b\rangle \cong \mathrm{C}_2^r \rtimes \mathrm{D}_r \tag{3.2}$$

acts arc-transitively on W_r. It was shown in [5] that if $r \neq 4$, then $\mathrm{Aut}(\mathrm{W}_r) = G_r$. Furthermore, $\mathrm{W}_4 \cong K_{4,4}$ and hence $\mathrm{Aut}(\mathrm{W}_4) \cong \mathrm{Sym}(4) \mathrm{~wr~} \mathrm{Sym}(2)$.

Since G_r permutes the $(s-1)$-paths of W_r containing at most one vertex from each fibre V_j, the group G_r acts as a group of automorphisms of $\mathrm{C}(r, s)$. The stabilizer in $G = G_r$ of the vertex v of $\mathrm{C}(r, s)$ corresponding to the $(s-1)$-path $(r-s, 0)$ $(r-s+1, 0) \cdots (r-1, 0)$ in W_r is the group $H = \langle x_0, x_1, \ldots, x_{r-s-1}, b_s\rangle$, where b_s is the element of G acting as $(v, i)^{b_s} = (r-s-1-v, i)$. Note that $G = \langle H, a\rangle$. Using Theorem 2.1, it is now easy to see that the graphs $\mathrm{C}(r, s)$ can be defined in terms of coset graphs as follows.

LEMMA 3.1. *The graph* $\mathrm{C}(r, s)$ *is isomorphic to the coset graph* $\mathrm{Cos}(G_{r,s}, H_{r,s}, a)$ *where*

$$\begin{aligned} G_{r,s} = \langle x_0, \ldots, x_{r-1}, a, b \mid\ & x_0^2 = \cdots = x_{r-1}^2 = a^r = b^2 = (ab)^2 = 1,\\ & x_i^a = x_{i+1}, x_i^b = x_{r-s-1-i}\rangle,\\ H_{r,s} = \langle x_0, \ldots, &\, x_{r-1}, b\rangle \leq G_{r,s}. \end{aligned}$$

Let us finish this section by reporting the following result from [5] regarding the automorphism group of $\mathrm{C}(r, s)$.

LEMMA 3.2 [5, Lemma 2.12]. *Let* $\Gamma = \mathrm{C}(r, s)$ *with* $1 \leq s \leq r-1$. *If* $r \neq 4$, *then* $\mathrm{Aut}(\Gamma) = G_{r,s}$. *Moreover,*

$$\mathrm{Aut}(\mathrm{C}(4, 1)) \cong \mathrm{Aut}(\mathrm{K}_{4,4}) \cong \mathrm{Sym}(4) \mathrm{~wr~} \mathrm{Sym}(2),$$
$$\mathrm{Aut}(\mathrm{C}(4, 2)) \cong \mathrm{Sym}(2) \mathrm{~wr~} \mathrm{Sym}(4)$$

https://doi.org/10.1017/S0004972710002078 Published online by Cambridge University Press

and

$$\mathrm{Aut}(\mathrm{C}(4,3)) \cong (\mathrm{Sym}(2) \mathrm{\ wr\ } \mathrm{D}_4) \cdot \mathrm{Sym}(2).$$

4. The graphs attaining the bound (1.1)

In this section, for every $t \geq 2$, we construct two locally-D_4 pairs (Γ_t^+, G_t^+) and (Γ_t^-, G_t^-) with $|\mathrm{V}\Gamma_t^{\pm}| = t2^{t+2}$ and $|G_t^{\pm}| = t2^{2t+3}$. Since $|(G_t^{\pm})_v| = |G_t^{\pm}|/|\mathrm{V}\Gamma_t^{\pm}| = 2^{t+1}$, we see that the pairs $(\Gamma_t^{\pm}, G_t^{\pm})$ indeed meet the bound (1.1) stated in Section 1.

