

# EMBEDDINGS, NORMAL INVARIANTS AND FUNCTOR CALCULUS

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**Abstract.** This paper investigates the space of codimension zero embeddings of a Poincaré duality space in a disk. One of our main results exhibits a tower that interpolates from the space of Poincaré immersions to a certain space of “unlinked” Poincaré embeddings. The layers of this tower are described in terms of the coefficient spectra of the identity appearing in Goodwillie’s homotopy functor calculus. We also answer a question posed to us by Sylvain Cappell. The appendix proposes a conjectural relationship between our tower and the manifold calculus tower for the smooth embedding space.

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## §1. Introduction

### 1.1 Background

Suppose that  $P$  and  $N$  are compact smooth  $n$ -manifolds, possibly with boundary. Let  $E^{\text{sm}}(P, N)$  be the space of smooth ( $C^\infty$ ) embeddings from  $P$  into the interior of  $N$ . The *manifold calculus* of Goodwillie and Weiss produces a tower of fibrations

$$\cdots \rightarrow E_2^{\text{sm}}(P, N) \rightarrow E_1^{\text{sm}}(P, N)$$

and compatible maps  $E^{\text{sm}}(P, N) \rightarrow E_j^{\text{sm}}(P, N)$ . If we assume that  $P$  admits a handle decomposition with handles of index at most  $n - 3$ , then the maps

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$E^{\text{sm}}(P, N) \rightarrow E_j^{\text{sm}}(P, N)$  have connectivity given by a linear function of  $j$  with positive slope, so in this case the tower strongly converges [8], [7]. Furthermore,  $E_1^{\text{sm}}(P, N)$  has the homotopy type of the space of smooth immersions from  $P$  to  $N$ . For  $j \geq 2$ , the layers of the tower, that is, the homotopy fibers of the maps  $E_j^{\text{sm}}(P, N) \rightarrow E_{j-1}^{\text{sm}}(P, N)$ , have an explicit description in terms of configuration spaces.

In essence, the strong convergence result relies on the following schematic passage:

$$E^{\text{sm}}(P, N) \rightarrow E^{\text{bl}}(P, N) \rightarrow E^{\text{pd}}(P, N),$$

where  $E^{\text{bl}}(P, N)$  is the space of smooth block embeddings of  $P$  in  $N$  and  $E^{\text{pd}}(P, N)$  is the corresponding space of Poincaré embeddings. Convergence is proved by establishing certain higher excision statements, which are known as “multiple disjunction” results for spaces of smooth embeddings. One achieves such results by first proving analogous ones for spaces of Poincaré embeddings. The Poincaré statements were proved in [6] using homotopy theory. One then lifts the Poincaré statements to the block setting using surgery theory. The final step is to lift the block statements to the smooth ones using concordance theory. Given the method of proof, it seems appropriate to ask:

QUESTION 1.1. Is there an analogue of Goodwillie–Weiss manifold calculus in the Poincaré duality space setting?

More precisely, suppose now that  $P$  and  $N$  are Poincaré spaces of dimension  $n$  (possibly with boundary). One then has a space of Poincaré embeddings  $E^{\text{pd}}(P, N)$  and we wish to construct a Goodwillie–Weiss calculus for it. Unfortunately, we do not know how to proceed. The problem here is that the set-up of [22] does not properly translate over: in the manifold case one considers the poset of subsets of the interior of  $P$  which are diffeomorphic to finite collections of open balls. This poset has good properties because a manifold is locally Euclidean. In the Poincaré case there does not seem to be a sensible replacement for this, as Poincaré spaces are not necessarily locally well-behaved.

A related but perhaps more accessible question is

QUESTION 1.2. Is there a version of the Goodwillie–Weiss tower in the Poincaré embedding case?

We propose to attack Question (1.2) from a point-of-view arising out of the surgery school in conjunction with one of the other functor calculi: Goodwillie’s *homotopy functor calculus*.

To simplify the presentation, we will only consider the case when  $N = D^n$  is an  $n$ -disk, and we will assume that  $P$  is “sectioned” in the sense defined below. We will see that a certain space

$$\mathfrak{L}E^{\text{pd}}(P, D^n)$$

of “unlinked” Poincaré embeddings of  $P$  in  $D^n$  does have a tower associated with it. A point in this space consists of a Poincaré embedding of  $P$  in  $D^n$  together with a choice of Null homotopy of the “link” of the embedding. We will also see that the tower associated with this space strongly converges under mild hypotheses, and its first stage coincides with the space of “Poincaré immersions” of  $P$  in  $D^n$ . Furthermore, we will identify the homotopy fibers of the canonical map  $\mathfrak{L}E^{\text{pd}}(P, D^n) \rightarrow E^{\text{pd}}(P, D^n)$  as spaces of “unlinkings” of  $P$  in  $D^n$ .

In what follows we simplify notation by setting

$$E(P, D^n) := E^{\text{pd}}(P, D^n).$$

### 1.2 Sectioned Poincaré spaces

Our goal will be to say something sensible about the embeddings of the following class of Poincaré spaces:

DEFINITION 1.3. A *sectioning* of a Poincaré space  $P$  with boundary  $\partial P$  is a triple

$$\xi = (K, f, s)$$

in which

- $K$  is a cofibrant space;
- $f: K \xrightarrow{\sim} P$  is a homotopy equivalence;
- $s: K \rightarrow \partial P$  is a map such that the composition

$$K \xrightarrow{s} \partial P \rightarrow P$$

coincides with  $f$ .

We refer to  $\xi$  as *sectioning data*. For the sake of brevity, we will say that  $P$  is *sectioned* when the sectioning data are understood.

EXAMPLE 1.4. Let  $D^1 = [-1, 1]$  be the 1-disk. If  $Q$  is a Poincaré space, possibly with boundary  $\partial Q$ , then  $Q \times D^1$  is sectioned by means of the homotopy equivalence  $Q \times \{-1\} \subset Q \times D^1$  and the inclusion  $Q \times \{-1\} \subset \partial(Q \times D^1)$ .

EXAMPLE 1.5. Suppose  $\eta$  is a  $(j - 1)$ -spherical fibration over a Poincaré space  $Q$  of dimension  $d$  having empty boundary. Let  $S(\eta)$  be its total space. Suppose  $\eta$  comes equipped with a section  $s: Q \rightarrow S(\eta)$ . Let  $D(\eta)$  be the mapping cylinder of  $\eta$ , and let  $f: Q \rightarrow D(\eta)$  be the inclusion. Then  $\xi := (Q, f, s)$  is sectioning data for the  $(n + j)$ -dimensional Poincaré space  $D(\xi)$ .

DEFINITION 1.6. (Generalized Thom Space) If  $P$  is sectioned by  $\xi = (K, f, s)$ , then we define

$$P^\xi := \partial P \cup_s CK,$$

that is, the mapping cone of the map  $s: K \rightarrow \partial P$ . This is a based space.

The justification for this notation/terminology is that the spherical fibration case in Example 1.5 gives the Thom space in the usual sense.

LEMMA 1.7. Assume that  $P$  is sectioned by  $\xi = (K, f, s)$  and  $\partial P \rightarrow P$  is a cofibration. Then there is a preferred weak homotopy equivalence

$$\Sigma P^\xi \simeq P/\partial P.$$

That is,  $P^\xi$  is a preferred desuspension of  $P/\partial P$ .

*Proof.* One has a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow[f]{\sim} & P & \longrightarrow & P \cup_K CK \\ s \downarrow & & \parallel & & \downarrow \\ \partial P & \longrightarrow & P & \longrightarrow & P \cup_{\partial P} C\partial P \\ \downarrow & & \downarrow & & \downarrow \sim \\ P^\xi & \longrightarrow & CP & \longrightarrow & CP \cup_{P^\xi} CP^\xi \end{array}$$

in which  $CX$  denotes the cone on a space  $X$ . The rows and columns of the diagram form homotopy cofiber sequences (cf. 2.1; the null homotopies in this case are evident). The space in the upper right corner is contractible so the map  $P \cup_{\partial P} C\partial P \rightarrow CP \cup_{P^\xi} CP^\xi$  is a weak equivalence. The domain of

this map is identified with  $P/\partial P$  up to a preferred weak equivalence (given by collapsing  $CP$  to a point) and the codomain is identified with  $\Sigma P^\xi$ .  $\square$

In particular, the lemma gives a preferred isomorphism of singular homology groups  $\tilde{H}_{k-1}(P^\xi) \cong H_k(P, \partial P)$ .

**1.3 Homotopy codimension**

A Poincaré space  $P$  of dimension  $n$  is said to have *homotopy codimension*  $\geq j$  if the map  $\partial P \rightarrow P$  is  $(j - 1)$ -connected.

EXAMPLE 1.8. Let  $\eta: S(\eta) \rightarrow Q$  be a  $(j - 1)$ -spherical fibration over a Poincaré space without boundary. Then  $D(\eta)$  has homotopy codimension  $\geq j$ .

EXAMPLE 1.9. If  $P$  is a compact smooth manifold, possibly with boundary, which admits a handle decomposition whose handles all have index  $\leq k$ , then the homotopy codimension of  $P$  is  $\geq n - k$ .

EXAMPLE 1.10. Suppose  $P$  is an  $n$ -dimensional Poincaré space,  $n - k \geq 3$ . Then  $P$  has homotopy codimension  $\geq n - k$  if the map  $\partial P \rightarrow P$  is 2-connected and  $P$  has the weak homotopy type of a CW complex of dimension  $\leq k$ . This is a consequence of duality, the relative Hurewicz theorem and a result of Wall [20, Theorem E].

We will assume the following throughout the paper.

*Hypothesis 1.11.* The Poincaré space  $P$  has homotopy codimension  $\geq 3$ .

