LELONG NUMBERS OF m-SUBHARMONIC FUNCTIONS ALONG SUBMANIFOLDS

JIANCHUN CHU[1] AND NICHOLAS MCCLEEREY[2]

[1]*School of Mathematical Sciences, Peking University, Yiheyuan Road 5, Beijing 100871, P.R. China*
(jianchunchu@math.pku.edu.cn)
[2]*Department of Mathematics, Univeristy of Michigan, Ann Arbor, 530 Church St, Ann Arbor, MI 48109*
(njmc@umich.edu)

Abstract We study the possible singularities of an m-subharmonic function φ along a complex submanifold V of a compact Kähler manifold, finding a maximal rate of growth for φ which depends only on m and k, the codimension of V. When $k < m$, we show that φ has at worst log poles along V, and that the strength of these poles is moreover constant along V. This can be thought of as an analogue of Siu's theorem.

1. Introduction

Let (X^n, ω) be a closed Kähler manifold, and let $m\mathrm{SH}(X,\omega)$ be the space of (m,ω)-sh functions on X, $1 \leqslant m \leqslant n$. m-sh functions were introduced by Błocki [9] as the natural space of weak solutions to the complex m-Hessian equation on domains in $\mathbb{C}^n$, and they naturally interpolate between plurisubharmonic (psh) and subharmonic (sh) functions as m varies.

As such, m-sh functions share many similarities with psh functions. Both support a robust potential theory which has been the subject of much interest in recent years (see, e.g., [7, 8, 9, 1, 19, 28, 12]). One major difference however is the lack of a strong relation between m-sh functions and analytic geometry, something which can be seen,

for instance, in our inability to solve the $\overline{\partial}$-problem with generic m-sh weights. This geometric connection is utilized to great effect in the study of psh functions, and is largely responsible for our understanding of what singularities of psh functions can look like, for example, Demailly approximation [16] and Siu's theorem [32].

Our goal in this paper is to develop a better understanding of the singularities of m-sh functions, proceeding in a more ad-hoc manner. A natural starting place is to study their behaviour along complex submanifolds:

Theorem 1.1. *Let $V \subset X$ be a closed, complex submanifold of codimension k and Ω a sufficiently small m-hyperconvex neighborhood of V.*

For each $m \leqslant k$, there exists an (m,ω)-sh function ψ_V on X, locally bounded and maximal on $\Omega \setminus V$, with the same singularity type as

$$G_m(r) := \begin{cases} \log r & \text{when } m = k \\ -r^{-2\left(\frac{k}{m}-1\right)} & \text{when } m < k, \end{cases}$$

where here $r := \mathrm{dist}_\omega(\cdot,V)$ is the ω-distance to V.

With the function ψ_V in hand, we can study generalized Lelong numbers, in the sense of Demailly [17], and relative types, in the sense of Rashkovskii [30], of arbitrary m-sh functions along V.

Corollary 1.2. *Let V, ψ_V and r be as in Theorem 1.1. Then for any $\varphi \in m\mathrm{SH}(X,\omega)$, the limit*

$$\nu(\varphi,\psi_V) := \lim_{s \to 0} \frac{C_{k,m}}{s^{2k-2k/m}} \int_{\{r<s\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}r^2)^{m-1} \wedge \omega^{n-m}$$

exists and is both finite and nonnegative. Here, $C_{k,m}$ is a constant depending only on k and m.

Corollary 1.3. *Let V and ψ_V be as in Theorem 1.1. Let $W_s := \{\psi_V < s\}$. Then for any $\varphi \in m\mathrm{SH}(X,\omega)$, the limit*

$$\sigma(\varphi,\psi_V) := \lim_{s \to -\infty} \frac{\max_{W_s} \varphi}{s},$$

exists and is both finite and nonnegative.

Note that, if $m > k$, then we can still apply Corollaries 1.2 and 1.3 by just considering $\varphi \in m\mathrm{SH}(X,\omega) \subset k\mathrm{SH}(X,\omega)$.

Corollaries 1.2 and 1.3 are the first such results about the singularities of m-sh functions along submanifolds of positive dimension – for more information about Lelong numbers of m-sh functions at points, we refer the reader to [19], and also [22, 23, 11]. As one sees from Theorem 1.1 and Corollary 1.2, the natural scaling changes with the codimension k when $k \geqslant m$, something quite different from the psh case.

The scaling we obtain agrees with what one might naively expect if they were to assume that the restriction of φ to any k-dimensional submanifold transverse to V were still

m-sh. This assumption is very much not true; φ will generally lose positivity when restricted to a submanifold since $\sqrt{-1}\partial\bar{\partial}\varphi$ can have negative eigenvalues. That we still recover this optimal scaling may seem surprising in light of this.

This loss of positivity also means that we cannot use a slicing argument to reduce our situation to the case of studying the singularity of φ at the origin, as one does in the psh case. Instead, we show Theorem 1.1 by constructing smooth sub- and supersolutions (essentially) to the equation $H^m(\varphi) = 0$ on $\Omega \setminus V$. This is done by taking small perturbations of $G_m(r)$, using the computations of Tam-Yu [33] for the Hessian of r. The function ψ_V is then realized as a singularity type envelope; the sub- and supersolutions guarantee that ψ_V has the correct singularity type.

We show that our construction of subsolutions can be modified in Proposition 3.7 to produce m-sh functions whose behaviour near V is roughly like

$$-\theta G_m(r)$$

for any smooth $\theta \geqslant 0$ (see Proposition 3.7 for a precise statement). This behaviour is predicted by Åhag-Cegrell-Czyż-Hiệp's [1], and Hung-Phu's [24], solutions to certain highly degenerate complex Monge-Ampère (resp. complex Hessian) equations. Their solutions are constructed as a sequence of envelopes, and the resulting behaviour of the solutions near V is not easily seen. Our examples provide constraints on what that behaviour can be (which can be quite strong in fact – see Remark 3.8), but without corresponding supersolutions, we cannot deduce the exact singularity types of the solutions.

In the context of this paper, the examples in Proposition 3.7 are interesting in that they show that $\sigma(\varphi,\psi_V) \neq \nu(\varphi,\psi_V)$ in general, unlike the more classical case when the weight is singular only at a point. Indeed, these examples show that it is possible to have $\sigma(\varphi,\psi_V) = 0$ but $\nu(\varphi,\psi_V) > 0$ – the reverse is impossible (see Corollary 2.12).

We conclude that we cannot expect constraints which are significantly stronger than Corollaries 1.2 and 1.3 when V has codimension $\geqslant m$. When the codimension is less than m, however, our next result finds that the situation is much closer to the psh case:

Theorem 1.4. *Suppose that $V \subset X$ is a complex submanifold of codimension k and $\varphi \in m\mathrm{SH}(X,\omega)$. Suppose that $k < m$, and let $\psi_V \in k\mathrm{SH}(X,\omega)$ be the function constructed in Theorem 1.1. Then the function*

$$L_{\psi_V}(\varphi)(z) := \liminf_{\substack{z' \to z \\ z' \notin V}} \frac{\varphi(z')}{\psi_V(z')} \quad z \in V$$

is constant along V.

Theorem 1.4 can be seen as a version of Siu's theorem in our context. Note in particular that it implies a certain propagation of singularities along V since the function $L_{\psi_V}(\varphi)$ is defined pointwise.

Since the analyticity of V is already assumed, the proof of Theorem 1.4 is pleasantly elementary and boils down to essentially the weak Harnack inequality and Corollary 1.3.

Taken together, our results indicate a stark dichotomy in the behaviour of m-sh functions, depending on the size of their singular set. The situation where the singular set is small appears to be easier since it can be 'selected' by assuming that φ has a well-defined complex Hessian measure – specifically, one sees that, if we assume that $\varphi \in \mathcal{E}^m$ (the m-subharmonic version of Cegrell's class) then $\nu(\varphi, \psi_V) = 0$ for all $k \leqslant m$. The small codimension case is likely to be more interesting from a geometric point of view however, and it is unclear if there is a similar potential theoretic assumption one can make to limit to this situation. Finding conditions which work well in both cases appears challenging.

We conclude this introduction with an outline of the rest of the paper. In Section 2, we recall some background results, which should be more-or-less standard to experts. Since we will need to utilize the lack of boundary of V at several points, it is important that we do not restrict ourselves to domains in $\mathbb{C}^n$ and work rather on abstract m-hyperconvex manifolds. In Section 3, we prove Theorem 1.1 (Theorem 3.4), as well as Corollary 1.2 (Proposition 3.6). The examples in Proposition 3.7 are constructed in Subsection 3.4. Theorem 1.4 is finally shown in Section 4 (Theorem 4.2).

2. Background and notation

Throughout, we shall assume that (X^n, ω) is a closed Kähler manifold of complex dimension n, with Kähler form ω. (Ω^n, ω) will always denote a compact Kähler manifold with boundary. In this paper, we will always assume that $\Omega \subset X$, mostly as a matter of convenience. We always assume that $V^{n-k} \subset \Omega$ is a compact submanifold without boundary and (complex) codimension k (so that V has complex dimension $n-k$).