Let t be an integer satisfying $t \geq 2$. We start by considering the *extraspecial group* E_t of order 2^{2t+1} of *plus type*, which has the following presentation:

$$\begin{aligned} E_t = \langle x_0, \ldots, x_{2t-1}, z \mid x_i^2 = z^2 = [x_i, z] = 1 \text{ for } 0 \leq i \leq 2t-1, \\ [x_i, x_j] = 1 \text{ for } |i-j| \neq t, [x_i, x_{t+i}] = z \text{ for } 0 \leq i \leq t-1\rangle. \end{aligned} \tag{4.1}$$

We will now extend the group E_t by the dihedral group

$$\mathrm{D}_{2t} = \langle a, b \mid a^{2t} = b^2 = 1, a^b = a^{-1}\rangle, \tag{4.2}$$

using two different 2-cocycles. In both extensions, the generators a and b will act upon the generators of E_t according to the rules

$$x_i^a = x_{i+1} \quad \text{and} \quad x_i^b = x_{t-1-i} \qquad \text{for } 0 \leq i \leq 2t-1 \text{ (with indices taken mod } 2t).$$

To obtain the split extension G_t^+, we let $a^{2t} = b^2 = 1$, and thus define

$$G_t^+ = E_t \rtimes \mathrm{D}_{2t}, \quad a^{2t} = b^2 = 1, x_i^a = x_{i+1}, x_i^b = x_{t-1-i}. \tag{4.3}$$

The second extension G_t^- is nonsplit, and we have $a^{2t} = z$ and $b^2 = 1$:

$$G_t^- = E_t \cdot \mathrm{D}_{2t}, \quad a^{2t} = z, b^2 = 1, x_i^a = x_{i+1}, x_i^b = x_{t-1-i}. \tag{4.4}$$

Finally, let

$$H_t^{\pm} = \langle x_0, \ldots, x_{t-1}, b\rangle \leq G_t^{\pm}, \tag{4.5}$$

and observe that $H_t^+ \cong H_t^- \cong \mathrm{C}_2^t \rtimes \mathrm{C}_2$. The graphs Γ_t^+ and Γ_t^- are now defined as coset graphs on the groups G_t^+ and G_t^-, respectively.

DEFINITION 4.1. Let $t \geq 2$ and let a, $H_t^{\pm}$ and $G_t^{\pm}$ be as above. Then $\Gamma_t^+ = \mathrm{Cos}(G_t^+, H_t^+, a)$ and $\Gamma_t^- = \mathrm{Cos}(G_t^-, H_t^-, a)$.

Before stating the main theorem of this section, let us first show that for any $t \geq 2$, a triple $(G, H, a) = (G_t^{\pm}, H_t^{\pm}, a)$ satisfies the conditions of Theorem 2.1 and thus gives rise to a connected G-arc-transitive graph $\Gamma_t^{\pm}$. Observe first that since a cyclically permutes the elements of the generating set $\{x_0, \ldots, x_{2t-1}\}$ of E, and since $x_0 \in H$, the group $\langle H, a\rangle$ contains the subgroup E. Since $b \in H$, we see that $\langle H, a\rangle = \langle H, b, a\rangle = \langle E, b, a\rangle = G$. The graph $\Gamma_t^{\pm}$ is therefore connected.

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To see that H is core-free in G, observe that

$$H \cap H^{a^t} = \langle x_0, \ldots, x_{t-1}, b\rangle \cap \langle x_t, \ldots, x_{2t-1}, b\rangle = \langle b\rangle.$$

Since $b \notin H^a$, we see that $H \cap H^{a^t} \cap H^a = 1$, implying that the core of H in G is trivial. Finally, since $b \in H$, it follows that $HaH = HbabH = Ha^{-1}H$, and hence $a^{-1} \in HaH$. In view of Theorem 2.1, this implies that the graph $\Gamma_t^{\pm}$ is indeed connected and G-arc-transitive.