**1.4 Poincaré embeddings**

We recall the notion of codimension zero Poincaré embedding (see e.g., [13], [6]). We will restrict ourselves to the case when the ambient space is an  $n$ -disk.

A *Poincaré embedding* of a Poincaré space  $P$  of dimension  $n$  in  $D^n$  consists of a “complement” space  $C$  equipped with a (gluing data) map

$$\partial P \amalg S^{n-1} \rightarrow C$$

making  $C$  into an  $n$ -dimensional Poincaré space with boundary  $\partial P \amalg S^{n-1}$ . Furthermore, we require the homotopy pushout of

$$P \leftarrow \partial P \rightarrow C$$

to have the weak homotopy type of  $D^n$ , that is, it is required to be weakly contractible. The set of all such Poincaré embeddings comes equipped with a topology (cf. Section 3).

We denote this space by  $E(P, D^n)$ . We typically specify a Poincaré embedding by writing its complement, that is, we write  $C \in E(P, D^n)$ .

**1.5 Unstable normal invariants**

Assume  $P$  is sectioned by  $\xi$ .

DEFINITION 1.12. An *unstable normal invariant* for  $P$  is a based map

$$\alpha: S^{n-1} \rightarrow P^\xi$$

such that

$$\alpha_*([S^{n-1}]) \in \tilde{H}_{n-1}(P^\xi) \cong H_n(P, \partial P)$$

is a fundamental class for  $P$ .

REMARK 1.13. The stable version of the normal invariant appeared in the context of surgery theory [16]. Applications of unstable normal invariants to embedding theory were investigated in [23], [18] and [13].

PROPOSITION 1.14. Assume  $P$  is sectioned. Then an unstable normal invariant for  $P$  gives rise to a Poincaré embedding of  $P$  in  $D^n$ .

*Proof.* The proof harkens back to a construction of Browder [2]. Let  $\xi$  be the sectioning data. Set  $C := P^\xi$ . We have an evident map

$$P^\xi \vee S^{n-1} \rightarrow C$$

where on the second wedge summand we use  $\alpha$ . Let  $\partial P \rightarrow P^\xi$  be the evident map. Consider the composite

$$\partial P \amalg S^{n-1} \rightarrow P^\xi \vee S^{n-1} \rightarrow C.$$

It follows from [11, Lemma 2.3] that this map gives  $C$  the structure of a Poincaré space with boundary  $\partial P \amalg S^{n-1}$  and defines a Poincaré embedding of  $P$  in  $D^n$ . (Note: since  $(P, \partial P)$  is 2-connected and  $P \cup_{\partial P} C$  is  $\infty$ -connected, it follows that  $C$  is 1-connected. So Poincaré duality for  $C$  only needs to be verified with constant  $\mathbb{Z}$  coefficients.) □

*Notation 1.15.* If  $P$  is sectioned by  $\xi$ , then the space of its unstable normal invariants will be denoted by

$$\Omega_{\perp}^{n-1}P^{\xi}.$$

This is to be topologized as a subspace of the  $(n-1)$ -fold based loop space  $\Omega^{n-1}P^{\xi}$ .

REMARK 1.16. The subspace  $\Omega_{\perp}^{n-1}P^{\xi} \subset \Omega^{n-1}P^{\xi}$  is a collection of connected components: the Hurewicz map gives a (homotopy) Cartesian square

$$\begin{array}{ccc} \Omega_{\perp}^{n-1}P^{\xi} & \xrightarrow{\subset} & \Omega^{n-1}P^{\xi} \\ \downarrow & & \downarrow \\ H_n(P, \partial P)^{\times} & \xrightarrow[\subset]{} & H_n(P, \partial P) \end{array}$$

where  $H_n(P, \partial P)^{\times}$  is the set of fundamental classes of  $P$ . This set is nonempty if and only if  $P$  is orientable. If  $P$  is connected and orientable then  $H_n(P, \partial P)^{\times}$  has precisely two elements.

The proof of Proposition 1.14 yields a map

$$(1) \quad \Omega_{\perp}^{n-1}P^{\xi} \rightarrow E(P, D^n)$$

which we henceforth call the *Browder construction*.

**1.6 The link; the first main result**

We continue to assume that  $P$  is sectioned by  $\xi = (K, f, s)$ . Given a point  $C \in E(P, D^n)$ , we have a weak map

$$(2) \quad P \xleftarrow{f} K \xrightarrow{s} \partial P \rightarrow C.$$

Using the basepoint of  $S^{n-1}$  we obtain a preferred basepoint for  $C$ . Let

$$(3) \quad \ell_0(C) \in [P_+, C]_* \cong [P, C]$$

be the homotopy class of the weak map (2), where  $P_+$  is  $P$  with a disjoint basepoint. We call  $\ell_0(C)$  the *link* of the Poincaré embedding. The next example motivates the terminology.

EXAMPLE 1.17. Let  $f: M^p \rightarrow D^{2p+1}$  be a smooth framed embedding, where  $M$  is connected. This means that  $f$  admits a preferred extension to a smooth embedding  $F: M^p \times D^{p+1} \rightarrow D^{2p+1}$ . By Alexander duality and the Hurewicz isomorphism we have a preferred isomorphism  $H_p(D^{2p+1} \setminus f(M^p)) \cong \mathbb{Z}$ . Let  $* \in S^p$  be the basepoint. Then the homology class induced by

$$M^p \times * \xrightarrow{F} D^{2p+1} \setminus f(M^p)$$

is the *self-linking number* of  $f$ .

THEOREM A. Assume  $P$  is sectioned by  $\xi$ . Given a Poincaré embedding  $C \in E(P, D^n)$ , then the homotopy fiber of the Browder construction (1) taken at  $C$  is non-empty if and only if the link  $\ell_0(C)$  is trivial.

Furthermore, if the link is trivial, then the Browder construction sits in a homotopy fiber sequence

$$F_*(\Sigma(P_+), C) \rightarrow \Omega_{\perp}^{n-1} P^{\xi} \rightarrow E(P, D^n),$$

where the homotopy fiber is taken at  $C$  and  $F_*(\Sigma(P_+), C)$  is the function space of based maps  $\Sigma(P_+) \rightarrow C$ .

Remarks 1.18. (1). The null homotopy yielding the homotopy fiber sequence will be made explicit in the proof of Theorem A.

(2). Theorem A answers a question posed to me by Sylvain Cappell about how far the Browder construction is from being a homotopy equivalence.

(3). Let  $Q$  be a Poincaré space of dimension  $n - 1$  and homotopy codimension  $\geq n - k \geq 4$ , but not necessarily sectioned. Then as in Example (1.4),  $Q \times D^1$  is sectioned with generalized Thom space  $Q/\partial Q$ .

The “decompression” map  $E(Q, D^{n-1}) \rightarrow E(Q \times D^1, D^n)$  (cf. (8) below) factors as

$$E(Q, D^{n-1}) \rightarrow \Omega_{\perp}^{n-1} Q/\partial Q \rightarrow E(Q \times D^1, D^n),$$

in which the first map is given by the Pontryagin–Thom construction, and the second is the one of Theorem A. The results of [13] imply that the first map is  $(2n - 3k - 6)$ -connected.

(4). The map  $F_*(\Sigma(P_+), C) \rightarrow \Omega_{\perp}^{n-1} P^{\xi}$  can be described as an orbit map of an “action” of  $\Omega F_*(P_+, P^{\xi})$  on  $\Omega_{\perp}^{n-1} P^{\xi}$ : fix an unstable normal invariant  $\alpha: S^{n-1} \rightarrow P^{\xi}$ . Then the Browder construction applied to  $\alpha$  gives a Poincaré embedding with complement  $C = P^{\xi}$ .



Let  $c: P^\xi \rightarrow P^\xi \vee \Sigma(P_+)$  be the Barratt–Puppe coaction map for the based cofiber sequence  $P_+ \rightarrow (\partial P)_+ \rightarrow P^\xi$ . Given  $\phi \in F_*(\Sigma(P_+), P^\xi) = \Omega F_*(P_+, P^\xi)$ , we obtain a new normal invariant  $\phi \star \alpha$  by taking the composition

$$S^{n-1} \xrightarrow{\alpha} P^\xi \xrightarrow{c} P^\xi \vee \Sigma(P_+) \xrightarrow{\text{id}+\phi} P^\xi.$$

Then the operation  $\phi \mapsto \phi \star \alpha$  yields the desired description.

EXAMPLE 1.19. (Embeddings of the  $n$ -disk) Let  $P = D^n$  where  $n \geq 3$ . Then  $P$  is sectioned by the basepoint of  $S^{n-1}$ . In this case, Theorem A gives a homotopy fiber sequence

$$\Omega S^{n-1} \rightarrow F_{n-1} \rightarrow E(D^n, D^n),$$

where  $F_{n-1}$  is the space of based self homotopy equivalences of  $S^{n-1}$ . In fact,  $E(D^n, D^n) \simeq G_n$ , the unbased self homotopy equivalences of  $S^{n-1}$ . Furthermore, the above homotopy fiber sequence is principal and with respect to the identifications is just a shift to the left of the evident fibration  $F_{n-1} \rightarrow G_n \rightarrow S^{n-1}$ .

EXAMPLE 1.20. (Disjoint unions of  $n$ -disks) Assume  $n \geq 3$ . Let  $T$  be a finite set and let  $P = D^n \times T$ . Then  $P$  is sectioned by choosing a basepoint  $* \in S^{n-1}$ . Theorem A gives in this case a homotopy fiber sequence

$$(4) \quad \Omega \prod_T S^{n-1} \wedge T_+ \rightarrow \Omega_{\perp}^{n-1}(S^{n-1} \wedge T_+) \rightarrow E(D^n \times T, D^n).$$

For each  $x \in T$  we have a projection map  $p_x: S^{n-1} \wedge T_+ \rightarrow S^{n-1}$ . The condition for a based map  $f: S^{n-1} \rightarrow S^{n-1} \wedge T_+$  to be a normal invariant is that every composite  $p_x \circ f$  should lie in  $F_{n-1}$ .