Definition 2.1. Suppose that (Ω^n, ω) is a Kähler manifold, with boundary, and let m be an integer between 1 and n. We say a smooth $(1,1)$-form α is m-subharmonic (or m-sh) if

$$\alpha^k \wedge \omega^{n-k} \geqslant 0 \text{ for each } k \in \{1, \ldots, m\}.$$

We denote the set of all m-sh forms by $\Gamma^m(\Omega)$, or $\Gamma_\omega^m(\Omega)$ if the metric needs to be specified.

In [9], Błocki defines a real $(1,1)$-current T to be m-sh if

$$T \wedge \alpha_2 \wedge \ldots \wedge \alpha_m \wedge \omega^{n-m} \geqslant 0 \text{ for all } \alpha_2, \ldots, \alpha_m \in \Gamma^m(\Omega).$$

By Gårding's inequality [20], the above definitions are consistent. Additionally, we shall say that T is strictly m-sh if $T - \delta\omega$ is m-sh for some $\delta > 0$.

We say a function ρ on Ω is m-sh if the $(1,1)$-current $dd^c\rho$ is m-sh and write $m\mathrm{SH}(\Omega)$ for the space of m-sh functions on Ω (or $m\mathrm{SH}_\omega(\Omega)$, if the metric needs to be specified). We will also say that ρ is strictly m-sh if $i\partial\bar{\partial}\rho$ is.

Given a closed, real $(1,1)$-form θ, we say that a function u is (m,θ)-sh if $\theta_u := \theta + dd^c u$ is m-sh and write $m\mathrm{SH}(X,\theta)$ for the set of all (m,θ)-sh functions on X.

Classically, the case which has been most studied is when $\theta = \omega$; however, it is easy to check that essentially all standard results hold when θ is any closed, strictly m-sh form.

2.1. *m*-hyperconvex manifolds

We have the following *m*-sh analogue of hyperconvex manifolds.

Definition 2.2. We say a compact Kähler manifold (Ω, ω) is *m*-hyperconvex if there exists a strictly *m*-sh exhaustion function on Ω° which is smooth up to the boundary, that is, a strictly *m*-sh function $0 > \rho \in C^{\infty}(\overline{\Omega})$ such that $\Omega_c := \{\rho < c\} \Subset \Omega$ for each $c < 0$.

It is easy to see that *m*-hyperconvex manifolds can contain nontrivial closed subvarieties. This basic observation is key to our setting, so it is important that we do not restrict our definition to domains in $\mathbb{C}^n$, which has been the primary case considered in much of the literature. Many of those previous results do not heavily utilize this assumption, however, and can be shown to hold for arbitrary hyperconvex domains with only minor changes to their proofs.

One difference between the two settings which is worth pointing out is that it is crucial for ρ to be strictly *m*-subharmonic in Definition 2.2. Without this, the complex Hessian operator may fail to be well-defined; for example, consider a neighborhood U of a smooth, ample divisor $D \subset \mathbb{CP}^n$. If we let s be a holomorphic section of $\mathcal{O}_{\mathbb{CP}^n}(D)$ with $\{s = 0\} = D$, and h a positive metric on $\mathcal{O}_{\mathbb{CP}^n}(D)$, then it is well-known that the product $(\sqrt{-1}\partial\overline{\partial} \log |s|_h)^2 \wedge \omega^{n-m}$ is not well-defined, even though there exists a smooth, psh exhaustion function on U. For positive results in this direction, see the recent papers [3, 6, 4, 10, 5], which define product currents which retain some of the singular nature of $\log |s|_h$.

Since we repeatedly make use of the fact that codimension *m*-subvarieties admit *m*-hyperconvex neighborhoods, we record this fact here:

Proposition 2.3. *Suppose that (X^n, ω) is a compact Kähler manifold and that $V \subset X$ is a compact submanifold of codimension m. Then there exists an m-hyperconvex neighborhood U of V with smooth boundary.*

Proof. Let $r(z) := \text{dist}_{\omega}(z, V)$ be the Riemannian distance function to V. When r is sufficiently small, r^2 is smooth. By [33],

$$\sqrt{-1}\partial\overline{\partial} r^2 = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} + o(1),$$

where $o(1)$ denotes a term satisfying $\lim_{r \to 0} o(1) = 0$. It is clear that the leading term belongs to Γ^m. Then we can take any sublevel set of r^2 to be the desired *m*-hyperconvex neighborhood of V. $\qquad\square$

2.2. The complex Hessian operator

For later use, we recall some basic facts about the complex Hessian operator on *m*-hyperconvex domains – see, for example, [34], which draws heavily from the work of Demailly [13, 14, 15, 17, 18] in the psh case. The above papers phrase their results in terms of a weight function ψ, and they required the unbounded locus of ψ to be a discrete set. However, it is easy to see that this assumption is superfluous for the results we will

need below – all that is needed is for the unbounded locus of ψ to be compact and that it not intersect the boundary of Ω, as in [18].

Throughout, we will be assuming that (Ω^n, ω) is an m-hyperconvex manifold (which recall we assume to be compact with boundary). We start by defining the classes of weights which we will study:

Definition 2.4. We say that ψ is a **weight** on Ω if ψ satisfies the following:

(1) ψ is bounded above.

(2) $\mathrm{Sing}(\psi) := \{\psi = -\infty\}$ is closed.

(3) the level sets $W_s := \{z \in \Omega \mid \psi(z) < s\}$ are connected and relatively compact for all s sufficiently negative.

Additionally, we will require our weights to satisfy a differential inequality.

- We say ψ is an m-**subweight** if ψ is m-sh.
- We say ψ is an m-**superweight** if $\psi \in C^\infty(\Omega \setminus \mathrm{Sing}(\psi))$ and $(\sqrt{-1}\partial\overline{\partial}\psi)^m \wedge \omega^{n-m} \leqslant 0$ there.
- We say ψ is a **maximal** m-**weight** if ψ is m-sh such that $(\sqrt{-1}\partial\overline{\partial}\psi)^m \wedge \omega^{n-m} = 0$ on $\Omega \setminus \mathrm{Sing}(\psi)$.

Sub- and superweights are suited to different measurements of the singularities of m-sh functions; subweights work well for Lelong numbers, while superweights are required when working with the relative type. Maximal m-weights are precisely those weights for which these two types of measurements can be compared.

Now, by [18, 34], for any m-subweight ψ, the complex Hessian operator

$$(\sqrt{-1}\partial\overline{\partial}\psi)^m \wedge \omega^{n-m}$$

is a well-defined, positive Borel measure on Ω, with locally finite mass, which is continuous along decreasing sequences (that is to say that $\psi \in \mathcal{E}^m(\Omega)$, the m-sh version of the Cegrell class [26]). More generally, if $\psi_1, \ldots, \psi_{m-1}$ are all m-subweights and $\varphi \in m\mathrm{SH}(\Omega)$ is arbitrary, then the mixed measure:

$$\sqrt{-1}\partial\overline{\partial}\varphi \wedge \sqrt{-1}\partial\overline{\partial}\psi_1 \wedge \ldots \wedge \sqrt{-1}\partial\overline{\partial}\psi_{m-1} \wedge \omega^{n-m}$$

shares these same properties and is moreover multilinear in each argument.

Definition 2.5. [18, 34] Suppose that ψ is an m-subweight on Ω such that $\{\psi < -1\} \Subset \Omega$. Let $\varphi \in m\mathrm{SH}(\Omega)$ be such that $\varphi \leqslant -1$. For any $s < -1$, set

$$W_s := \{\psi < s\}.$$

Then we define:

$$\nu_m(\varphi, \psi) := \lim_{s \to -\infty} \int_{W_s} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{m-1} \wedge \omega^{n-m} \qquad (2.1)$$

$$= \int_{\{\psi = -\infty\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{m-1} \wedge \omega^{n-m}.$$

We call $\nu_m(\varphi,\psi)$ the **generalized m-Lelong number** of φ with respect to ψ or, more compactly, the (m,ψ)-Lelong number of φ. By standard results, $\nu_m(\varphi,\psi)$ is always finite and nonnegative, and the sequence in Equation (2.1) is monotone decreasing.

The following comparison theorem for m-polar measures is standard, and can be obtained by following the proof of [1, Lemma 4.1] (see also [17, 24]):

Proposition 2.6. *Suppose that ψ_1,ψ_2 are m-subweights and $\varphi_1,\varphi_2 \in mSH(\Omega)$ are such that $\psi_1 \leqslant \psi_2 + C$ and $\varphi_1 \leqslant \varphi_2 + C$ for some constant C. Let $S := \mathrm{Sing}(\psi_2) \cap \mathrm{Sing}(\varphi_2)$. Then*

$$\chi_S \sqrt{-1}\partial\overline{\partial}\varphi_2 \wedge (\sqrt{-1}\partial\overline{\partial}\psi_2)^{m-1} \wedge \omega^{n-m} \leqslant \chi_S \sqrt{-1}\partial\overline{\partial}\varphi_1 \wedge (\sqrt{-1}\partial\overline{\partial}\psi_1)^{m-1} \wedge \omega^{n-m}.$$

2.3. Relative types

Relative-types were first studied by Rashkovskii in [30]. The natural generalization of his original definition is the following lower semicontinuous function defined on $\mathrm{Sing}(\psi)$:

Definition 2.7. Suppose that $\psi \leqslant -1$ is an m-superweight (or a maximal m-weight) and $\varphi \in mSH(\Omega)$. We define

$$L_\psi(\varphi)(z_0) := \liminf_{\substack{z \to z_0 \\ z \notin \mathrm{Sing}(\psi)}} \frac{\varphi(z)}{\psi(z)}.$$

It is easy to see that $L_\psi(\varphi)$ is a lower-semicontinous function on $\mathrm{Sing}(\psi)$ (Proposition 2.11).