THEOREM 4.2. *Let t be an integer with $t \geq 2$ and let (Γ, G) be either (Γ_t^+, G_t^+) or (Γ_t^-, G_t^-). Then the following statements hold:*

(i) *$|V\Gamma| = t2^{t+2}$ and $|G_v| = 2^{t+1}$, and hence the bound (1.1) in Section 1 is met;*
(ii) *(Γ, G) is a central C_2-cover of the pair $(C(2t, t), G_{2t,t})$, where $G_{2t,t}$ is as in Lemma 3.1;*
(iii) *(Γ, G) is a locally-D_4 pair;*
(iv) *the girth of Γ is 4 if $\Gamma = \Gamma_2^+$, is 6 if $\Gamma = \Gamma_3^+$, and is 8 otherwise;*
(v) *if $(\Gamma, G) \neq (\Gamma_2^-, G_2^-)$, then $\mathrm{Aut}(\Gamma) = G$, while $|\mathrm{Aut}(\Gamma_2^-) : G_2^-| = 9$.*

PROOF. Let $E = E_t$ and let $H = H_t^+$ or H_t^-, so that $\Gamma = \mathrm{Cay}(G, H, a)$. To prove part (i), note that

$$|G| = |E_t||D_{2t}| = 2^{2t+1}4t = t2^{2t+3}$$

and that

$$H \cong \langle x_0, \ldots, x_{t-1}\rangle \rtimes \langle b\rangle \cong C_2^t \rtimes C_2,$$

implying that $|G_v| = |H| = 2^{t+1}$. Therefore, $|V\Gamma| = |G|/|H| = t2^{t+2}$, as claimed.

To prove part (ii), first observe that $\langle z\rangle \cong C_2$ is contained in the centre of G, that $G/\langle z\rangle \cong G_{2t,t}$ and that the natural isomorphism between $G/\langle z\rangle$ and $G_{2t,t}$ maps the group H bijectively onto the group $H_{2t,t} \leq G_{2t,t}$ (defined in Lemma 3.1), and the element $a \in G$ from the definition of the graph $\Gamma = \mathrm{Cos}(G, H, a)$ to the element a from the definition of the graph $C(2t, t) = \mathrm{Cos}(G_{2t,t}, H_{2t,t}, a)$.

We now show that $z \notin H^aH$. Suppose (to the contrary) that $z \in H^aH$. Every element of H is of the form eb^ϵ for some $e \in \langle x_0, \ldots, x_{t-1}\rangle \leq E$ and $\epsilon \in \{0, 1\}$, hence z can be written in the form $(eb^\epsilon)^a(e'b^{\epsilon'})$. Since E is normal in G, we have $z = c(b^\epsilon)^a b^{\epsilon'}$, for some $c \in E$. Since $z \in E$, it follows that $(b^\epsilon)^a b^{\epsilon'} = 1$ and hence $\epsilon = \epsilon' = 0$. It follows that z can be written in the form

$$(x_0^{\epsilon_0} \cdots x_{t-1}^{\epsilon_{t-1}})^a (x_0^{\epsilon'_0} \cdots x_{t-1}^{\epsilon'_{t-1}}) = (x_1^{\epsilon_0} \cdots x_t^{\epsilon_{t-1}})(x_0^{\epsilon'_0} \cdots x_{t-1}^{\epsilon'_{t-1}}).$$

If $\epsilon_{t-1} = 0$, then the latter belongs to the elementary abelian group $\langle x_0, \ldots, x_{t-1}\rangle$, which does not contain z. Similarly, if $\epsilon'_0 = 0$, then the latter belongs to the elementary abelian group $\langle x_1, \ldots, x_t\rangle$, which does not contain z. Hence we may assume