Since  $S^{n-1} \wedge T_+$  is a finite wedge of spheres, the Hilton–Milnor theorem implies that the homotopy groups of the two spaces on the left of (4) can be expressed explicitly in terms of the homotopy groups of spheres.

For example, if  $n$  is even, then these two spaces are rationally equivalent to generalized Eilenberg–Mac Lane spaces with finitely generated homotopy groups that only occur in even degrees. These rational homotopy groups can be explicitly computed in terms of a Hall basis for the free Lie algebra on  $|T|$ -generators (see e.g., [1, Theorem 4.7]). For parity reasons, the long exact homotopy sequence of rational homotopy groups splits into short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_{2k+1}(E(D^n \times T, D^n))_{\mathbb{Q}} & \longrightarrow & \oplus_T \pi_{2k+1}(S^{n-1} \wedge T_+)_{\mathbb{Q}} & \longrightarrow & \\
 & & & & \searrow & & \\
 & & \pi_{2k+n-1}(S^{n-1} \wedge T_+)_{\mathbb{Q}} & \longrightarrow & \pi_{2k}(E(D^n \times T, D^n))_{\mathbb{Q}} & \longrightarrow & 0,
 \end{array}$$

where we are assuming  $k > 0$  (this is not a serious restriction: it can be shown that  $\pi_0(E(D^n \times T, D^n))$  is a set of cardinality  $2^{|T|}$  and  $\pi_1(E(D^n \times T, D^n))$  is the direct sum of  $|T|$  copies of the cyclic group of order two). From this we immediately obtain a crude bound for the rank of the rational homotopy groups of  $E(D^n \times T, D^n)$ . To obtain finer information would require explicit knowledge of the curved arrow in the diagram.

**1.7 Unlinked embeddings; the second main result**

DEFINITION 1.21. Assume  $P$  is sectioned. The space of *unlinked embeddings*

$$\mathfrak{L}E(P, D^n)$$

consists of those points  $C \in E(P, D^n)$  such that the gluing data

$$\partial P \amalg S^{n-1} \rightarrow C$$

comes equipped with a factorization

$$\partial P \amalg S^{n-1} \rightarrow P^\xi \vee S^{n-1} \rightarrow C$$

where the first map is evident.

It is clear that the Browder construction (1) factors as

$$(5) \quad \Omega_{\perp}^{n-1} P^\xi \rightarrow \mathfrak{L}E(P, D^n) \rightarrow E(P, D^n).$$

We call the map

$$(6) \quad \Omega_{\perp}^{n-1} P^\xi \rightarrow \mathfrak{L}E(P, D^n)$$

the *refined Browder construction*.

THEOREM B. *The refined Browder construction (6) is a homotopy equivalence.*

*Remarks 1.22.* (1). Theorem B is essentially a “space-ification” of a result Williams [23, Theorem A(i)] who considered a version of the map (6) on the level of path components. However, there are some minor differences:

- Williams restricts himself to sectioned Poincaré spaces of the type appearing in Example 1.5.
- Williams studies Poincaré embeddings in the  $n$ -sphere rather than in the  $n$ -disk. This distinction does not appear on the level of path components.
- Williams equips his Poincaré spaces and their embeddings with orientations. His unstable normal invariants are of degree one.

The disadvantage with the  $n$ -sphere is that Williams’ result does not extend to higher homotopy groups without modifying the domain of the refined Browder construction: the correct replacement is the space of “fiberwise unstable normal invariants over  $S^n$ ” (cf. [13]). Another disadvantage is that Williams has to work much harder than we do to prove his result.

(2). Let  $Q$  be as in Remark 1.18(1). Then Theorem B implies that the decompression map

$$E(Q, D^n) \rightarrow \mathfrak{L}E(Q \times D^1, D^{n+1})$$

is  $(2n - 3k - 4)$ -connected.

**1.8 Poincaré Immersions**

The space of (Poincaré) *immersions*  $I(P, D^n)$  is defined to be the homotopy colimit of the diagram

$$E(P, D^n) \rightarrow E(P_1, D^{n+1}) \rightarrow E(P_2, D^{n+2}) \rightarrow \dots$$

where  $P_j = P \times D^j$ . Note by construction

$$I(P, D^n) \simeq I(P_1, D^{n+1}) \simeq \dots$$

We view this as a reasonable definition, since the analogous statement is valid in the case of smooth block embeddings.

We will exhibit below a homotopy equivalence

$$I(P, D^n) \simeq \Omega^n Q(P/\partial P),$$

where the right-hand side denotes the space of *stable normal invariants* of  $P$ : this is the space whose points are stable maps  $\alpha: S^n \rightarrow P/\partial P$  such that  $\alpha_*([S^n]) \in H_n(P, \partial P)$  is a fundamental class. We topologize this as a subspace of  $\Omega^n Q(P/\partial P) := \Omega^n \Omega^\infty \Sigma^\infty(P/\partial P)$ .

REMARK 1.23. Assuming  $I(P, D^n)$  is non-empty, we will also exhibit (Lemma 3.4) a homotopy equivalence

$$I(P, D^n) \simeq F(P, G),$$

where  $G$  is the topological monoid of stable self-equivalences of the sphere. The equivalence depends on choosing a basepoint in  $I(P, D^n)$ .

REMARK 1.24. Assume  $P$  is orientable. The *Spivak normal fibration* of  $P$  is an orientable stable spherical fibration  $\xi$  over  $P$  that is equipped with a stable map  $\alpha: S^n \rightarrow P^\xi/(\partial P)^\xi$  representing a fundamental class for  $P$  by means of the Thom isomorphism. The data  $(\xi, \alpha)$  are well-defined up to contractible choice [19], [12].

Note that if  $\xi$  is fiber homotopically trivial, then  $\alpha$  amounts to a stable normal invariant for  $P$ . Furthermore,  $I(P, D^n)$  is identified with the space of fiber homotopy trivializations  $\xi$  (in particular, it is non-empty if and only if  $\xi$  is trivializable). The fiber homotopy triviality of  $\xi$  is the Poincaré analog of stable parallelizability.

### 1.9 A tower for unlinked embeddings; the third main result

We introduce some notation. If  $V$  is an orthogonal representation of a group  $G$ , we let  $S^V$  be the based  $G$ -sphere given by the one-point compactification of  $V$ . If  $V$  and  $W$  are two orthogonal representations, then we write  $V + W$  for the direct sum and  $nV$  will denote the direct sum of  $n$ -copies of  $V$ . If  $W \subset V$  is an orthogonal sub-representation, then we let  $V - W$  be its orthogonal complement. Let  $1$  denote the trivial representation of rank one. Let  $\Sigma_j$  be the symmetric group on the standard basis for  $\mathbb{R}^j$ . Then we obtain the *standard representation* of  $\Sigma_j$  on  $\mathbb{R}^j$ . The diagonal gives an embedding of the trivial representation  $1$  inside the standard representation. Let  $V_j$  be denote its orthogonal complement. Call this the *reduced standard representation*; it has rank  $j - 1$ . For example,  $V_2$  is the rank one sign representation.

If  $X$  is a based (cofibrant) space and  $E$  is a (fibrant) spectrum, then we let  $F_*(X, E)$  be the (stable) function spectrum whose  $j$ th space is given by the based maps  $X \rightarrow E_j$ . If  $X$  and  $E$  are equipped with  $G$ -actions then  $F_*(X, E)$  inherits a  $G$ -action by conjugation. In particular, one can consider the homotopy orbit spectrum  $F_*(X, E)_{hG}$  and its associated infinite loop space  $\Omega^\infty F_*(X, E)_{hG}$ , the latter which will be denoted by  $F_*^{\text{st}}(X, E)_{hG}$ .

Let  $\mathscr{W}_j$  denote the  $j$ -coefficient spectrum of the identity functor from based spaces to based spaces in the sense of the calculus of homotopy

functors [10]. In particular,  $\mathscr{W}_j$  is a spectrum with  $\Sigma_j$ -action which is unequivariantly weak equivalent to a wedge of  $(j - 1)!$  copies of the  $(1 - j)$ -sphere.

**THEOREM C.** *Assume  $P$  is sectioned and assume that the homotopy codimension of  $P$  is  $\geq n - k \geq 3$ . Then there is a tower of fibrations*

$$\cdots \rightarrow \mathfrak{L}E_j(P, D^n) \rightarrow \mathfrak{L}E_{j-1}(P, D^n) \rightarrow \cdots \rightarrow \mathfrak{L}E_1(P, D^n)$$

*equipped with compatible maps*

$$\phi_j: \mathfrak{L}E(P, D^n) \rightarrow \mathfrak{L}E_j(P, D^n)$$

*such that*

- *the map  $\phi_j$  is  $(2 - n + (j + 1)(n - k - 2))$ -connected; in particular, the induced map*

$$\mathfrak{L}E(P, D^n) \rightarrow \lim_{j \rightarrow \infty} \mathfrak{L}E_j(P, D^n)$$

*is a weak equivalence;*

- *there is a preferred homotopy equivalence*

$$\mathfrak{L}E_1(P, D^n) \simeq I(P, D^n);$$

- *if  $j \geq 2$  and  $x \in \mathfrak{L}E_{j-1}(P, D^n)$  is a point, then there is an obstruction*

$$\ell_{j-1}(x) \in \pi_0(F_*(P_+^{\times j}, \mathscr{W}_j \wedge S^{(n-1)V_j+1})_{h\Sigma_j})$$

*which is trivial if and only if the homotopy fiber of the map  $\mathfrak{L}E_j(P, D^n) \rightarrow \mathfrak{L}E_{j-1}(P, D^n)$  at  $x$  is non-empty;*

- *if  $\ell_{j-1}(x)$  is trivial, then there is a homotopy fiber sequence*

$$F_*^{\text{st}}((P^{\times j})_+, \mathscr{W}_j \wedge S^{(n-1)V_j})_{h\Sigma_j} \rightarrow \mathfrak{L}E_j(P, D^n) \rightarrow \mathfrak{L}E_{j-1}(P, D^n).$$

*where the displayed homotopy fiber is taken at  $x$ .*

*Remarks 1.25.* (1). The first part of Theorem C implies that if  $(j + 1)k + 2j \leq jn$  and  $\mathfrak{L}E_j(P, D^n)$  is non-empty, then  $\mathfrak{L}E(P, D^n)$  is also non-empty.