For a general ψ, this definition is somewhat lacking on its own. One of the main applications of the relative type is to get bounds of the form $\varphi \leqslant L_\psi(\varphi) \cdot \psi + C$ on a neighbourhood of $\mathrm{Sing}(\psi)$ – but since $L_\psi(\varphi)$ is nonconstant, the right-hand side is only defined up to fixing an extension of $L_\psi(\varphi)$ to a neighborhood of $\mathrm{Sing}(\psi)$. Since $L_\psi(\varphi)$ is only lower semicontinuous and $\mathrm{Sing}(\psi)$ can be very poorly behaved, it seems possible to us that this inequality may fail to hold if $L_\psi(\varphi)$ is extended haphazardly.

A definition which can be more easily applied to get upper bounds for φ is the following, which recovers the minimum of $L_\psi(\varphi)$ (although we will see later that this definition has drawbacks of its own):

Definition 2.8. Suppose that $(\Omega^n,\omega) \subset (X^n,\omega)$ is an m-hyperconvex manifold. Let ψ be an m-superweight (or maximal m-weight) on Ω. Suppose that $W_0 = \{z \in \Omega \mid \psi(z) < 0\} \Subset \Omega$.

There exists a constant $A > 0$ such that for any $\varphi \in mSH(X,\omega)$, we have

$$\varphi + A\rho \in mSH(\Omega).$$

Define

$$M_s(\varphi + A\rho) := \max_{W_s}(\varphi + A\rho).$$

Then we define the **relative type** of φ with respect to ψ to be

$$\sigma(\varphi,\psi) := \lim_{s\to-\infty} \frac{M_s(\varphi + A\rho)}{s} = \lim_{s\to-\infty} \frac{\max_{W_s}(\varphi)}{s}. \tag{2.2}$$

That $\sigma(\varphi,\psi)$ is well-defined comes from the following three-circles type result (see, e.g., [25] for similar results in the psh case):

Proposition 2.9. *The function $M_s(\varphi + A\rho)$ in Definition 2.8 is convex on $(-\infty, -1)$. It follows that the limit (2.2) exists, and $\sigma(\varphi,\psi)$ is always finite and nonnegative.*

Proof. Suppose first that ψ is a superweight and that φ is also smooth on Ω. Let $\rho \leqslant 0$ be a strictly m-sh function on Ω, and define $\varphi_\varepsilon := \varphi + (A+\varepsilon)\rho$. Let $-\infty < s_1 < s_2 < -1$, and consider the function

$$F(z) := \frac{s_2 - \psi(z)}{s_2 - s_1} M_{s_1}(\varphi_\varepsilon) + \frac{\psi(z) - s_1}{s_2 - s_1} M_{s_2}(\varphi_\varepsilon)$$

defined on $W := W_{s_2} \setminus \overline{W}_{s_1}$. We seek to show that

$$\varphi_\varepsilon(z) \leqslant F(z) \text{ on } W.$$

It is clear this inequality holds on ∂W, so suppose for the sake of a contradiction that $\varphi_\varepsilon - F$ admits an interior maximum at some $z_0 \in W$. It follows that

$$0 < (\sqrt{-1}\partial\overline{\partial}\varphi_\varepsilon(z_0))^m \wedge \omega^{n-m} \leqslant (\sqrt{-1}\partial\overline{\partial}F(z_0))^m \wedge \omega^{n-m}.$$

But this is impossible, as F is a superweight. Taking the limit as $\varepsilon \to 0$ finishes the proof in this case.

The case of a general φ follows now by using a decreasing sequence of smooth m-sh functions which converge to φ [27, 29].

If ψ is instead a maximal subweight, we may bypass the smooth approximation argument and instead appeal to the comparison principle directly since in this case, F will be a maximal m-sh function on $W_{s_2} \setminus \overline{W}_{s_1}$. $\qquad\square$

An immediate consequence of Proposition 2.9 is the following alternative definition for $\sigma(\varphi,\psi)$:

Proposition 2.10. $\sigma(\varphi,\psi) = \max\{\gamma \geqslant 0 \mid \varphi \leqslant \gamma\psi + O(1)\}.$

Proof. The proof is standard – it suffices to check that

$$\sigma(\varphi,\psi) \in \{\gamma \geqslant 0 \mid \varphi \leqslant \gamma\psi + O(1)\}.$$

Let $A > 0$ be such that $\varphi + A\rho \in m\mathrm{SH}(\Omega)$. Convexity of $M_s(\varphi + A\rho)$ implies that $\sigma(\varphi,\psi)$ can be computed by the slopes of the secant lines

$$\sigma(\varphi,\psi) = \lim_{s\to-\infty} \frac{M_{s_0}(\varphi + A\rho) - M_s(\varphi + A\rho)}{s_0 - s} \tag{2.3}$$

for any fixed $s_0 < -1$. Since $M_s(\varphi + A\rho)$ is also decreasing as $s \to -\infty$, the sequence in Equation (2.3) is decreasing, and we have

$$M_s(\varphi + A\rho) \leqslant \sigma(\varphi,\psi)s + M_{s_0}(\varphi + A\rho) - \sigma(\varphi,\psi)s_0,$$

for all $s < s_0$. From this, we see that

$$\varphi \leqslant \sigma(\varphi,\psi)\psi + O(1) \quad \text{on } W_{s_0}. \qquad \square$$

For later use, we record the following facts:

Proposition 2.11. *The function $L_\psi(\varphi)$ is lower semicontinuous on $S_\psi := \text{Sing}(\psi)$, and*

$$\sigma(\varphi,\psi) = \min_{\text{Sing}(\psi)} L_\psi(\varphi).$$

Proof. We first show the lower semicontinuity of $L_\psi(\varphi)$. Fix a point $z_0 \in S_\psi$, and let $z_i \in S_\psi$ be a sequence of points converging to z_0. For any $\varepsilon > 0$, there exists $x_i \in X \setminus S_\psi$ such that

$$\text{dist}_\omega(x_i, z_i) \leqslant i^{-1}, \quad \frac{\varphi(x_i)}{\psi(x_i)} \leqslant L_\psi(\varphi)(z_i) + \varepsilon.$$

It is clear that $x_i \to z_0$. By definition,

$$L_\psi(\varphi)(z_0) \leqslant \liminf_{i \to \infty} \frac{\varphi(x_i)}{\psi(x_i)} \leqslant \liminf_{i \to \infty} L_\psi(\varphi)(z_i) + \varepsilon.$$

Letting $\varepsilon \to 0$ concludes.

Now, it is clear from the definitions that $L_\psi(\varphi) \geqslant \sigma(\varphi,\psi)$. To see that it is the minimum, using that φ is upper semicontinuous, we can find a sequence of points $z_i \in W_{-i} = \{\psi \leqslant -i\}$ such that $\varphi(z_i) = \max_{W_{-i}} \varphi$. It follows from compactness of S_ψ that there exists a convergent subsequence $z_i \to z_0 \in S_\psi$, and so we see that $L_\psi(\varphi)(z_0) \leqslant \sigma(\varphi,\psi)$, finishing the proof. $\qquad \square$

Again, following Demailly [18], Proposition 2.6 can be used to compare $\sigma(\varphi,\psi)$ and $\nu_m(\varphi,\psi)$. For notational convenience, we set

$$S_\psi := \text{Sing}(\psi) \quad \text{and} \quad \mu_\psi := \chi_{S_\psi}(\sqrt{-1}\partial\overline{\partial}\psi)^m \wedge \omega^{n-m}.$$

Note that $\text{Supp}(\mu_\psi) \subseteq S_\psi$, but the inclusion may be proper; indeed, as Example 2.1 of [2] shows, $\text{Supp}(\mu_\psi)$ can be quite small inside S_ψ.

Corollary 2.12. *Suppose that $\psi \leqslant -1$ is a maximal m-weight such that $\mu_\psi(S_\psi) > 0$. Then*

$$\sigma(\varphi,\psi) \leqslant \frac{1}{\mu_\psi(S_\psi)}\nu_m(\varphi,\psi).$$

Proof. By Corollary 2.10, we have that

$$\varphi \leqslant \sigma(\varphi,\psi)\psi + C$$

in some neighborhood of S_ψ. By Proposition 2.6, it follows that

$$\sigma(\varphi,\psi)(\sqrt{-1}\partial\overline{\partial}\psi)^m \wedge \omega^{n-m} \leqslant \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{m-1} \wedge \omega^{n-m}.$$

Integrating over S_ψ finishes the proof. $\qquad \square$

Remark 2.13. Proposition 2.6 allows us to define the Radon–Nikodym derivative of $\chi_{S_\psi}\sqrt{-1}\partial\overline{\partial}\varphi\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{m-1}\wedge\omega^{n-m}$ with respect to μ_ψ in certain cases, for example, when $\varphi\in\mathcal{E}^m(\Omega)$ (see [1, Lem. 4.4]). It is interesting to ask how different this function is from $L_\psi(\varphi)$.

3. Construction of m-Weights along submanifolds

In this section, we construct our maximal m-weights ψ_V, associated to the complex submanifold V.