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that $\epsilon_{t-1} = \epsilon'_0 = 1$. Now, since $x_t x_0 = x_0 x_t z$, it follows that

$$\begin{aligned} z &= x_1^{\epsilon_0} \cdots x_{t-1}^{\epsilon_{t-2}} x_t x_0 x_1^{\epsilon'_1} \cdots x_{t-1}^{\epsilon'_{t-1}} \\ &= x_1^{\epsilon_0} \cdots x_{t-1}^{\epsilon_{t-2}} x_0 x_t z x_1^{\epsilon'_1} \cdots x_{t-1}^{\epsilon'_{t-1}} \\ &= x_1^{\epsilon_0} \cdots x_{t-1}^{\epsilon_{t-2}} x_0 x_1^{\epsilon'_1} \cdots x_{t-1}^{\epsilon'_{t-1}} x_t z = d x_t z \end{aligned}$$

with $d \in \langle x_0, \dots x_{t-1}\rangle$. Therefore we have $x_t \in \langle x_0, \dots, x_{t-1}\rangle$, which is clearly a contradiction. Thus we have shown that $z \notin H^a H$. It follows by Lemma 2.2 that $\Gamma/\langle z\rangle \cong \mathrm{C}(2t, t)$ and that Γ is a $\langle z\rangle$-cover of $\mathrm{C}(2t, t)$. Part (ii) of the theorem is thus proved.

Moreover, since the pair $(\mathrm{C}(2t, t), G_{2t,t})$ is locally-D_4, so is the covering pair (Γ, G), thus proving part (iii).

In the proof of parts (iv) and (v) we will need detailed information about the spheres of radius 2 and 3 around the vertex H. For $i \geq 1$ and $v \in \mathrm{V}\Gamma$, let $\Gamma_i(v)$ denote the set of vertices in Γ at distance i from v. To determine the neighbourhood $\Gamma_1(H)$, observe that Ha, $Ha^{-1} = Hab$, $Hx_{2t-1}a = Hax_0$ and $Hx_t a^{-1} = Ha^{-1}x_{t-1} = Habx_{t-1}$ are four pairwise distinct cosets of the form Hah with $h \in H$. Since Γ has valency four, this implies that

$$\Gamma_1(H) = \{Hg : g \in X_1\} \quad \text{where } X_1 = \{a, x_{2t-1}a, a^{-1}, x_t a^{-1}\}. \tag{4.6}$$

Further, observe that

$$\Gamma_2(H) = \{Hg : g \in X_2\} \setminus \{H\} \quad \text{where } X_2 = \{gh : g, h \in X_1\}. \tag{4.7}$$

An easy computation shows that

$$\begin{aligned} X_2 = {} & \{x_{2t-2}^{\epsilon_1} x_{2t-1}^{\epsilon_2} a^2 : \epsilon_1, \epsilon_2 \in \{0, 1\}\} \cup \{x_t^{\epsilon_1} x_{t+1}^{\epsilon_2} a^{-2} : \epsilon_1, \epsilon_2 \in \{0, 1\}\} \\ & \cup \{x_t, x_t z, x_{2t-1}, x_{2t-1} z\}. \end{aligned} \tag{4.8}$$

Similarly, note that

$$\Gamma_3(H) = \{Hg : g \in X_3\} \setminus \Gamma_1(H) \quad \text{where } X_3 = \{gh : g, h \in X_1, X_2\}. \tag{4.9}$$

By a straightforward computation,

$$\begin{aligned} X_3 = {} & \{x_{2t-3}^{\epsilon_1} x_{2t-2}^{\epsilon_2} x_{2t-1}^{\epsilon_3} a^3 : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \\ & \cup \{x_t^{\epsilon_1} x_{t+1}^{\epsilon_2} x_{t+2}^{\epsilon_3} a^{-3} : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \\ & \cup \{x_t^{\epsilon_1} x_{2t-1}^{\epsilon_2} z^{\epsilon_3} a : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \\ & \cup \{x_{2t-2} x_{2t-1}^{\epsilon_1} z^{\epsilon_2} a : \epsilon_1, \epsilon_2 \in \{0, 1\}\} \\ & \cup \{x_t^{\epsilon_1} x_{t+1}^{\epsilon_2} z^{\epsilon_3} a^{-1} : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} \\ & \cup \{x_t^{\epsilon_1} x_{2t-1} z^{\epsilon_2} a : \epsilon_1, \epsilon_2 \in \{0, 1\}\}. \end{aligned} \tag{4.10}$$

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Having computed the second and the third neighbourhood of the vertex H, it is now easy to determine the girth of the graph Γ. Recall first that Γ is a twofold cover of the graph $\mathrm{C}(2t, t)$, which is bipartite and of girth four. This implies that Γ is also bipartite and of girth not exceeding eight.