(2). It follows from the last two parts of the theorem that the map  $\mathfrak{L}E_j(P, D^n) \rightarrow \mathfrak{L}E_{j-1}(P, D^n)$  is  $(2 - n + j(n - k - 2))$ -connected.

(3). Modulo torsion, a transfer argument shows that the class  $\ell_{j-1}(x)$  is detected in the singular cohomology group  $H^s(P^{\times j}; \mathbb{Q}^{(j-1)!})$ , where  $s = (n - 2)(j - 1) + 1$ .

(4). The layers of the tower depend only on the homotopy type of  $P$  and in particular do not depend on the choice of sectioning data.

*Outline.* The material of Section 2 is mostly language. The literate reader can skip it and refer back to it as needed. In Section 3 we provide constructions of the Poincaré embedding space, its unlinked variant and the space of Poincaré immersions. We prove Theorems A and B in Section 4. In Section 5 we prove Theorem C. The final section, Appendix A, is conjectural: it poses a connection between the tower of Theorem C and the Goodwillie–Weiss tower for smooth embeddings.

## §2. Preliminaries

### 2.1 Spaces

Our ground category is  $T$ , the category of compactly generated weak Hausdorff spaces. A non-empty space  $X$  is  $r$ -connected if  $\pi_j(X, x)$  is trivial for  $j \leq r$ , for all base points  $x \in X$ . The empty space is  $(-2)$ -connected and every non-empty space is  $(-1)$ -connected. A map  $X \rightarrow Y$  of non-empty spaces is  $r$ -connected if each of its homotopy fibers is  $(r - 1)$ -connected (every map of non-empty spaces is  $(-1)$ -connected; a weak homotopy equivalence is an  $\infty$ -connected map).

For unbased spaces  $X$  and  $Y$  we let  $F(X, Y)$  be the unbased function space and if  $X$  and  $Y$  are based we let  $F_*(X, Y)$  be the based function space. When we write  $[X, Y]$ , we mean homotopy classes of based maps  $X^c \rightarrow Y$ , where  $X^c$  is a cofibrant replacement for  $X$ . When  $X$  and  $Y$  are based, then the based homotopy classes are to be written as  $[X, Y]_*$ . We use the usual notation for the smash product:  $X \wedge Y$ , and the iterated smash product of  $j$ -copies of  $X$  is denoted  $X^{[j]}$ .

We equip  $T$  with the Quillen model category structure given by the Serre fibrations, Serre cofibrations and weak homotopy equivalences [17], [9, Theorem 2.4.23]. Note that  $T$  is enriched over itself. We let  $T_*$  denote the model category of based spaces.

A commutative square of spaces

$$\begin{array}{ccc} X_0 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_{12} \end{array}$$

is *homotopy co-Cartesian* if the map

$$\mathrm{hocolim}(X_1 \leftarrow X_0 \rightarrow X_2) \rightarrow X_{12} \rightarrow X_{12}$$

is a weak equivalence, where the domain of this map is given by the homotopy pushout of the diagram obtained from the square by removing its terminal vertex. In the special case when  $X_2$  is contractible, we abuse notation and refer to  $X_\emptyset \rightarrow X_1 \rightarrow X_{12}$  as a *homotopy cofiber sequence*. This is the same as equipping the composition  $X_\emptyset \rightarrow X_1 \rightarrow X_{12}$  with a preferred choice of null homotopy such that the induced map  $X_2 \cup_{X_\emptyset} CX_\emptyset \rightarrow X_{12}$  is required to be a weak equivalence.

Similarly, the above square is *homotopy Cartesian* if the map from  $X_\emptyset$  to the homotopy pullback of  $X_1 \rightarrow X_{12} \leftarrow X_2$  is a weak equivalence. When  $X_2$  is contractible, we refer to  $X_\emptyset \rightarrow X_1 \rightarrow X_{12}$  as a *homotopy fiber sequence*. The latter is equivalent to describing a null homotopy of the composition  $X_\emptyset \rightarrow X_1 \rightarrow X_{12}$  such that the map from  $X_\emptyset$  to the homotopy fiber of the map  $X_1 \rightarrow X_{12}$  is a weak equivalence.

In each of these notions, when the null homotopy is understood, we typically omit it from the notation to avoid clutter.

## 2.2 Factorization categories

Fix a map of spaces  $f : A \rightarrow B$ . Define a category

$$T(f) = T(f : A \rightarrow B)$$

whose objects are spaces  $X$  and a factorization  $A \rightarrow X \rightarrow B$  by continuous maps. A morphism  $X \rightarrow X'$  is a map of spaces that is compatible with the factorizations. When  $f$  is understood, we usually write this category as  $T(A \rightarrow B)$ .

Here are some important special cases:

EXAMPLE 2.1. Let  $A = B$  and use the identity map. Then  $T(B \rightarrow B)$  is the category of spaces which contain  $B$  as a retract.

EXAMPLE 2.2. Let  $A = \emptyset$  be the empty space. Then  $T(\emptyset \rightarrow B)$  is the category of spaces over  $B$ .

EXAMPLE 2.3. Let  $B = *$  be the one-point space. Then  $T(A \rightarrow *)$  is the category of spaces under  $A$ .

The forgetful functor  $T(A \rightarrow B) \rightarrow T$  induces a model structure on  $T(A \rightarrow B)$  by declaring a morphism to be a cofibration, fibration or weak equivalence if and only if it is one in  $T$  [17, 2.8, Proposition 6]. This model structure is enriched over  $T$ . The category of weak equivalences is denoted by

$$wT(A \rightarrow B).$$

REMARK 2.4. We use the notation  $|\mathcal{C}|$  for the *realization* (of the nerve) of a small category  $\mathcal{C}$ . The functor  $\mathcal{C} \mapsto |\mathcal{C}|$  enables one to transfer homotopical properties of spaces over to small categories. For example, we declare a functor  $f: \mathcal{C} \rightarrow \mathcal{D}$  to be  $r$ -connected if and only if it is so upon taking realization. Likewise, it makes sense to ask whether a commutative square of small categories is homotopy Cartesian.

In this paper the categories  $\mathcal{C}$  that we will want to apply realization to are full subcategories of  $wT(A \rightarrow B)$  – but they are not small. This is not a major dilemma; for a discussion of the options on how to deal with the matter, see [6, p. 766].

### 2.3 Spectra

The spectra appearing in this paper are formed from objects of  $T_*$ . For us, a spectrum will be a sequence of based spaces  $E_j$  and (structure) maps  $\Sigma E_j \rightarrow E_{j+1}$ . We say that  $E$  is *cofibrant* if each of the spaces  $E_j$  is cofibrant and each structure map is a cofibration.  $E$  is *fibrant* if each adjoint  $E_j \rightarrow \Omega E_{j+1}$  is a weak equivalence.

A map of spectra  $f: E \rightarrow E'$  is a collection of maps  $f_j: E_j \rightarrow E'_j$  that are compatible with the structure maps. Any spectrum  $E$  has a *fibrant replacement*, which is a spectrum  $E^f$  equipped with a natural map of spectra  $E \rightarrow E^f$ , where  $E_j^f := \text{colim}_k \Omega^k(S^j \wedge E_k)$ . The map  $f: E \rightarrow E'$  is a (*stable*) *weak equivalence* if the associated map  $E^f \rightarrow (E')^f$  is such that for each  $j$  the map of based spaces  $E_j^f \rightarrow (E')_j^f$  is a weak equivalence. If  $E$  is a spectrum, we write  $\Omega^\infty E$  for the associated infinite loop space given by the zeroth space of its fibrant replacement. If  $X$  is a based space, then we let  $\Sigma^\infty X$  be its suspension spectrum whose  $j$ th space is  $S^j \wedge X$ . For it to have the correct homotopy type we should assume that  $X$  is cofibrant. The zeroth space of  $\Sigma^\infty X$  is denoted  $Q(X)$ .

Given a based space  $X$  and a spectrum  $E$  we can form  $X \wedge E$  which is the spectrum whose  $j$ th space is  $X \wedge E_j$ . This has the correct homotopy type if both  $X$  and  $E$  are cofibrant. Similarly we can form the functions  $F_*(X, E)$  which is the spectrum whose  $j$ th space is  $F_*(X, E_j)$ . This has the correct homotopy type when  $X$  is cofibrant and  $E$  is fibrant (when  $E$  fails to be fibrant, we will implicitly replace it by its fibrant model). The associated *stable function space* is  $\Omega^\infty F_*(X, E)$ . We will typically be sloppy and omit the  $\Omega^\infty$  from the notation. Thus,  $F_*(X, E)$  can mean either the spectrum or its associated infinite loop space. If  $X$  is unbased then we set  $F(X, E) = F_*(X_+, E)$  where  $X_+ = X \amalg *$ . If  $X$  and  $Y$  are based spaces, then



a *stable map*  $X \rightarrow Y$  is an element of the stable function space  $F_*(X, \Sigma^\infty Y)$ , that is, a point of the function space  $F_*(X, Q(Y))$ , where  $Q(Y) = \Omega^\infty \Sigma^\infty Y$ . We let  $\{X, Y\}_*$  denote the stable homotopy classes of maps from  $X$  to  $Y$ ; this is the same as  $\pi_0 ( F_*(X, Q(Y))$  when  $X$  and  $Y$  are cofibrant.