3.1. Construction of the m-subweight associated to V

Let $V\subseteq X$ be a smooth submanifold of codimension k. Let $r(z):=\operatorname{dist}_\omega(z,V)$ be the Riemannian distance function to V, and choose $0<s_V<1$ sufficiently small so that $\Omega:=\{z\in X\mid r(z)<s_V\}$ does not intersect the cut-locus of V.

It follows that r will be smooth on $\Omega\setminus V$, so that

$$r_\varepsilon:=\sqrt{r^2+\varepsilon}$$

will be smooth on all of Ω for any $\varepsilon>0$.

Define $G_m:\mathbb{R}_{\geqslant 0}\to\mathbb{R}\cup\{-\infty\}$:

$$G_m(t):=\begin{cases}\log t & \text{if } m=k\\ -t^{2-\frac{2k}{m}} & \text{if } m<k.\end{cases}$$

G_m will be the natural scaling of a maximal m-weight near V – by perturbing it slightly, we will be able to produce sub- and superweights. Our perturbations will be of the form

$$h(s):=s+As^{1+\delta},$$

where $\max\left\{1,2\left(\frac{k}{m}-1\right)\right\}<\delta$ will be a uniform constant for all s sufficiently small. We will also have $A=\pm 1$, depending on if we are constructing the sub- or supersolution. Note that as long as $\max\left\{1,2\left(\frac{k}{m}-1\right)\right\}<\delta$, $G_m(h(r))$ and $G_m(r)$ will have the same singularity type on Ω, for either choice of A.

We will need to choose suitable coordinates for our computations, which we fix once and for all. Let $z\in\Omega\setminus V$, and let $x\in V$ be the unique nearest point to z. Consider the geodesic connecting these points, which will be normal to V at x. Choose a unitary frame $\{e_i\}_{i=1}^n$ at x such that

$$e_1=\frac{1}{\sqrt{2}}(\nabla r-\sqrt{-1}J\nabla r),$$

the $e_2,\ldots,e_k$ are perpendicular to V, and the $e_{k+1},\ldots,e_n$ are tangent to V; we also write $\{e_i\}_{i=1}^n$ for the parallel transport of this frame to z. Then, according to [33, Lemma 2.2], in these coordinates we have that

$$r_{i\bar{j}}=\begin{pmatrix}\frac{1}{2r} & 0 & 0\\ 0 & \frac{1}{r}\cdot\operatorname{Id}_{k-1} & 0\\ 0 & 0 & 0\end{pmatrix}+o(1),$$

where $o(1)$ denotes a term satisfying $\lim_{r \to 0} o(1) = 0$ (see in particular Equation (2.22) in [33]).

In what follows, it suffices to restrict to the case $m \leqslant k$, since, if $m > k$, then the m-sh functions we construct will also be k-sh.

Proposition 3.1. *Suppose that $z \in \Omega \setminus V$ and that $m \leqslant k$. Let $D_{k,m} = \left(\frac{2k}{m} - 2 \right)^{-1}$ if $k \neq m$ and $D_{k,m} = 1$ if $k = m$.*

Then for any $\varepsilon > 0$, we have

$$D_{k,m} h^{\frac{2k}{m}} \left(\sqrt{-1} \partial \overline{\partial} G_m(h(r_\varepsilon)) \right)_{i\bar{j}} = \begin{pmatrix} 1 - \frac{k}{m} + AB_1 r_\varepsilon^\delta & 0 & 0 \\ 0 & \left(1 + AB_2 r_\varepsilon^\delta \right) \mathrm{Id}_{k-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ \frac{\varepsilon}{r_\varepsilon^2} \begin{pmatrix} \frac{k}{m} + AB_3 r_\varepsilon^\delta + o(r_\varepsilon^\delta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(r_\varepsilon^\delta), \qquad (3.1)$$

where we define the constants

$$B_1 = 2 \left[1 - \frac{k}{m} + \left(1 - \frac{k}{m} \right) \delta + \frac{\delta^2}{4} \right]$$

and

$$B_2 := 2 + \delta, \qquad B_3 := 2 \left[\frac{k}{m} + \delta \left(\frac{k}{m} - \frac{1}{2} \right) - \frac{\delta^2}{4} \right].$$

The terms $o(r_\varepsilon^\delta)$ satisfy $o(r_\varepsilon^\delta) \leqslant c r_\varepsilon^\delta$ for any $0 < c$ and all $0 < r_\varepsilon^\delta < r_c$, where r_c depends on both c and an upper bound for $\delta > \max \left\{ 1, 2 \left(\frac{k}{m} - 1 \right) \right\}$.

Since the constant $D_{k,m} > 0$, it will have no bearing on the positivity/negativity of $\sqrt{-1} \partial \overline{\partial} G_m(h(r_\varepsilon))$, and we will omit it in all of the below computations.

For later use, we also remark that it will be clear from the below proof that (3.1) still holds if either $\varepsilon > 0$ and $z \in V$ or if $\varepsilon = 0$ and $z \notin V$ (replacing $o(r_\varepsilon)$ with $o(r)$ in this case).

Proof. The proof is a computation. To start, if $m < k$, then we have

$$\left(G_m(h(r_\varepsilon)) \right)_{i\bar{j}} = \left(\frac{2k}{m} - 2 \right) \cdot h^{-\frac{2k}{m}} \cdot \left(hh'(r_\varepsilon)_{i\bar{j}} + \left(hh'' + \left(1 - \frac{2k}{m} \right) (h')^2 \right) (r_\varepsilon)_i (r_\varepsilon)_{\bar{j}} \right),$$

while if $m = k$

$$\left(G_m(h(r)) \right)_{i\bar{j}} = h^{-2} \cdot \left(hh'(r_\varepsilon)_{i\bar{j}} + \left(hh'' - (h')^2 \right) (r_\varepsilon)_i (r_\varepsilon)_{\bar{j}} \right).$$

Thus, we may deal with both cases simultaneously by computing

$$hh'(r_\varepsilon)_{i\bar{j}} + \left(hh'' + \left(1 - \frac{2k}{m} \right) (h')^2 \right) (r_\varepsilon)_i (r_\varepsilon)_{\bar{j}}. \qquad (3.2)$$

We now compute

$$(r_\varepsilon)_i = \frac{rr_i}{\sqrt{r^2 + \varepsilon}} = \frac{rr_i}{r_\varepsilon}, \quad (r_\varepsilon)_i(r_\varepsilon)_{\bar{j}} = \frac{r^2 r_i r_{\bar{j}}}{r_\varepsilon^2}$$

and

$$\begin{aligned}
(r_\varepsilon)_{i\bar{j}} &= \frac{rr_{i\bar{j}}}{\sqrt{r^2 + \varepsilon}} + \frac{r_i r_{\bar{j}}}{\sqrt{r^2 + \varepsilon}} - \frac{r^2 r_i r_{\bar{j}}}{(r^2 + \varepsilon)^{\frac{3}{2}}} \\
&= \frac{rr_{i\bar{j}}}{r_\varepsilon} + \frac{\varepsilon r_i r_{\bar{j}}}{r_\varepsilon^3}.
\end{aligned}$$

Hence, in our coordinates, we have

$$(r_\varepsilon)_i(r_\varepsilon)_{\bar{j}} = \frac{r^2 r_i r_{\bar{j}}}{r_\varepsilon^2} = \begin{pmatrix} \frac{r^2}{2r_\varepsilon^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$(r_\varepsilon)_{i\bar{j}} = \frac{rr_{i\bar{j}}}{r_\varepsilon} + \frac{\varepsilon r_i r_{\bar{j}}}{r_\varepsilon^3} = \begin{pmatrix} \frac{1}{2r_\varepsilon} + \frac{\varepsilon}{2r_\varepsilon^3} & 0 & 0 \\ 0 & \frac{1}{r_\varepsilon} \cdot \mathrm{Id}_{k-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{r}{r_\varepsilon} \cdot o(1).$$

Applying this to Equation (3.2) gives

$$hh'(r_\varepsilon)_{i\bar{j}} + \left(hh'' + \left(1 - \frac{2k}{m} \right) (h')^2 \right) (r_\varepsilon)_i(r_\varepsilon)_{\bar{j}}$$

$$= \begin{pmatrix} (hh')\left(\frac{1}{2r_\varepsilon} + \frac{\varepsilon}{2r_\varepsilon^3} \right) + \frac{r^2}{2r_\varepsilon^2} hh'' + \left(\frac{1}{2} - \frac{k}{m} \right) \frac{r^2}{r_\varepsilon^2} (h')^2 & 0 & 0 \\ 0 & (hh')\frac{1}{r_\varepsilon} \cdot \mathrm{Id}_{k-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(r)$$

as $hh' = r_\varepsilon + o(r_\varepsilon)$.