Now, if $t = 2$ and $\Gamma = \Gamma_t^+$, then the order of a is four, and hence the elements a^{-2}, a^2, listed in (4.8) coincide. In particular, Γ contains the 4-cycle (H, Ha, Ha^2, Ha^{-1}), and thus the girth of Γ is four.

In all the other cases (that is, if $\Gamma = \Gamma_2^-$ or if $t \geq 3$), the 12 elements of X_2 listed in (4.8) are representatives of 12 pairwise distinct H-cosets. This implies that the girth of Γ is at least six. If $t = 3$ and $\Gamma = \Gamma_t^+$, then $a^3 = a^{-3}$ and therefore Γ contains the 6-cycle $(H, Ha, H^2, Ha^3, Ha^{-2}, Ha^{-1})$. In particular, the girth of Γ_3^+ is six.

In all the other cases (that is, if $\Gamma = \Gamma_3^-$ or if $t \geq 4$), the 36 elements of X_3, listed in (4.10), are pairwise distinct, and in fact are representatives of 36 pairwise distinct H-cosets. The girth of Γ is thus at least (and therefore exactly) eight. This proves part (iv) of the theorem.

Let us now prove part (v). The automorphism groups of $\Gamma_t^\pm$ for $t \leq 3$ can be determined using `Magma` [1]. We will therefore assume that $t \geq 4$ and set $\Gamma = \Gamma_t^\pm$ and $A = \mathrm{Aut}(\Gamma)$.

We will first show that the orbits of $\langle z \rangle$ on $\mathrm{V}\Gamma$ form a system of imprimitivity for the action of A on $\mathrm{V}\Gamma$. Set

$$\mathcal{C} = \bigcap_{u \in \Gamma(H)} \Gamma_3(u) = \Gamma_3(H)a \cap \Gamma_3(H)x_{2t-1}a \cap \Gamma_3(H)a^{-1} \cap \Gamma_3(H)x_t a^{-1} \tag{4.11}$$

and set $\mathcal{B} = \mathcal{C} \cup \{H\}$. Using (4.10), a straightforward calculation shows that

$$\mathcal{B} = \{H, Hx_t x_{2t-1}, Hz, Hx_t x_{2t-1} z\}. \tag{4.12}$$

We claim that $\mathcal{B}$ is a block of imprimitivity for A. Observe first that the setwise stabilizer $A_{\mathcal{B}}$ of the set $\mathcal{B}$ in A acts transitively on $\mathcal{B}$ and contains the vertex stabilizer A_H. The latter follows directly from the definition of the set $\mathcal{B}$, while the former follows from the observation that the group $\langle x_t x_{2t-1}, z \rangle$ preserves $\mathcal{B}$ and acts transitively upon it. In particular, $A_{\mathcal{B}} = A_H \langle x_t x_{2t-1}, z \rangle$. Hence $\mathcal{B}$ is an orbit of a subgroup of A which strictly contains the vertex-stabilizer A_H. This shows that $\mathcal{B}$ is a block of imprimitivity for A.