Smash products of spectra are barely used in this paper, and are confined to the proof of Theorem C. It is for this reason that we are content to work in the above category of spectra. The reader is free to use a more modern approach.

**2.4 Spectra with group action**

Fix a discrete group  $G$ . We say that a spectrum  $E$  has a  $G$ -action if each  $E_j$  has the structure of a based  $G$ -space and each structure map  $\Sigma E_j \rightarrow E_{j+1}$  is equivariant, where  $G$  acts trivially on the suspension coordinate. A map  $E \rightarrow E'$  of spectra with  $G$ -action is just a map of underlying spectra which is  $G$ -equivariant. A map of spectra is a *weak equivalence* if it is when considered as a map of spectra without action. We say that  $E$  is fibrant if its underlying spectrum (without action) is. Call a based  $G$ -space  $X$   *$G$ -cofibrant* if it is built up from the basepoint by attaching free  $G$ -cells along equivariant maps; a free  $G$ -cell has the form  $D^n \times G$ .

If  $X$  is a  $G$ -space and  $E$  is a spectrum with  $G$ -action, then  $G$  acts diagonally on  $X \wedge E$ . We write  $X \wedge_G E$  for the orbit spectrum. This has the correct homotopy type if  $X$  and  $E$  are both  $G$ -cofibrant. The *homotopy orbits* of  $G$  acting on  $E$  is the spectrum

$$E_{hG} = EG_+ \wedge_G E$$

where  $EG$  the universal contractible  $G$ -space. This has the correct homotopy type of the underlying spectrum of  $E$  is cofibrant.

**2.5 Poincaré spaces**

The Poincaré spaces of this paper are orientable. An orientable *Poincaré space* of dimension  $d$  consists of a homotopy finite space  $P$  for which there exists a fundamental class  $[P] \in H_d(P; \mathbb{Z})$  such that the cap product

$$\cap [P]: H^*(P; \mathcal{M}) \rightarrow H_{d-*}(P; \mathcal{M})$$

is an isomorphism in all degrees for any locally constant sheaf  $\mathcal{M}$ .

If  $\pi: \tilde{P} \rightarrow P$  is a choice of universal cover, then the cap product is an isomorphism for all  $\mathcal{M}$  if and only if it is an isomorphism for the locally

constant sheaf  $\Lambda$  whose stalk at  $x \in P$  is given by the free abelian group with basis  $\pi^{-1}(x)$  (cf. [21, Lemma 1.1]).

Poincaré spaces  $P$  with boundary  $\partial P$ , also known as Poincaré pairs, are defined similarly, where now  $[P] \in H_d(P, \partial P; \mathbb{Z})$ , the cap product

$$H^*(P; \mathcal{M}) \xrightarrow{\cap [P]} H_{d-*}(P, \partial P; \mathcal{M})$$

is an isomorphism, and the class  $[\partial P] \in H_{d-1}(\partial P; \mathbb{Z})$ , obtained by applying the boundary homomorphism to  $[P]$ , equips  $\partial P$  the structure of a Poincaré space of dimension  $d - 1$  (this assumes in particular that  $\partial P$  is homotopy finite). We will be relaxed about language and refer to a Poincaré space with or without boundary simply as a Poincaré space.

We will also sometimes omit the condition that the map  $\partial P \rightarrow P$  is an inclusion. The definition of a Poincaré space still makes sense in this instance since we can replace any map by its mapping cylinder inclusion.

**§3. Poincaré embeddings**

Let  $P$  be a Poincaré space of dimension  $n$ . We will assume here that  $\partial P \rightarrow P$  is a cofibration. An (*interior*) *Poincaré embedding* of  $P$  in  $D^n$  consists of a space  $C$  and a map  $\partial P \amalg S^{n-1} \rightarrow C$  such that

- $C$  is a Poincaré space with boundary  $\partial P \amalg S^{n-1}$ ;
- the amalgamated union

$$P \cup_{\partial P} C$$

is weakly contractible.

In what follows we set

$$A := \partial P \amalg S^{n-1}.$$

Then

$$C \in wT(A \rightarrow *)$$

is an object. Let

$$\mathcal{E}(P, D^n) \subset wT(A \rightarrow *)$$

be the full subcategory whose objects give Poincaré embeddings of  $P$  in  $D^n$ . The space of Poincaré embeddings of  $P$  in  $D^n$  is then defined as the realization

$$E(P, D^n) = |\mathcal{E}(P, D^n)|.$$

This is an open and closed subspace of  $|wT(A \rightarrow *)|$ .

REMARK 3.1. The version of the Poincaré embedding space appearing here is slightly different from the one in [6, Definition 2.8]. There it is defined to be the homotopy fiber of the functor

$$(7) \quad \mathcal{I}(A) \rightarrow \mathcal{I}(S^{n-1})$$

given by “gluing in  $P$ ,” where for a Poincaré space  $\partial$  without boundary, the category  $\mathcal{I}(\partial)$  has objects Poincaré spaces  $X$  with  $\partial$  as boundary, and morphisms are weak homotopy equivalences  $X \rightarrow X'$  which restrict to the identity on  $\partial$ . In the definition of [6, Definition 2.8], the homotopy fiber of (7) is taken at  $D^n \in \mathcal{I}(S^{n-1})$ . Our definition here amounts to taking a certain open and closed subspace of  $\mathcal{I}(A)$  rather than a homotopy fiber. This definition is equivalent to the one in [6] because the component of  $D^n \in \mathcal{I}(S^{n-1})$  is contractible.

The *decompression functor*

$$\mathcal{E}(P, D^n) \rightarrow \mathcal{E}(P \times D^1, D^{n+1})$$

is defined by mapping  $C \in \mathcal{E}(P, D^n)$  to its unreduced suspension  $SC$ . On realizations it defines the *decompression map*

$$(8) \quad E(P, D^n) \rightarrow E(P \times D^1, D^{n+1}).$$

### 3.1 Unlinked embeddings

If  $P$  is sectioned, then we set

$$A' = P^\xi \vee S^{n-1}.$$

There is then a cofiber sequence  $P_+ \rightarrow A \rightarrow A'$ . The map  $A \rightarrow A'$  induces a (forgetful) functor

$$wT(A' \rightarrow *) \rightarrow wT(A \rightarrow *).$$

DEFINITION 3.2. The space of *unlinked embeddings*  $\mathfrak{L}\mathcal{E}(P, D^n)$  is the realization of the full subcategory

$$\mathfrak{L}\mathcal{E}(P, D^n) \subset wT(A' \rightarrow *)$$

consisting of objects  $C$  which become Poincaré embeddings when considered in  $wT(A \rightarrow *)$ .

Unraveling the definition, we see that an unlinked embedding consists of a space  $C$  and a map  $P^\xi \vee S^{n-1} \rightarrow C$  such that the composition

$$\partial P \amalg S^{n-1} \rightarrow P^\xi \vee S^{n-1} \rightarrow C$$

defines a Poincaré embedding of  $P$  in  $D^n$ .

By definition, there is a homotopy Cartesian square

$$(9) \quad \begin{array}{ccc} \mathfrak{L}\mathcal{E}(P, D^n) & \longrightarrow & wT(A' \rightarrow *) \\ \downarrow & & \downarrow \\ \mathcal{E}(P, D^n) & \longrightarrow & wT(A \rightarrow *). \end{array}$$

### 3.2 Poincaré Immersions

Recall  $P_j := P \times D^j$ . The *immersion space*  $I(P, D^n)$  is defined as the homotopy colimit of the sequence of decompression maps

$$E(P, D^n) \rightarrow E(P_1, D^{n+1}) \rightarrow E(P_2, D^{n+2}) \dots$$

LEMMA 3.3. *There is a homotopy equivalence*

$$I(P, D^n) \simeq \Omega_{\perp}^n Q(P/\partial P).$$

*Proof.* The Browder construction gives a factorization of the filtration defining  $I(P, D^n)$  as

$$E(P, D^n) \rightarrow \Omega_{\perp}^n P/\partial P \rightarrow E(P_1, D^{n+1}) \rightarrow \Omega_{\perp}^{n+1} P_1/\partial P_1 \rightarrow E(P_2, D^{n+2}) \rightarrow \dots$$

The homotopy colimit of the odd terms appearing in the sequence yields  $I(P, D^n)$  by definition, whereas the homotopy colimit of the even terms gives the space of stable normal invariants

$$\Omega_{\perp}^n Q(P/\partial P),$$

since  $\Omega^{n+j}(P_j/\partial P_j) = \Omega^{n+j}\Sigma^j(P/\partial P)$ . □

LEMMA 3.4. (Smale–Hirsch for Poincaré Spaces) *If  $I(P, D^n)$  is non-empty and equipped with a basepoint, then there is preferred weak homotopy equivalence*

$$I(P, D^n) \simeq F(P, G),$$

where the right side denotes the function space of unbased maps from  $P$  to the topological monoid of stable self equivalences of the sphere.

REMARK 3.5. The corresponding statement in the smooth case is that the smooth immersions of  $P$  to  $D^n$  is weak equivalent to the function space  $F(P, O_n)$ , where  $O_n$  is the group of orthogonal  $n \times n$  matrices. Note that the smooth version depends on  $n$ .