We simplify this matrix. Recalling that $h(s) = s + As^{1+\delta}$, we have

$$h'(s) = 1 + A(1+\delta)s^\delta, \quad \text{and} \quad h''(s) = A\delta(1+\delta)s^{\delta-1}.$$

It follows that

$$hh' = s + (2+\delta)As^{1+\delta} + (1+\delta)A^2 s^{1+2\delta}$$
$$hh'' = (1+\delta)\delta As^\delta + (1+\delta)\delta A^2 s^{2\delta}$$
$$(h')^2 = 1 + 2(1+\delta)As^\delta + (1+\delta)^2 A^2 s^{2\delta}$$

so that the first entry simplifies to

$$(hh')\left(\frac{1}{2r_\varepsilon} + \frac{\varepsilon}{2r_\varepsilon^3} \right) + \frac{r^2}{2r_\varepsilon^2} hh'' + \left(\frac{1}{2} - \frac{k}{m} \right) \frac{r^2}{r_\varepsilon^2} (h')^2$$

$$= \left(r_\varepsilon + (2+\delta)Ar_\varepsilon^{1+\delta} + (1+\delta)A^2 r_\varepsilon^{1+2\delta} \right) \left(\frac{1}{2r_\varepsilon} + \frac{\varepsilon}{2r_\varepsilon^3} \right)$$

$$+\left(\frac{1}{2}-\frac{\varepsilon}{2r_\varepsilon^2}\right)\left((1+\delta)\delta Ar_\varepsilon^\delta+(1+\delta)\delta A^2r_\varepsilon^{2\delta}\right)$$

$$+\left(\frac{1}{2}-\frac{\varepsilon}{2r_\varepsilon^2}\right)\left(1-\frac{2k}{m}\right)\left(1+2(1+\delta)Ar_\varepsilon^\delta+(1+\delta)^2A^2r_\varepsilon^{2\delta}\right)$$

$$=\left(\frac{1}{2}+\frac{\varepsilon}{2r_\varepsilon^2}\right)\left(1+(2+\delta)Ar_\varepsilon^\delta+(1+\delta)A^2r_\varepsilon^{2\delta}\right)$$

$$+\left(\frac{1}{2}-\frac{\varepsilon}{2r_\varepsilon^2}\right)\left((1+\delta)\delta Ar_\varepsilon^\delta+(1+\delta)\delta A^2r_\varepsilon^{2\delta}\right)$$

$$+\left(\frac{1}{2}-\frac{\varepsilon}{2r_\varepsilon^2}\right)\left(1-\frac{2k}{m}\right)\left(1+2(1+\delta)Ar_\varepsilon^\delta+(1+\delta)^2A^2r_\varepsilon^{2\delta}\right)$$

$$=1-\frac{k}{m}+2A\left[1-\frac{k}{m}+\left(1-\frac{k}{m}\right)\delta+\frac{\delta^2}{4}\right]r_\varepsilon^\delta+o(r_\varepsilon^\delta)$$

$$+\frac{\varepsilon}{r_\varepsilon^2}\left(\frac{k}{m}+2A\left(\frac{k}{m}+\delta\left(\frac{k}{m}-\frac{1}{2}\right)-\frac{\delta^2}{4}\right)r_\varepsilon^\delta+o(r_\varepsilon^\delta)\right)$$

$$=1-\frac{k}{m}+AB_1r_\varepsilon^\delta+\frac{\varepsilon}{r_\varepsilon^2}\left(\frac{k}{m}+AB_3r_\varepsilon^\delta+o(r_\varepsilon^\delta)\right)+o(r_\varepsilon^\delta).$$

As $\frac{hh'}{r_\varepsilon}=1+AB_2r_\varepsilon^\delta+o(r_\varepsilon^\delta)$ and $0<\max\{r,\sqrt{\varepsilon}\}\leqslant r_\varepsilon$, we conclude Equation (3.1). $\square$

We may now construct our m-subweights.

Proposition 3.2. *Suppose that $m\leqslant k$. Then for any $\delta>\max\left\{1,2\left(\frac{k}{m}-1\right)\right\}$, there exists a $0<\underline{s}<s_V/2$ such that for all $0<\varepsilon<s_V/2$ we have that*

$$\underline{\psi}_{V,\varepsilon}:=G_m(\underline{h}(r_\varepsilon))\in m\mathrm{SH}(W_{\underline{s}}),$$

where $\underline{h}(s):=s+s^{1+\delta}$. Moreover, as $\varepsilon\to 0$, we have that

$$\underline{\psi}_{V,\varepsilon}\searrow G_m(\underline{h}(r)):=\underline{\psi}_V\in m\mathrm{SH}(W_{\underline{s}}).$$

The constant $\underline{s}$ depends on k,m,ω,V, and an upper bound for δ.

Proof. Since $G_m(\underline{h}(r_\varepsilon))$ is smooth on Ω, it will be sufficient to check that it is m-sh at an arbitrary $z\in\Omega\setminus V$ (though the computations translate easily to the case when $z\in V$). Choose coordinates as before, so that Proposition 3.1 with $A=1$ gives

$$D_{k,m}\underline{h}^{\frac{2k}{m}}(\sqrt{-1}\partial\overline{\partial}\underline{\psi}_{V,\varepsilon})_{i\bar{j}}=\begin{pmatrix}1-\frac{k}{m}+B_1r_\varepsilon^\delta & 0 & 0\\ 0 & \left(1+B_2r_\varepsilon^\delta\right)\mathrm{Id}_{k-1} & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$+\frac{\varepsilon}{r_\varepsilon^2}\begin{pmatrix}\frac{k}{m}+B_3r_\varepsilon^\delta+o(r_\varepsilon^\delta) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}+o(r_\varepsilon^\delta). \qquad (3.3)$$

B_3 can be bounded using an upper bound on δ, so long as $\underline{s}$ is sufficiently small the second matrix will be positive semidefinite for any $\varepsilon>0$ and can be dropped from all further computations.

When $k = m$, Equation (3.3) now simplifies to

$$\underline{h}^2(\sqrt{-1}\partial\overline{\partial}\underline{\psi}_{V,\varepsilon})_{i\overline{j}} \geqslant \begin{pmatrix} \frac{\delta^2}{2}r_\varepsilon^\delta & 0 & 0 \\ 0 & \mathrm{Id}_{k-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(r_\varepsilon^\delta).$$

The error term can be controlled by shrinking $\underline{s}$ if necessary, again depending on δ, completing this case.

We now deal with the case when $m < k$. We compute the leading term of the j-th symmetric polynomial $(1 \leqslant j \leqslant m)$ of Equation (3.3) to be

$$\left(1 - \frac{k}{m} + B_1 r_\varepsilon^\delta\right)(1 + B_2 r_\varepsilon^\delta)^{j-1} + \left(\frac{k}{j} - 1\right)(1 + B_2 r_\varepsilon^\delta)^j$$

$$= \frac{k}{j} - \frac{k}{m} + \left[B_1 + \left(\left(1 - \frac{k}{m}\right)(j-1) + \left(\frac{k}{j} - 1\right)j\right)B_2\right]r_\varepsilon^\delta + o(r_\varepsilon^\delta),$$

omitting a multiplicative factor of $j \cdot (k-1) \ldots (k-j+1)$. When $j < m$, the leading term is positive. When $j = m$, this term is zero, so the leading order term becomes

$$\left[B_1 + \left(\left(1 - \frac{k}{m}\right)(m-1) + \left(\frac{k}{m} - 1\right)m\right)B_2\right]r_\varepsilon^\delta$$

$$= \left[B_1 + \left(\frac{k}{m} - 1\right)B_2\right]r_\varepsilon^\delta$$

$$= \delta\left[\frac{\delta}{2} - \left(\frac{k}{m} - 1\right)\right]r_\varepsilon^\delta$$

since we have assumed

$$\delta > 2\left(\frac{k}{m} - 1\right).$$

By taking $\underline{s}$ sufficiently small depending on the above constant, we can control the error term, showing that $\sqrt{-1}\partial\overline{\partial}\underline{\psi}_{V,\varepsilon}$ is an m-subweight. Letting $\varepsilon \to 0$ shows the secondary statement in the proposition immediately. $\qquad\square$

3.2. Construction of the m-superweight

We may now construct our m-superweight by very similar considerations:

Proposition 3.3. *Suppose that $m \leqslant k$. Then for any $\delta > \max\left\{1, 2\left(\frac{k}{m} - 1\right)\right\}$, there exists a $0 < \overline{s} < s_V/2$, depending on k, m, ω, V, and an upper bound for δ so that if $\overline{h}(s) := s - s^{1+\delta}$, then*

$$\overline{\psi}_V := G_m(\overline{h}(r))$$

is an m-superweight on $W_{\overline{s}}$.

Proof. By the remark immediately following Proposition 3.1, we have

$$\overline{h}^{\frac{2k}{m}}(\sqrt{-1}\partial\overline{\partial}\psi_V)_{i\overline{j}} = \begin{pmatrix} 1 - \frac{k}{m} - B_1 r^\delta & 0 & 0 \\ 0 & (1 - B_2 r^\delta)\,\mathrm{Id}_{k-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(r^\delta). \quad (3.4)$$

When $k = m$, Equation (3.4) simplifies to

$$\overline{h}^2(\sqrt{-1}\partial\overline{\partial}\psi_V)_{i\overline{j}} = \begin{pmatrix} -\frac{\delta^2}{2} r^\delta & 0 & 0 \\ 0 & (1 - (2+\delta)r^\delta)\mathrm{Id}_{k-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(r^\delta).$$

Choosing $\overline{s}$ sufficiently small relative to δ again controls the error term so that this is a superweight.