Now observe that each of the vertices H and Hz has a neighbour in each of the four translates $\mathcal{B}a$, $\mathcal{B}a^{-1}$, $\mathcal{B}x_{2t-1}a$, $\mathcal{B}x_t a^{-1}$ of the block $\mathcal{B}$; this is obviously true for H, and follows easily for Hz from the fact that z is contained in the centre of G. On the other hand, direct inspection shows that $Hx_t x_{2t-1}$ and $Hx_t x_{2t-1} z$ have no neighbours in these four translates of $\mathcal{B}$. In particular, the stabilizer A_H cannot map the vertex Hz to either of the other two vertices $Hx_t x_{2t-1}$ and $Hx_t x_{2t-1} z$ in the block $\mathcal{B}$. In particular, A_H fixes the vertex Hz, and hence $A_H = A_{Hz}$. It follows that the group $\langle A_H, z \rangle$ preserves the set $\{H, Hz\}$, acts upon it transitively, and contains the vertex-stabilizer A_H. This implies that its orbit $\{H, Hz\}$ is a block of imprimitivity for A, as claimed.

https://doi.org/10.1017/S0004972710002078 Published online by Cambridge University Press

Now consider the kernel K of the action of A on $\langle z\rangle$-orbits. Since $\Gamma \to \Gamma/\langle z\rangle$ is a covering projection, we know that K acts semiregularly on $V\Gamma$. On the other hand, $\langle z\rangle \le K$ has the same orbits on $V\Gamma$ as K, and hence $K = K_v\langle z\rangle = \langle z\rangle$. In particular, $\langle z\rangle$ is normal in A. Therefore $A/\langle z\rangle \le \mathrm{Aut}(\Gamma/\langle z\rangle) \cong \mathrm{Aut}(\mathrm{C}(2t, t)) \cong G_{2t,t}$ (see Lemma 3.2). In particular, $|A| = 2|G_{2t,t}| = |G|$, and therefore $A = G$. □

REMARK 4.3. Since G_t^+ and G_t^- are nonisomorphic groups, it follows from Theorem 4.2 that $\mathrm{Aut}(\Gamma_t^+) \ncong \mathrm{Aut}(\Gamma_t^-)$. In particular, Γ_t^+ and Γ_t^- are nonisomorphic graphs. Moreover, since the girth of the graphs $\mathrm{C}(r, s)$ is four, none of the graphs $\Gamma_t^{\pm}$, other than possibly Γ_2^+, is isomorphic to any of the graphs $\mathrm{C}(r, s)$. On the other hand, it can be easily checked that $\Gamma_2^+ \cong \mathrm{C}(4, 3)$.

5. A family of Sym(n)-arc-transitive graphs

In this section, we introduce another interesting family of tetravalent G-arc-transitive graphs with arbitrarily large vertex-stabilizers. Unlike the graphs $\Gamma_t^{\pm}$, which have soluble automorphism groups, the graphs we describe here have an almost simple arc-transitive group of automorphisms.

Let $m \ge 2$ be an integer and let G be the symmetric group $\mathrm{Sym}(4m)$ acting on the set $\{1, 2, \dots, 4m\}$. Define the following permutations of G:

$$
\begin{aligned}
x_i &= (2i-1, 2i) \quad \text{for } 1 \le i \le 2m-1,\\
h &= (4m-1, 4m)\prod_{i=1}^{m-1}(2i-1, 4m-2i-1)(2i, 4m-2i)\\
a &= (4m-2, 4m)\prod_{i=1}^{m-1}(2i-1, 4m-2i-3)(2i, 4m-2i-2)\\
g &= (1, 3, 5, \dots, 4m-3)(2, 4, 6, \dots, 4m-4, 4m-2, 4m-1, 4m).
\end{aligned}
$$

We can now define the graphs Δ_m.

DEFINITION 5.1. For $G = \mathrm{Sym}(4m)$ and $H = \langle x_1, x_2, \dots, x_{2m-1}, h\rangle$, let $\Delta_m = \mathrm{Cos}(G, H, a)$.

Before proving that the graphs Δ_m are indeed connected tetravalent graphs, we first observe that the following hold:

$$
\begin{aligned}
g &= ah, \\
x_i^h &= x_{2m-i} && \text{for } 1 \le i \le 2m-1, \\
x_i^g &= x_{i+1} && \text{for } 1 \le i \le 2m-2, \\
x_{2m-1}^g &= (4m-3, 4m-2)^g = (1, 4m-1).
\end{aligned}
\tag{5.1}
$$

In particular, the group H is isomorphic to a semidirect product $\mathrm{C}_2^{2m-1} \rtimes \mathrm{C}_2$ and has order 2^{2m}.

https://doi.org/10.1017/S0004972710002078 Published online by Cambridge University Press

THEOREM 5.2. *For any $m \geq 2$, the graph Δ_m is nonbipartite, connected, tetravalent and G-arc-transitive. Moreover,* $|\mathrm{V}\Delta_m| = (4m)!/2^{2m}$ *and* $|G_v| = 2^{2m}$.