To obtain a smooth statement which does not depend on  $n$ , one should replace the smooth immersion space by its block analogue. In this instance one obtains a weak equivalence to the function space  $F(P, O)$ .

*Proof of Lemma 3.4.* If  $I(P, D^n)$  is non-empty then the Spivak fibration for  $P$  is trivializable, implying that  $P/\partial P$  is  $n$ -dual to  $P_+$ . By S-duality, we have a weak equivalence

$$F_*(P_+, Q(S^0)) \simeq \Omega^n Q(P/\partial P).$$

Restricting to stable normal invariants on the right corresponds to replacing  $Q(S^0)$  on the left by its units, namely  $G$ . The result now follows by Lemma 3.3. □

**§4. Proof of Theorems A and B**

Consider the following situation: fix a map of based spaces

$$f: A \rightarrow X$$

and let  $Y$  denote its reduced mapping cone. Consider the forgetful functor

$$wT(Y \rightarrow *) \rightarrow wT(X \rightarrow *).$$

Let  $Z \in wT(X \rightarrow *)$  be an object; in particular,  $Z$  has the structure of a based space. The map  $X \rightarrow Z$  factors through  $Y$  precisely when the composition

$$A \rightarrow X \rightarrow Z$$

is null homotopic. In what follows we fix based null homotopy  $CA \rightarrow Z$ .

LEMMA 4.1. *With these assumptions, there is a homotopy fiber sequence*

$$F_*(\Sigma A, Z) \rightarrow |wT(Y \rightarrow *)| \rightarrow |wT(X \rightarrow *)|,$$

where the displayed fiber is taken at the basepoint  $Z \in |wT(X \rightarrow *)|$ .

*Proof.* By [6, Proposition 2.14], the homotopy fiber of  $|wT(Y \rightarrow *)| \rightarrow |wT(X \rightarrow *)|$  taken at  $Z$  identified up to preferred weak equivalence with the function space of liftings/extensions

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & \nearrow & \downarrow \\ Y & \longrightarrow & * \end{array}$$

We employ the notation

$$F_X(Y, Z)$$

for this space. The given extension equips  $F_X(Y, Z)$  with a basepoint and the restriction map  $F_X(Y, Z) \rightarrow F_A(CA, Z)$  is a homeomorphism. Let  $\rho: CA \rightarrow Z$  be the given null homotopy. The cofiber sequence  $A \rightarrow A \rightarrow CA$  has a coaction map  $\delta: CA \rightarrow CA \vee \Sigma A$ . Given a map  $g: \Sigma A \rightarrow Z$ , we form

$$\rho \star g: CA \xrightarrow{\delta} CA \vee \Sigma A \xrightarrow{\rho \star g} Z.$$

Then  $g \mapsto \rho \star g$  defines a weak equivalence  $F_*(\Sigma A, Z) \simeq F_A(CA, Z)$ . □

*Proof of Theorem B.* Recall that  $A = \partial P \amalg S^{n-1}$  and  $A' = P^\xi \vee S^{n-1}$ . Consider the full subcategory

$$wT(A' \rightarrow *; \sim P^\xi) \subset wT(A' \rightarrow *)$$

with objects  $C$  such that the composite

$$P^\xi \subset A' \rightarrow C$$

is a weak homotopy equivalence. We claim there is a homotopy equivalence

$$|wT(A' \rightarrow *; \sim P^\xi)| \simeq \Omega^{n-1} P^\xi.$$

To see this, note that  $wT(A' \rightarrow *; \sim P^\xi)$  is the right fiber taken at  $P^\xi \in wT(* \rightarrow *)$  of the forgetful functor

$$wT(S^{n-1} \rightarrow *) \rightarrow wT(* \rightarrow *)$$

and by [6, Proposition 2.19] we may identify this right fiber with  $\Omega^{n-1} P^\xi$ . This gives the claim.

We restrict our attention to the full subcategory

$$wT_{\perp}(A' \rightarrow *; \sim P^{\xi}) \subset wT(A' \rightarrow *; \sim P^{\xi})$$

of those objects  $C$  such that the weak map  $S^{n-1} \rightarrow C \xleftarrow{\sim} P^{\xi}$  corresponds to an unstable normal invariant. This additional constraint yields a homotopy equivalence

$$|wT_{\perp}(A' \rightarrow *; \sim P^{\xi})| \simeq \Omega_{\perp}^{n-1} P^{\xi}.$$

The refined Browder construction defines a functor

$$F : wT_{\perp}(A' \rightarrow *; \sim P^{\xi}) \rightarrow \mathcal{L}\mathcal{E}(P, D^n).$$

On the other hand, the identity defines a functor

$$G : \mathcal{L}\mathcal{E}(P, D^n) \rightarrow wT_{\perp}(A' \rightarrow *; \sim P^{\xi}).$$

It is tautological that these functors are inverses to each other. □

*Proof of Theorem A.* By Theorem B it suffices to consider the map

$$\mathcal{L}E(P, D^n) \rightarrow E(P, D^n).$$

We will make use of the cofiber sequence

$$P_+ \rightarrow A \rightarrow A'.$$

If  $C \in \mathcal{E}(P, D^n)$  is an object, then clearly the obstruction to lifting it to an object of  $\mathcal{L}\mathcal{E}(P, D^n)$  up to weak equivalence is that the composite

$$P_+ \rightarrow A \rightarrow C$$

is null homotopic. This proves the first part. Now suppose a null homotopy  $P_+ \rightarrow C$  has been chosen. Using Lemma 4.1, we have a homotopy fiber sequence

$$F_*(\Sigma(P_+), C) \rightarrow |wT(A' \rightarrow *)| \rightarrow |wT(A \rightarrow *)|.$$

One completes the proof using the homotopy Cartesian square (9). □

§5. Proof of Theorem C

5.1 Principal fibrations

We recall a basic result about principal fibrations from [14, Lemma 6.1]. Suppose  $p: E \rightarrow Z$  is a fibration. We say that  $p$  is *principal* if there exists a commutative homotopy Cartesian square of spaces

$$\begin{array}{ccc} E & \longrightarrow & C \\ p \downarrow & & \downarrow \\ Z & \longrightarrow & B \end{array}$$

such that  $C$  is contractible. Note that the property of being a principal fibration is preserved under base changes. Choose a basepoint for  $C$ . This gives a basepoint for  $B$ .

Suppose that  $Z$  is connected. If  $p: E \rightarrow Z$  is principal, there is an “action”  $\Omega B \times E \rightarrow E$ . If there exists a section  $Z \rightarrow E$ , one can combine it with this action to produce a map of fibrations  $\Omega B \times Z \rightarrow E$  covering the identity map of  $Z$ . This implies that  $p$  is weak fiber homotopically trivial. Let  $\text{sec}(p)$  denote the space of sections of  $p$ . Then we have shown

LEMMA 5.1. *Assume  $p: E \rightarrow Z$  is principal. Assume that  $\text{sec}(p)$  is non-empty and comes equipped with basepoint. Then there is a preferred weak equivalence  $\text{sec}(p) \simeq F(Z, \Omega B)$ .*

Let  $p: E \rightarrow Z$  be a principal fibration and suppose that  $A \rightarrow Y$  is a cofibration. Then given a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ Y & \xrightarrow{f} & Z \end{array}$$

we let  $\text{lifts}(f|p)$  be the solution space: the space of maps  $Y \rightarrow E$  of  $f$  making the diagram commute.

COROLLARY 5.2. *If  $\text{lifts}(f|p)$  is non-empty then a choice of lift determines a weak equivalence  $\text{lifts}(f|p) \simeq F_*(Y/A, \Omega B)$ .*

*Proof.* Observe that  $f^*E \rightarrow Y$  is principal. Furthermore  $\text{lifts}(f|p) \cong \text{sec}(f^*E \rightarrow Y)$ . Hence if  $\text{lifts}(f|p)$  is nonempty we can identify  $f^*E$  with



the trivial fibration  $\Omega B \times Y \rightarrow Y$  once a basepoint lift has been chosen. With respect to the identification, the given map  $A \rightarrow f^*E$  corresponds to the inclusion  $* \times A \rightarrow \Omega B \times Y$ .

Hence,  $\text{lifts}(f|_p)$  is then identified up to weak equivalence with the space of sections of the trivial fibration  $\Omega B \times Y \rightarrow Y$  which are fixed on  $A$ . But this is just  $F_*(Y/A, \Omega B)$ . □

**5.2 The Goodwillie tower of the identity**

Let  $\mathbb{I}: \mathbf{Top}_* \rightarrow \mathbf{Top}_*$  be the identity functor. We recall some of the basic properties of its Goodwillie tower. (cf. [3], [4], [5], [10]).

**THEOREM 5.3.** *There is a tower of fibrations of homotopy functors on based spaces*

$$\cdots \rightarrow P_2\mathbb{I}(X) \rightarrow P_1\mathbb{I}(X)$$

and compatible natural transformations  $X \rightarrow P_j\mathbb{I}(X)$  such that

- if  $X$  is 1-connected, then the natural map

$$X \rightarrow \lim_j P_j\mathbb{I}(X)$$

is a weak equivalence.

- There is a natural weak equivalence  $P_1\mathbb{I}(X) \simeq Q(X)$ ;
- For  $j \geq 2$ , the fibration

$$P_j\mathbb{I}(X) \rightarrow P_{j-1}\mathbb{I}(X)$$

is principal (cf. [5, Lemma 2.2]);

- the  $j$ th layer  $L_j\mathbb{I}(X) := \text{fib}(P_j\mathbb{I}(X) \rightarrow P_{j-1}\mathbb{I}(X))$  is naturally weak equivalent to the functor

$$X \mapsto \Omega^\infty(\mathscr{W}_j \wedge_{h\Sigma_j} X^{[j]});$$

where the spectrum with  $\Sigma_j$ -action  $\mathscr{W}_j$  is as in [10].