When $m < k$, we again compute the leading terms of the j-th symmetric polynomials of the matrix in Equation (3.4). When $j < m$, the leading order term is the same as the leading term in Proposition 3.2; when $m = j$, it becomes the negation of the term in Proposition 3.2 and hence will be negative as long as $\delta > 2\left(\frac{k}{m} - 1\right)$. Again, the error terms are an order of magnitude smaller than the leading order term and so can be controlled assuming an upper bound for δ. $\qquad\square$

3.3. The maximal m-weight along V

We now show Theorem 1.1, and construct ψ_V from $\underline{\psi}_V$ and $\overline{\psi}_V$ by using an envelope to make the complex Hessian measure vanish.

Theorem 3.4. *Suppose that $m \leqslant k$, and $\delta > \max\left\{1, 2\left(\frac{k}{m} - 1\right)\right\}$. Let $\underline{\psi}_V$ and $\overline{\psi}_V$ be the m-subweight and m-superweights constructed in Propositions 3.2 and 3.3, respectively, for the given δ. Set $s_0 := \min\{\underline{s}, \overline{s}\}$. Then the function*

$$\psi_V := \sup\{\varphi \in m\mathrm{SH}(X, \omega) \mid \varphi \leqslant 0, \varphi(z) \leqslant G_m(r(z)) + O(1)\}^*$$

is a maximal m-weight on W_{s_0}, which moreover has the same singularity type as $\underline{\psi}_V, \overline{\psi}_V$, and $G_m(r)$.

Proof. As mentioned at the beginning of this section, the three functions $\underline{\psi}_V, \overline{\psi}_V$, and $G_m(r)$ have the same singularity type so that

$$\underline{\psi}_V - C_0 \leqslant G_m(r) \leqslant \overline{\psi}_V + C_0 \leqslant \underline{\psi}_V + C_0$$

for some sufficiently large constant C_0. Since it is clear that the definition of ψ_V only depends on the singularity type of $G_m(r)$, we have that

$$\psi_V := \sup\{\varphi \in m\mathrm{SH}(X, \omega) \mid \varphi \leqslant 0, \varphi(z) \leqslant \overline{\psi}_V(z) + O(1)\}^*.$$

Since $\underline{\psi}_V \leqslant 0$, the envelopes

$$\psi_{V,C} := \sup\{\varphi \in m\mathrm{SH}(X, \omega) \mid \varphi \leqslant \min\{0, \overline{\psi}_V + C\}\}^*$$

satisfy

$$\underline{\psi}_V \leqslant \psi_{V,C}$$

for all $C \geqslant C_0$. Recall that the $\psi_{V,C}$ increase to ψ_V as $C \to \infty$ so that $\underline{\psi}_V \leqslant \psi_V$ as well.

Let $A > 0$ be such that $\overline{\psi}_V + A \geqslant 0$ on ∂W_{s_0}. By Lemma 3.5 below, we have that

$$\psi_{V,C} \leqslant \overline{\psi}_V + A \text{ on } W_{s_0}.$$

Letting $C \to \infty$ now shows that $\psi_V \leqslant \overline{\psi}_V + A$, so ψ_V has the same singularity type as $G_m(r)$.

The conclusion about vanishing mass follows from a standard balayage argument – see, for example, [7]. □

The following lemma is just a restatement of [31, Lemma 4.1].

Lemma 3.5. *Suppose that $\psi \leqslant 0$ is an m-superweight (or a maximal m-weight) and $\varphi \in m\mathrm{SH}(X,\omega)$, $\varphi \leqslant 0$, is such that $\sigma(\varphi,\psi) \geqslant 1$. If $\varphi \leqslant \psi$ on $\partial\Omega$, then there exists a constant, independent of φ so that $\varphi \leqslant \psi + C$ on all of Ω.*

Proof. Let $s_0 \leqslant 0$ be such that $W_{s_0} := \{\psi(z) < s_0\} \Subset \Omega$. Since $\sigma(\varphi,\psi) \geqslant 1$, we can use Proposition 2.9 to see that

$$(\varphi + A\rho)(z) \leqslant \psi(z) - s_0 + M_{s_0}(\varphi + A\rho) \leqslant \psi(z) - s_0,$$

for all $z \in W_{s_0}$.

It follows that $\varphi + A\rho \leqslant \psi - s_0$ on $\partial(\Omega \setminus W_{s_0})$. We can now conclude by the maximum/comparison principle, as in Proposition 2.9 (depending on if ψ is a super/maximal weight). □

We now show Corollary 1.2, proving a formula for $\nu(\varphi,\psi_V)$ as a density of a weighted Laplacian of φ, which has been averaged over V.

Proposition 3.6. *Suppose that we are in the setting of the previous section so that r is the geodesic distance to $V^{n-k} \subset \Omega^n$. Let $\varphi \in m\mathrm{SH}(\Omega)$ with $m \leqslant k$. Then*

$$\nu_m(\varphi,\psi_V) = \lim_{s \to 0} \frac{C_{k,m}}{s^{2k-2k/m}} \int_{\{r<s\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}r^2)^{m-1} \wedge \omega^{n-m}$$

for some constant $C_{k,m}$ depending only on k,m.

Proof. For the sake of convenience, we only give the proof when $k > m$. Let $H_m(t) := (-t)^{-\frac{m}{k-m}}$. Note that H_m is convex and increasing on $(-\infty,0)$ and that

$$H_m(G_m(t)) = t^2.$$

Since $\{\underline{\psi}_V < G_m(s)\} = \{\underline{h}(r) < s\}$, by [18, Chapter III, Formula (5.5)], we have that

$$
\int_{\{\underline{h}<s\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\underline{h}^2)^{m-1}\wedge\omega^{n-m}
$$

$$
= (H'm(G_m(s)))^{m-1}\int_{\{\underline{\psi}_V \leqslant G_m(s)\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\underline{\psi}_V)^{m-1}\wedge\omega^{n-m}
$$

$$
= \left(\frac{m}{k-m}\right)^{m-1}\left(s^{\frac{2(m-k)}{m}}\right)^{\frac{-k(m-1)}{k-m}}\int_{\{\underline{\psi}_V \leqslant G_m(s)\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\underline{\psi}_V)^{m-1}\wedge\omega^{n-m}
$$

so that

$$
\nu_m(\varphi,\underline{\psi}_V) = \left(\frac{k-m}{m}\right)^{m-1}\lim_{s\to 0}\frac{1}{s^{2k-2k/m}}\int_{\{\underline{h}<s\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\underline{h}^2)^{m-1}\wedge\omega^{n-m}.
$$

Since $\underline{h}$ is only a small perturbation of r, we can replace it with r in the above expression by the following argument. By [33], $\sqrt{-1}\partial\overline{\partial}r^2 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} + o(1)$ is smooth and m-sh on any sufficiently small neighborhood of V. By possibly increasing δ, we may assume that it is a large integer so that $\underline{h}^2$ is also smooth.

Let $T := \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}\underline{h}^2)^{m-2}\wedge\omega^{n-m}$. We have that

$$
\sqrt{-1}\partial\overline{\partial}\underline{h}^2 = \sqrt{-1}\partial\overline{\partial}\left(r+r^{1+\delta}\right)^2 = \sqrt{-1}\partial\overline{\partial}\left(r^2 + 2r^{2+\delta} + r^{2+2\delta}\right),
$$

so away from V

$$
\sqrt{-1}\partial\overline{\partial}r^{2+\delta} = (2+\delta)\sqrt{-1}\partial\left(r^{1+\delta}\overline{\partial}r\right)
$$

$$
= (2+\delta)r^{1+\delta}\sqrt{-1}\partial\overline{\partial}r + (2+\delta)(1+\delta)r^{\delta}\sqrt{-1}\partial r\wedge\overline{\partial}r
$$

$$
= \frac{2+\delta}{2}r^{\delta}\sqrt{-1}\partial\overline{\partial}r^2 + \delta(2+\delta)r^{\delta}\sqrt{-1}\partial r\wedge\overline{\partial}r.
$$

Since both sides are continuous across V, however, the above expression holds there as well. Then

$$
\sqrt{-1}\partial\overline{\partial}\underline{h}^2 - \sqrt{-1}\partial\overline{\partial}r^2 = \left((2+\delta)r^{\delta} + (1+\delta)r^{2\delta}\right)\sqrt{-1}\partial\overline{\partial}r^2
$$
$$
+ 2\delta\left((2+\delta)r^{\delta} + (2+2\delta)r^{2\delta}\right)\sqrt{-1}\partial r\wedge\overline{\partial}r
$$

and so

$$
0 \leqslant \sqrt{-1}\partial\overline{\partial}\underline{h}^2 - \sqrt{-1}\partial\overline{\partial}r^2 \leqslant C_\delta r^{\delta}\sqrt{-1}\partial\overline{\partial}\underline{h}^2,
$$

where the above $\leqslant$'s are understood as meaning that the difference between the two sides is m-sh. It follows that

$$
0 \leqslant \frac{\chi_{\{\underline{h}<s\}}}{s^{2k-2k/m}}\left(T\wedge\sqrt{-1}\partial\overline{\partial}\underline{h}^2 - T\wedge\sqrt{-1}\partial\overline{\partial}r^2\right)
$$
$$
\leqslant C_\delta s^{\delta}\left(\frac{\chi_{\{\underline{h}<s\}}}{s^{2k-2k/m}}T\wedge\sqrt{-1}\partial\overline{\partial}\underline{h}^2\right)
$$

since the measure $T \wedge \sqrt{-1}\partial\overline{\partial}\underline{h}^2 \geqslant 0$. Taking $s \to 0$ now implies that

$$\lim_{s\to 0} \frac{1}{s^{2k-2k/m}} \int_{\{\underline{h}<s\}} T \wedge \sqrt{-1}\partial\overline{\partial}\underline{h}^2 = \lim_{s\to 0} \frac{1}{s^{2k-2k/m}} \int_{\{\underline{h}<s\}} T \wedge \sqrt{-1}\partial\overline{\partial}r^2,$$

and it is easy to see that, by repeating the above argument $m-2$ more times, we have that

$$\nu(\varphi,\underline{\psi}_V) = \left(\frac{k-m}{m}\right)^{m-1} \lim_{s\to 0} \frac{1}{s^{2k-2k/m}} \int_{\{\underline{h}<s\}} \sqrt{-1}\partial\overline{\partial}\varphi \wedge (\sqrt{-1}\partial\overline{\partial}r^2)^{m-1} \wedge \omega^{n-m}.$$

By noting that $\{r < (1-\varepsilon)s\} \subset \{\underline{h} < s\} \subset \{r < s\}$ as long as s is sufficiently small (depending on ε), we can also replace the set $\{\underline{h} < s\}$ with $\{r < s\}$ in the above limit.