PROOF. Let us first prove that the triple (G, H, a) from Definition 5.1 satisfies the conditions stated in Theorem 2.1. In other words, let us prove that the core of H in G is trivial, that $G = \langle H, a\rangle$, and that $Ha^{-1}H = HaH$. Since a is an involution, the latter condition is automatically fulfilled. Furthermore, since the only nontrivial proper normal subgroup of $G = \mathrm{Sym}(4m)$ is the group $\mathrm{Alt}(4m)$ and since $H \neq \mathrm{Alt}(4m)$, it follows that the core of H in G is trivial. It remains to see that G is generated by H and a.

Set $K = \langle H, a\rangle$, and observe by (5.1) that $g \in K$ and hence G is a transitive subgroup of $\mathrm{Sym}(4m)$. Furthermore, g is a product of two cycles of lengths $2m - 1$ and $2m + 1$. Since $2m - 1$ and $2m + 1$ are coprime, it follows that K is a primitive subgroup of $\mathrm{Sym}(4m)$. As K contains the transposition $x_1 = (1, 2)$, we see from [3, Theorem 3.3A(ii)] that $K = \mathrm{Sym}(4m) = G$. Theorem 2.1 now implies that Δ_m is a connected G-arc-transitive graph with $G_v \cong H$.

Finally, recall that $x_i^a = x_{2m-i-1}$ for $i = 1, \ldots, 2m - 2$. In particular, $H \cap H^a \geq \langle x_1, \ldots, x_{2m-2}\rangle$ and $|H : H \cap H^a|$ divides 4. Since $H \cap H^a$ is the stabilizer in G of the arc (H, Ha) and since Δ_m is G-arc-transitive, we find that the valency of Δ_m divides 4. As G is almost simple and acts faithfully on Δ_m, we see that Δ_m has valency four. Since $H \not\leq \mathrm{Alt}(4m)$, the group $\mathrm{Alt}(4m)$ is transitive on $\mathrm{V}\Delta_m$. As $\mathrm{Alt}(4m)$ is the only subgroup of index two in G, we see that Δ_m is nonbipartite. □

Let $\Gamma = \Delta_m$ as in Definition 5.1 and let $G = \mathrm{Sym}(4m) \leq \mathrm{Aut}(\Gamma)$. We consider the growth rate of the quantity $x = |G_v| = 2^{2m}$ with respect to $y = |\mathrm{V}\Gamma| = (4m)!/2^{2m}$. Using Stirling's formula for the factorial term in y, one easily obtains that

$$y \approx \sqrt{4\pi \log(x)}\, x^{1-2\log(e)} \log(x)^{\log(x^2)},$$

with all the logarithms having base 2. Since $\log(x)^{\log(x^2)} = x^{2\log(x)\log(\log(x))}$, this shows that $x = |G_v|$ grows more rapidly than any logarithmic function on $y = |V\Gamma|$ but more slowly than $|\mathrm{V}\Gamma|^c$ for any $c > 0$. The graphs Δ_m are thus quite far from attaining the bound given in (1.1). We conjecture that a much faster growth of $|G_v|$ cannot be expected when G is almost simple.

CONJECTURE 5.3. For any positive constant c there exists a finite family of graphs $\mathcal{F}_c$, such that the following holds: if G is an almost simple group and Γ is a connected tetravalent G-arc-transitive graph such that Γ is not contained in $\mathcal{F}_c$, then $|G_v| < |\mathrm{V}\Gamma|^c$.

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