**5.3 The spaces  $E_j(P, D^n)$**

Recall that  $P$  is sectioned by  $\xi$ . We fix the natural identification  $P_1\mathbb{I}(X) \simeq Q(X)$ . Then we have a map

$$(10) \quad \Omega_{\perp}^{n-1}Q(P^\xi) \rightarrow \Omega^{n-1}P_1\mathbb{I}(P^\xi).$$

Note that the source of this map is identified with a collection of components of the target.

DEFINITION 5.4. The space  $\mathfrak{L}E_j(P, D^n)$  is defined to be the union of path components of  $\Omega^{n-1}P_j\mathbb{I}(P^\xi)$  given by taking the pullback of the diagram

$$\Omega_{\perp}^{n-1}Q(P^\xi) \rightarrow \Omega^{n-1}P_1\mathbb{I}(P^\xi) \leftarrow \Omega^{n-1}P_j\mathbb{I}(P^\xi).$$

*Proof of Theorem C.* It is a consequence of the definition that there is a tower of fibrations

$$(11) \quad \cdots \rightarrow \mathfrak{L}E_2(P, D^n) \rightarrow \mathfrak{L}E_1(P, D^n).$$

By definition  $\mathfrak{L}E_1(P, D^n) \cong \Omega_{\perp}^{n-1}Q(P^\xi)$  and by Lemma 3.3,  $\Omega_{\perp}^{n-1}Q(P^\xi) \simeq I(P, D^n)$ . Moreover, the square

$$(12) \quad \begin{array}{ccc} \lim_j \mathfrak{L}E_j(P, D^n) & \longrightarrow & \lim_j \Omega^{n-1}P_j\mathbb{I}(P^\xi) \\ \downarrow & & \downarrow \\ \mathfrak{L}E_1(P, D^n) & \longrightarrow & P_1\mathbb{I}(P^\xi) \end{array}$$

is homotopy Cartesian. The lower right corner of this diagram is identified with  $Q(P^\xi)$ . Since that map  $\partial P \rightarrow P$  is at least 2-connected, it follows that the section  $K \rightarrow \partial P$  is at least 1-connected. Hence,  $P^\xi$  is 1-connected and the upper right corner of diagram is identified with  $\Omega^{n-1}P^\xi$ . Substituting these identifications, we obtain a homotopy Cartesian square

$$(13) \quad \begin{array}{ccc} \lim_j \mathfrak{L}E_j(P, D^n) & \longrightarrow & \Omega^{n-1}P^\xi \\ \downarrow & & \downarrow \\ I(P, D^n) & \longrightarrow & \Omega^{n-1}Q(P^\xi). \end{array}$$

Clearly, if we replace the upper left corner by the space of unstable normal invariants  $\Omega_{\perp}^{n-1}P^\xi$  the square remains homotopy Cartesian, since a point of  $\Omega^{n-1}P^\xi$  yields an unstable normal invariant if and only if the associated point of  $\Omega^{n-1}Q(P^\xi)$  yields a stable normal invariant. It follows that the map

$$\Omega_{\perp}^{n-1}P^\xi \rightarrow \lim_j \mathfrak{L}E_j(P, D^n)$$

is a weak equivalence. Therefore, the composite map

$$\mathfrak{L}E(P, D^n) \xrightarrow{\sim} \Omega_{\perp}^{n-1}P^\xi \xrightarrow{\sim} \lim_j \mathfrak{L}E_j(P, D^n)$$

is also a weak equivalence.

We next identify the layers of the tower (11) whenever they are non-empty. The fiber of the map  $\Omega^{n-1}P_j\mathbb{I}(P^\xi) \rightarrow \Omega^{n-1}P_{j-1}\mathbb{I}(P^\xi)$  at any base-point is just the lifting space

$$(14) \quad \begin{array}{ccc} * & \longrightarrow & P_j\mathbb{I}(P^\xi) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ S^{n-1} & \longrightarrow & P_{j-1}\mathbb{I}(P^\xi). \end{array}$$

If this lifting space is non-empty, then Corollary 5.2 says that after making a choice of lift, the space of all such lifts is identified with the stable function space

$$F_*(S^{n-1}, \mathscr{W}_j \wedge_{h\Sigma_j} (P^\xi)^{[j]}).$$

We need to rewrite this stable function space up to homotopy in the requisite form. First identify it as the zeroth space of the homotopy orbit spectrum

$$(15) \quad (S^{1-n} \wedge \mathscr{W}_j \wedge (P^\xi)^{[j]})_{h\Sigma_j}.$$

The plan is to rewrite the latter in terms of  $P_+$  using Spanier–Whitehead duality. Assuming that  $I(P, D^n)$  is non-empty guarantees that  $P^\xi$  is  $(n - 1)$ -dual to  $P_+$ . We can write this as

$$\Sigma^\infty P^\xi \simeq \Sigma^{n-1}D(P_+),$$

where  $D(P_+)$  is the 0-dual of  $P_+$ , that is,  $F_*(P_+, S)$ , where  $S$  denotes the sphere spectrum. If we smash this identification with itself  $j$ -times, we obtain an equivariant weak equivalence of spectra with  $\Sigma_j$ -action

$$\Sigma^\infty (P^\xi)^{[j]} \simeq F_*((P^{\times j})_+, \Sigma^\infty S^{(n-1)(V_j+1)}).$$

Substituting this into (15), and doing some minor rewriting, we obtain the spectrum

$$F_*((P^{\times j})_+, \mathscr{W}_j \wedge S^{(n-1)V_j})_{h\Sigma_j}.$$

The zeroth space of this spectrum is thus identified with the homotopy fibers of  $\mathscr{L}E_j(P, D^n) \rightarrow \mathscr{L}E_{j-1}(P, D^n)$  whenever these are non-empty.

Lastly, we need to exhibit the obstruction  $\ell_{j-1}$ . According to [5, Lemma 2.2] there is a  $j$ -homogeneous functor  $X \mapsto R_j\mathbb{I}(X)$  and a homotopy Cartesian square

$$\begin{array}{ccc}
 P_j\mathbb{I}(X) & \longrightarrow & C \\
 \downarrow & & \downarrow \\
 P_{j-1}\mathbb{I}(X) & \longrightarrow & R_j\mathbb{I}(X)
 \end{array}$$

where  $C$  is contractible. By the classification of  $j$ -homogeneous functors [5], [3, p. 5], we infer

$$R_j\mathbb{I}(X) \simeq \Omega^\infty \mathcal{V}_j \wedge_{h\Sigma_j} X^{[j]}$$

for a some spectrum with  $\Sigma_j$ -action  $\mathcal{V}_j$ . Furthermore, the map

$$L_j\mathbb{I}(X) \rightarrow \Omega R_j\mathbb{I}(X)$$

is a weak equivalence of  $j$ -homogenous functors. It follows that there is a weak equivalence of spectra with  $\Sigma_j$ -action

$$\mathcal{V}_j \simeq \Sigma \mathcal{W}_j.$$

Consequently, setting  $X = P^\xi$ , we have a homotopy Cartesian square

$$\begin{array}{ccc}
 P_j\mathbb{I}(P^\xi) & \longrightarrow & C \\
 \downarrow & & \downarrow \\
 P_{j-1}\mathbb{I}(P^\xi) & \longrightarrow & \Omega^\infty \Sigma \mathcal{W}_j \wedge_{h\Sigma_j} (P^\xi)^{[j]}
 \end{array}$$

in which  $C$  is contractible. Hence, the obstruction up to homotopy to a lifting a based map  $x: S^{n-1} \rightarrow P_{j-1}\mathbb{I}(P^\xi)$  to a based map  $S^{n-1} \rightarrow P_j\mathbb{I}(P^\xi)$  is given by the homotopy class of the composition

$$(16) \quad S^{n-1} \rightarrow P_{j-1}\mathbb{I}(P^\xi) \rightarrow \Omega^\infty \Sigma \mathcal{W}_j \wedge_{h\Sigma_j} (P^\xi)^{[j]}.$$

In particular, if  $x \in E_{j-1}(P, D^n) \subset \Omega^{n-1} P_{j-1}\mathbb{I}(P^\xi)$  is a point, then we define  $\ell_{j-1}(x)$  to be the homotopy class of (16). Then  $\ell_{j-1}(x)$  *a priori* lies in the abelian group

$$\{S^{n-1}, \Sigma \mathcal{W}_j \wedge_{h\Sigma_j} (P^\xi)^{[j]}\}_*$$

Again by duality, we can rewrite the latter up to canonical isomorphism as

$$\pi_0(F_*(P_+^{\times j}, \mathcal{W}_j \wedge S^{(n-1)(V_j-1)+1})_{h\Sigma_j}). \quad \square$$

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### Appendix A. Relationship with manifold calculus

It is legitimate to ask what the tower of Theorem C has to do with the Goodwillie–Weiss manifold calculus [22]. Here is one possible scenario: suppose that  $Q$  is a compact smooth  $(n - 1)$ -manifold—which we assume admits a handle decomposition with handles of index at most  $k \leq n - 4$ . We consider the forgetful/decompression map

$$(A1) \quad E^{\text{sm}}(Q, D^{n-1}) \rightarrow \mathcal{L}E(Q \times D^1, D^n)$$

from the space of smooth embeddings of  $Q$  in  $D^n$  to the space of unlinked Poincaré embeddings of  $Q \times D^1$  in  $D^n$ .