Finally, since $\underline{\psi}_V$ and ψ_V have the same singularity type, we can conclude by using Proposition 2.6. $\qquad\square$

3.4. Subweights with localized singularities

We now construct explicit m-subweights, ψ_θ, which have 'nonconstant' behavior along V when $k \geqslant m$. It is easy to see from the construction that these subweights will satisfy $\sigma(\psi_\theta,\psi_V) = 0$ and $\nu(\psi_\theta,\psi_V) > 0$ (Remark 3.9).

For any real number $\nu \in \mathbb{R}$, define $F_\nu : \mathbb{R}_{\geqslant 0} \to \mathbb{R} \cup \{-\infty\}$ by

$$F_\nu(t) := \begin{cases} t^{2-2\nu} & \text{if } \nu < 1 \\ \log t & \text{if } \nu = 1 \\ -t^{2-2\nu} & \text{if } \nu > 1. \end{cases}$$

We additionally define $D_\nu := |2\nu - 2|$ if $\nu \neq 1$, and $D_\nu = 1$ if $\nu = 1$. If $m \leqslant k$, then for any $\nu < \frac{k}{m}$ we have that $G_m < F_\nu$.

Proposition 3.7. *Suppose we are in the setting of the previous subsections and δ is sufficiently large, depending only on k and m. Let θ be a smooth, nonnegative function on V, and suppose that $m \leqslant k$. Then for any $\frac{k}{m} - \frac{1}{2} \leqslant \nu < \frac{k}{m}$, there exists a $C > 0$ such that*

$$\underline{\psi}_\theta := \underline{\theta}\underline{\psi}_V + CF_\nu(\underline{h}(r)) \in m\mathrm{SH}(\Omega),$$

where $\underline{\theta}$ is a smooth extension of θ with compact support. When $m = 1$, we can take $k - 1 \leqslant \nu < k$.

The construction is local and so works if V is only locally defined in $\overline{\Omega}$.

Remark 3.8. The bounds on ν seem to be optimal. Note that when $k < \frac{3}{2}m$, we can choose $\nu < 1$ so that the F_ν term is bounded from below.

Proof. The proof is a computation which is very similar to Proposition 3.1, so we omit several details. Working in the same geodesic normal coordinates as before, it follows that

$$\sqrt{-1}\partial\overline{\partial}(\underline{\theta}\underline{\psi}_V(\underline{h}(r_\varepsilon))) = \underline{\theta}\sqrt{-1}\partial\overline{\partial}G_m(\underline{h}(r_\varepsilon)) + \sqrt{-1}\partial\underline{\theta}\wedge\overline{\partial}G_m(\underline{h}(r_\varepsilon))$$

$$+ \sqrt{-1}\partial G_m(\underline{h}(r_\varepsilon))\wedge\overline{\partial}\underline{\theta} + G_m(\underline{h}(r_\varepsilon))\sqrt{-1}\partial\overline{\partial}\underline{\theta}$$

$$\geqslant \begin{pmatrix} 0 & 0 & \frac{D_{k,m}^{-1}r}{r_\varepsilon^{\frac{2k}{m}}}\overline{\partial}\underline{\theta} \\ 0 & 0 & 0 \\ \frac{D_{k,m}^{-1}r}{r_\varepsilon^{\frac{2k}{m}}}\partial\underline{\theta} & 0 & \frac{-1}{r_\varepsilon^{\frac{2k}{m}-2}}\sqrt{-1}\partial\overline{\partial}\underline{\theta} \end{pmatrix},$$

where we again understand $\geqslant$ as meaning the difference of the two sides is m-sh in this proof. Here, we are also interpreting $\overline{\partial}\underline{\theta}$ as a $1\times k$ matrix and $\partial\underline{\theta}$ as its conjugate transpose.

From the previous proposition, we also have

$$(\sqrt{-1}\partial\overline{\partial}F_\nu(\underline{h}(r_\varepsilon)))_{i\bar{j}} \geqslant D_\nu^{-1}\underline{h}^{-2\nu}\begin{pmatrix} 1-\nu & 0 & 0 \\ 0 & \mathrm{Id}_{k-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(r_\varepsilon^{\delta-2\nu}).$$

This leading order matrix will be strictly m-sh as long as $\nu < \frac{k}{m}$, by previous computations.

We now have that

$$\sqrt{-1}\partial\overline{\partial}(\underline{\theta}\underline{\psi}_V + CF_\nu(\underline{h}(r))) \geqslant \begin{pmatrix} \frac{C}{r_\varepsilon^{2\nu}}(1-\nu) & 0 & \frac{r}{r_\varepsilon^{\frac{2k}{m}}}\overline{\partial}\underline{\theta} \\ 0 & \frac{C}{r_\varepsilon^{2\nu}}\mathrm{Id}_{k-1} & 0 \\ \frac{r}{r_\varepsilon^{\frac{2k}{m}}}\partial\underline{\theta} & 0 & \frac{-1}{r_\varepsilon^{\frac{2k}{m}-2}}\sqrt{-1}\partial\overline{\partial}\underline{\theta} \end{pmatrix} + O(r_\varepsilon^{\delta-2\nu}),$$

ignoring some multiplicative constants. This leading matrix will be (strictly) m-sh with a positive term of order $r_\varepsilon^{2\nu}$ if we choose ν appropriately. This can be done by choosing ν so that the strict m-subharmonicity of the F_ν term beats the powers coming from the $\underline{\theta}$ terms. This can be seen by noting that $r < r_\varepsilon$ so that, when $m > 1$, we need

$$2\nu \geqslant \frac{2k}{m} - 1,$$

giving the combined constraints

$$\frac{k}{m} - \frac{1}{2} \leqslant \nu < \frac{k}{m};$$

when $m = 1$, we only need the trace to be positive, so we only require

$$2\nu \geqslant 2k - 2 \implies k - 1 \leqslant \nu < k. \qquad \square$$

Remark 3.9. For φ in Cegrell's class $\mathcal{E}^m$, we denote the complex Hessian operator of φ by $\mathrm{H}^m(\varphi) := (\sqrt{-1}\partial\overline{\partial}\varphi)^m\wedge\omega^{n-m}$. We claim that the $\underline{\psi}_\theta\in\mathcal{E}^m$ constructed in Proposition 3.7 satisfies

$$\chi_V\sqrt{-1}\partial\overline{\partial}\underline{\psi}_\theta\wedge(\sqrt{-1}\partial\overline{\partial}\psi_V)^{m-1}\wedge\omega^{n-m} = \chi_V\theta\mathrm{H}^m(\psi_V).$$

Indeed, on any sufficiently small open B near $z \in V$, we have

$$(\theta(z) + \varepsilon)\psi_V - C \leqslant \underline{\psi}_\theta \leqslant (\theta(z) - \varepsilon)\psi_V + C,$$

so Proposition 2.6 implies that

$$\chi_{V \cap B}(\theta(z) - \varepsilon)^m \mathrm{H}^m(\psi_V) \leqslant \chi_{V \cap B} \mathrm{H}^m(\underline{\psi}_\theta) \leqslant \chi_{V \cap B}(\theta(z) + \varepsilon)^m \mathrm{H}^m(\psi_V).$$

Covering V with a disjoint collection of such B and then letting $\varepsilon \to 0$ produces the claimed equality.

Taking θ such that $\min_V \theta = 0$ but $\theta \neq 0$ now gives a function such that

$$\sigma(\underline{\psi}_\theta, \psi_V) = 0, \quad \text{but} \quad \nu(\underline{\psi}_\theta, \psi_V) > 0.$$

Additionally, we can use Proposition 2.6 to see that $\chi_V \mathrm{H}^m(\underline{\psi}_\theta) = \chi_V \theta^m \mathrm{H}^m(\psi_V)$. Solutions to $\mathrm{H}^m(v) = \chi_V \theta^m \mathrm{H}^m(\varphi)$ for any given $\varphi \in \mathcal{E}^m(\Omega)$ were constructed in [24], following the work of [1] in the psh case, as a nested sequence of envelopes. Our $\underline{\psi}_\theta$ can be seen as explicit contenders in these envelopes for $\varphi = \psi_V$.

Let ψ_θ be the solution to $\mathrm{H}^m(\psi_\theta) = \chi_V \theta^m \mathrm{H}^m(\psi_V)$ constructed in [24]. If $k \leqslant \frac{3}{2}m$, our examples show that ψ_θ is actually bounded on $V \setminus \mathrm{Supp}(\theta)$. We suspect our examples actually compute the exact singularity type of ψ_θ on all of V.

3.5. Subharmonic functions and minimal submanifolds

We conclude this section with the observation that the above method can be copied directly when V is only a minimal submanifold of X (which we still assume to be Kähler), provided that we take $m = 1$. Subharmonic functions have long been known to be intimately related with minimal submanifolds, and as such, we are unsure if a similar construction is already present in the literature.

Proposition 3.10. *Suppose that V is a minimal submanifold of X of (real) codimension k. Then there exists an ω-subharmonic function ψ_V on a neighborhood Ω of V, which is moreover ω-harmonic on $\Omega \setminus V$, such that ψ_V has the same singularity type as*

$$-\frac{1}{r^{k-2}}.$$

Proof. Construct real coordinates at a point z near V in the same way as done at the beginning of this section so that e_1 is in the direction of the geodesic connecting z and V. By [33, Lemma 2.2], we have

$$r_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r} \cdot \mathrm{Id}_{k-1} & 0 \\ 0 & 0 & h_{ij} \end{pmatrix} + o(1),$$

where here $(h_{ij})_{k+1 \leqslant i,j \leqslant 2n}$ are the components of the second fundamental form of V evaluated against e_1. All of the computations in Proposition 3.1 go through verbatim, with the exception that the second fundamental form term becomes $rh_{ij} + o(r_\varepsilon)$. Since V is minimal, this bad term disappears when we take the trace of $\sqrt{-1}\partial\bar{\partial}r_\varepsilon^{k-2}$, and the computations in Propositions 3.2 and 3.3 can then be repeated.

Once a sub- and superweight have been produced, Proposition 3.4 is standard. $\qquad\square$

4. A Siu-type theorem

We now present our Siu-type theorem, stated as Theorem 1.4 in the introduction. We will need the following weak Harnack inequality:

Proposition 4.1 (Theorem 8.18 of [21]). *Let $\Omega \subset \mathbb{R}^{2n}$ be a domain. Suppose that φ is a nonpositive smooth subsolution of a linear elliptic operator $L = a^{ij}\partial_i\partial_j$, that is,*

$$\varphi \leqslant 0, \quad L\varphi \geqslant 0 \quad in\,\Omega,$$

and $\lambda I_{2n} \leqslant (a^{ij}) \leqslant \Lambda I_{2n}$ for positive constants λ and Λ. Then for any ball $B_2(y) \subset \Omega$ and $p \in [1, n/(n-1))$, we have

$$\varphi(y) \leqslant -\frac{1}{C(\lambda, \Lambda, p, n)} \int_{B_1(y)} (-\varphi)^p \leqslant 0.$$

Theorem 4.2. *Suppose that V is a compact, complex submanifold of codimension k. If $\varphi \in m\mathrm{SH}(\Omega)$ is such that $k < m$, then $L_{\psi_V}(\varphi) \equiv \sigma(\varphi, \psi_V)$.*

Proof. For convenience, set $\sigma := \sigma(\varphi, \psi_V)$. We will show that the set of points where $L_{\psi_V}(\varphi) = \sigma$ is open in V – since $L_{\psi_V}(\varphi)$ is lower semicontinuous, it is clear that this set is closed and nonempty, by Proposition 2.11.

Let $z_0 \in V$ be such that $L_{\psi_V}(\varphi)(z_0) = \sigma$. Choose local holomorphic coordinates $z = (z', z'')$ centered at z_0 such that $V = \{z' = 0\}$ in these coordinates. Let B_R be the coordinate ball of radius $0 < R \leqslant 2$, centered at $z_0 = (0,0)$. Fix a uniform constant C such that

$$C^{-1}r(z) \leqslant |z'| \leqslant Cr(z) \quad \text{for all } z \in B_2. \tag{4.1}$$

Further, let $\rho(z)$ be a smooth potential for ω on B_2 such that $-C \leqslant \rho(z) \leqslant 0$.

By definition, there exists a sequence $z_i = (z_i', z_i'') \to z_0$, with $z_i \in \Omega \setminus V$, such that

$$\lim_{i \to \infty} \frac{\varphi(z_i)}{\psi_V(z_i)} = \sigma.$$

We assume without loss of generality that $\log|z_i'|, \log r(z_i), \psi_V(z_i) \leqslant -1$. Since ψ_V and $\log r$ have the same singularity type, by Equation (4.1), we have that

$$\lim_{i \to \infty} \frac{\varphi(z_i)}{\log|z_i'|} = \sigma$$

also. It follows that, for any $\varepsilon > 0$, we have

$$(\sigma + \varepsilon)\log|z_i'| \leqslant \varphi(z_i) \leqslant (\sigma - \varepsilon)\log|z_i'| \tag{4.2}$$

for all i sufficiently large.

Define the functions $\eta_{i,\varepsilon}$ on $V_2 := V \cap B_2$ by

$$\eta_{i,\varepsilon}(z'') := \frac{\varphi(z_i', z'') + \rho(z_i', z'') - \varphi(z_i) + 3\varepsilon\log|z_i'|}{-\log|z_i'|}.$$

Note that $\eta_{i,\varepsilon}$ is subharmonic on V_2 with respect to the metric $\omega|_{\{z'=z'i\}}$ since we have assumed that $m > k$.

We will show that $\eta_{i_j,\varepsilon_j} \to 0$ pointwise a.e. for some subsequence $i_j \to \infty$ of the i and an associated sequence of $\varepsilon_j \to 0$ by using the Harnack inequality. To utilize this, we need to first verify that $\sup \eta_{i,\varepsilon} \leqslant 0$. Increasing C if necessary, we have that

$$\{z' = z_i'\} \subseteq \{\log |z'| \leqslant \log |z_i'|\} \subseteq \{\psi_V(z) \leqslant \log |z_i'| + C\},$$

so by the definition of σ we have

$$\sup_{V_2} \varphi(z_i', \cdot) \leqslant \sup_{\{\psi_V(z) \leqslant \log |z_i'| + C\}} \varphi \leqslant (\sigma - \varepsilon)(\log |z_i'| + C).$$

Combining this with Equation (4.2) and our choice of ρ gives

$$\sup_{V_2} \eta_{i,\varepsilon} \leqslant -(\sigma - \varepsilon) + (\sigma + \varepsilon) - 3\varepsilon + \frac{(\sigma - \varepsilon)C}{-\log|z_i|} \leqslant -\varepsilon + \frac{(\sigma - \varepsilon)C}{-\log|z_i|} < 0 \qquad (4.3)$$

for ε fixed and i sufficiently large.

Let $\varepsilon_j > 0$ be a sequence such that $\varepsilon_j \to 0$ as $j \to \infty$. By Equation (4.3), for each j there exists a sufficiently large i_j such that $i_j \to \infty$ as $j \to \infty$ and

$$\sup_{V_2} \eta_{i_j,\varepsilon_j} < 0.$$

Letting $V_1 := V \cap B_1$, Proposition 4.1 now implies that

$$0 \leqslant \int_{V_1} (-\eta_{i_j,\varepsilon_j})^p \leqslant -C\eta_{i_j,\varepsilon_j}(z_{i_j}'') \leqslant C\varepsilon_j,$$

for any $p \in [1, n/(n-1))$ (up to increasing C again, depending on p and ω). Fixing some $p > 1$, it is clear that

$$\lim_{i \to \infty} \|\eta_{i_j,\varepsilon_j}\|_{L^p(V_1)} = 0.$$

Hence, we see that $\eta_{i_j,\varepsilon_j} \to 0$ in L^p as $j \to \infty$, and so after passing to a subsequence of the j, we have that

$$\eta_{i_j,\varepsilon_j} \to 0 \quad \text{pointwise a.e. on } V_1. \qquad (4.4)$$

Now, from our definition of $\eta_{i,\varepsilon}$, we see that

$$\frac{\varphi(z'_{i_j}, z'')}{\log|z_{i_j}'|} = \frac{\varphi(z_{i_j}', z_{i_j}'')}{\log|z_{i_j}'|} - \eta_{i_j,\varepsilon_j}(z'') - 3\varepsilon_j - \frac{\rho(z'_{i_j}, z'')}{\log|z'_{i_j}|}.$$

Using Equation (4.4) and taking the limit as $j \to \infty$ gives

$$\lim_{j \to \infty} \frac{\varphi(z'_{i_j}, z'')}{\log|z_{i_j}'|} = \sigma$$

for almost all $z'' \in V_1$. Then we conclude that

$$L_{\psi_V}(\varphi)(0, z'') \leqslant \lim_{j \to \infty} \frac{\varphi(z'_{i_j}, z'')}{\log|z_{i_j}'|} = \sigma$$

for almost every $z'' \in V_1$. It follows now from Proposition 2.11 that in fact $L_{\psi_V}(\varphi) \equiv \sigma$ on V_1, finishing the proof. $\qquad\qquad\square$

Competing interests. The authors have no competing interest to declare.

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