CONJECTURE A.1. *The map (A1) induces a map of towers from the Goodwillie–Weiss tower for  $E^{\text{sm}}(Q, D^{n-1})$  to the tower of Theorem C for  $\mathcal{L}E(Q \times D^1, D^n)$ .*

REMARK A.2. By Theorem B we can identify the map with the Pontryagin–Thom construction

$$E^{\text{sm}}(Q, D^{n-1}) \rightarrow \Omega_{\perp}^{n-1}(Q/\partial Q)$$

from a space of smooth embeddings to the space of unstable normal invariants, the latter which is collection of components of  $\Omega^{n-1}(Q/\partial Q)$ . Recall that the tower of Theorem C is arises from the Goodwillie tower of the identity functor  $\mathbb{I}$  applied to the based space  $Q/\partial Q$ . The conjecture therefore concerns how the manifold calculus tower for the embedding functor relates to the homotopy calculus for the functor  $\mathbb{I}$ .

We will give some evidence for the conjecture on the level of layers. In what follows we shall assume that the reader is familiar with [22]. Here is some notation: suppose  $T$  is a finite set. We write

$$\mathcal{P}(Q, T)$$

for the configuration space of the injective functions from  $T$  to the interior of  $Q$ . This has a free action of  $\Sigma_T$ , the symmetric group of automorphisms

of  $T$ . In the case when  $T = \underline{j} = \{1, \dots, j\}$ , we write  $\mathcal{P}(Q, j) := \mathcal{P}(Q, \underline{j})$ . Let

$$\binom{Q}{T} := \mathcal{P}(Q, T)_{\Sigma_T}$$

the orbit space of the action of  $\Sigma_T$  on  $\mathcal{P}(Q, T)$ ; this is the configuration space of (unordered) subsets of the interior of  $Q$  of cardinality  $|T|$ . Similarly, we write  $\binom{Q}{j} := \binom{Q}{\underline{j}}$ .

The quotient map  $\pi: \mathcal{P}(Q, j) \rightarrow \binom{Q}{j}$  is a principal covering space with respect to the group  $\Sigma_j$ ; we let  $i: \binom{Q}{j} \rightarrow B\Sigma_j$  denote its classifying map. Then  $i \circ \pi$  is the constant map to the basepoint. Let  $c: Q^{\times j} \rightarrow B\Sigma_j$  be the constant map. Then  $c$  restricted to  $\mathcal{P}(Q, j)$  is  $i \circ \pi$ . There is an inclusion  $\mathcal{P}(Q, j) \subset Q^{\times j}$  whose complement is the diagonal  $\Delta \subset Q^{\times j}$ .

The evidence we give for Conjecture A.1 is a diagram

$$\begin{array}{ccc} H_{\text{cs}}^\bullet(\binom{Q}{j}; \mathcal{E}_j) & \xrightarrow{(a)} & H_{\text{cs}}^\bullet(\binom{Q}{j}; \mathcal{F}_j) \\ & & \downarrow (b) \\ & & H_{\text{cs}}^\bullet(\binom{Q}{j}; i^*\mathcal{G}_j) \\ & & \simeq \uparrow (c) \\ & & H_{\text{cs}}^\bullet(\mathcal{P}(Q, j); (i \circ \pi)^*\mathcal{G}_j)_{h\Sigma_j} \\ & & \simeq \uparrow (d) \\ & & H^\bullet(Q^{\times j}, \Delta; c^*\mathcal{G}_j)_{h\Sigma_j} \xrightarrow{(e)} H^\bullet(Q^{\times j}; c^*\mathcal{G}_j)_{h\Sigma_j}. \end{array}$$

We first summarize what the maps of the diagram are about and thereafter we give some of the details. For  $j \geq 2$ , the source of (a) is the  $j$ th layer for the manifold calculus tower of the smooth embedding space  $E^{\text{sm}}(Q, D^{n-1})$ . The target of (a) is the  $j$ th layer for the manifold calculus tower of, in the terminology of [22, Definition 2.2], the good cofunctor  $\mathcal{O} \mapsto \Omega^{n-1}\Sigma^{n-1}\mathcal{O}^+$ , where  $\mathcal{O} \subset Q$  varies over the open subsets of the interior of  $Q$  and  $\mathcal{O}^+$  denotes one-point compactification. The map (a) is induced by the Pontryagin–Thom construction. The map (b) is a kind of stabilization map. The equivalence (c) is a version of the Adams isomorphism (which is valid since the source and target in this case are infinite loop spaces) and

the equivalence (d) is excision. The map (e) is a relaxation of constraints. The target of (e) coincides with the  $j$ th layer of the tower of Theorem C for  $\mathcal{L}E(Q \times D^1, D^n)$ .

We now proceed to give more detail. In the above,  $\mathcal{E}_j$  and  $\mathcal{F}_j$  are fibrations over  $\binom{Q}{j}$ , and  $\mathcal{G}_j$  is a fibration over  $B\Sigma_j$ . The notation  $H^\bullet(B; \mathcal{U})$  refers to the space of sections of a fibration  $\mathcal{U} \rightarrow B$  (which we feel compelled to indicate as “unstable” cohomology), and similarly,  $H_{cs}^\bullet(B; \mathcal{U})$  refers to the space of sections with compact support relative to a given fixed section; these are the sections which agree with the given one outside a compact subset of  $B$ .

The fibration  $\mathcal{E}_j \rightarrow \binom{Q}{j}$  may be described as follows: if  $T \in \binom{Q}{j}$ , then we form the  $j$ -cube of spaces

$$(A2) \quad U \mapsto \mathcal{P}(D^{n-1}, U), \quad U \subset T.$$

The total homotopy fiber of (A2) is the fiber at  $T$  of the fibration  $\mathcal{E}_j \rightarrow \binom{Q}{j}$  (we will leave it to the reader to provide the topology on  $\mathcal{E}_j$  as well as on  $\mathcal{F}_j$ ). By [22, sum. 4.2] we know that  $H_{cs}^\bullet(\binom{Q}{j}; \mathcal{E}_j)$  is the  $j$ th layer of the Goodwillie–Weiss tower for the space of smooth embeddings  $E^{sm}(Q, D^{n-1})$  when  $j \geq 2$ .

The fibration  $\mathcal{F}_j \rightarrow \binom{Q}{j}$  has fiber at  $T$  given by the total homotopy fiber of the  $j$ -cube

$$(A3) \quad U \mapsto \Omega^{n-1}\Sigma^{n-1}(U_+), \quad U \subset T.$$

The map  $\mathcal{E}_j \rightarrow \mathcal{F}_j$  is induced by the Pontryagin–Thom construction  $\mathcal{P}(D^{n-1}, U) \rightarrow \Omega^{n-1}\Sigma^{n-1}(U_+)$  with respect to the trivial framing. This induces the map (a). The map (a) is  $(2n - 3k - 5)$ -connected [13].

The fibration  $\mathcal{G}_j \rightarrow B\Sigma_j$  arises as follows: take the unreduced Borel construction of  $\Sigma_j$  acting on  $\mathcal{W}_j \wedge S^{(n-1)V_j}$ . This gives a fibered spectrum

$$(A4) \quad E\Sigma_j \times_{\Sigma_j} (\mathcal{W}_j \wedge S^{(n-1)V_j}) \rightarrow B\Sigma_j.$$

Then  $\mathcal{G}_j$  is the fiberwise zeroth space of (A4), i.e.  $\mathcal{G}_j$  is the unreduced Borel construction of  $\Sigma_j$  acting on  $\Omega^\infty(\mathcal{W}_j \wedge S^{(n-1)V_j})$ . The fibration  $i^*\mathcal{G}_j \rightarrow \binom{Q}{j}$  is obtained by taking the base change of  $\mathcal{G}_j$  along  $i$ .

The source of (c) is the homotopy orbits of  $\Sigma_j$  acting on the section space with compact supports of the fibration  $(i \circ \pi)^*\mathcal{G}_j \rightarrow \mathcal{P}(Q, j)$ . Here we are using the observation that this last map is  $\Sigma_j$ -equivariant ( $\Sigma_j$ -acts on  $(i \circ \pi)^*\mathcal{G}_j$  because it is a trivial fibration over  $\mathcal{P}(Q, j)$  whose fiber

$\Omega^\infty(\mathcal{W}_j \wedge S^{(n-1)V_j})$  comes equipped with a  $\Sigma_j$ -action). As already mentioned, the map (c) is a homotopy equivalence by the Adams isomorphism [15, Section 2] and the map (d) is an equivalence by excision. The map (e) is the map which forgets that a section is fixed along the diagonal; it is  $((j-1)(n-2)-k-1)$ -connected.

The map (b) is induced by a map  $\mathcal{F}_j \rightarrow i^*\mathcal{G}_j$  which on the level of fibers at  $T$  arises from the natural map

$$\Omega^{n-1}\Sigma^{n-1}(U_+) \rightarrow \Omega^{n-1}P_j\mathbb{I}(\Sigma^{n-1}(U_+))$$

which induces map of  $j$ -cubes. As noted above, the total homotopy fiber of the source is identified with the fiber of  $\mathcal{F}_j$  at  $T$ . The total homotopy fiber of the target cube can be obtained by replacing the  $j$ th stage  $P_j\mathbb{I}$  in the above with the  $j$ -layer in the tower for  $\mathbb{I}$ . An analysis which we omit shows that total homotopy fiber of the target is identified with the fiber of  $i^*\mathcal{G}_j$  at  $T$ .

In fact, the Hilton–Milnor theorem shows that the map  $\mathcal{F}_j \rightarrow i^*\mathcal{G}_j$  is  $((j+1)(n-2)+2-n)$ -connected. By subtracting the handle dimension of  $Q^{\times j}$  (i.e.,  $jk$ ) we infer that (b) is  $j(n-k-2)$ -connected.

Finally, observe that the connectivity of each of the maps (b) and (e) is a linear function of  $j$  with positive slope. Thus the map (a) is the only map of the diagram which does not tend to weak equivalence as  $j$  gets large.

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