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Laplace Transform Type Multipliers for Hankel Transforms

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Abstract. In this paper we establish that Hankel multipliers of Laplace transform type are bounded from $L^p(w)$ into itself when $1 < p < \infty$, and from $L^1(w)$ into $L^{1,\infty}(w)$, provided that w is in the Muckenhoupt class A^p on $((0,\infty),\ dx)$.

1 Introduction

If m is a bounded measurable function on $(0,\infty)$ we define the multiplier operator for the Hankel transform associated with m by

$$T_m f = h_\mu(m h_\mu(f)). \tag{1.1}$$

Here $h_\mu(f)(x) = \int_0^\infty \sqrt{xy}\, J_\mu(xy) f(y)\, dy$ is the Hankel transform defined by [28], where J_μ denotes, as usual, the Bessel function of the first kind and order μ. Throughout this paper we will always assume that $\mu > -1/2$. Since h_μ is an isometry of $L^2(0,\infty)$ [26, Ch. VIII], T_m is a bounded operator from $L^2(0,\infty)$ into itself. Conditions on the function m can be specified in order that T_m maps boundedly L^p-type spaces. Guy [14] established the first results on multipliers for Hankel transforms. More recently, Gosselin and Stempak [13], Betancor and Rodríguez-Mesa [4] and Kapelko [16, 17] obtained Mihlin–Hörmander type multipliers theorems for Hankel transforms. Also in [4], a Hankel version of a result of Córdoba and Fefferman [7] concerning the boundedness of multipliers on weighted L^p-spaces was established. Results on multipliers for Hankel transforms on Hardy spaces were shown in [3]. Other classes of results about multipliers for Hankel transforms were proved by Gasper and Trebels [8–12].

If f is a suitable function, it is easy to see by partial integration that

$$h_\mu(\Delta_\mu f)(x) = x^2 h_\mu(f)(x), \tag{1.2}$$

where Δ_μ denotes the Bessel operator $\Delta_\mu = -x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2}$. This operator Δ_μ is positive and self-adjoint in $L^2(0,\infty)$. According to (1.1) and (1.2) we can formally write that $T_m f = m(\Delta_\mu^{1/2}) f$.

In this paper we investigate the L^p-boundedness of the multiplier T_m when m is of Laplace transform type, *i.e.*, $m(x) = x\int_0^\infty e^{-xt} k(t)\, dt$, $x \in (0,\infty)$, where k is a

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bounded measurable function in $(0,\infty)$. Following Stein [22], we say that T_m is a Hankel multiplier of Laplace transform type.

We obtain a representation for T_m involving Poisson kernel associated with the operator Δ_μ. For every $x \in (0,\infty)$, the function $\varphi_x(y) = \sqrt{xy}J_\mu(xy)$, $y \in (0,\infty)$, is an eigenfunction of Δ_μ, since $\Delta_\mu\varphi_x(y) = y^2\varphi_x(y)$, $y \in (0,\infty)$. Hence the Poisson kernel in the Δ_μ-setting (see [27] and [19, (16.4)]) is defined by

(1.3)
$$P_\mu(t,x,y) = \int_0^\infty e^{-tz}\sqrt{xz}J_\mu(xz)\sqrt{yz}J_\mu(yz)\,dz$$
$$= \frac{2\mu+1}{\pi}\int_0^\pi \frac{t(xy)^{\mu+1/2}\sin^{2\mu}\theta}{[(x-y)^2+t^2+2xy(1-\cos\theta)]^{\mu+3/2}}\,d\theta, \quad t,x,y \in (0,\infty).$$

The corresponding Poisson integrals were studied by Philipp [21]. We get the following representation for the Hankel multiplier T_m as an integral operator including P_μ in its kernel.

Theorem 1.1 *Let T_m be a Hankel multiplier of Laplace transform type. For every $f \in L^2(0,\infty)$ and $x \notin \operatorname{supp} f$,*

$$T_m f(x) = \int_0^\infty K_m(x,y)f(y)\,dy, \quad K_m(x,y) = \int_0^\infty k(t)\Big(-\frac{d}{dt}\Big)P_\mu(t,x,y)\,dt.$$

In [2] some operators of harmonic analysis associated to Δ_μ were investigated using Calderón-Zygmund theory. These were inspired by the studies of Muckenhoupt and Stein about Riesz transforms associated with the Bessel type operator $S_\mu = -x^{-2\mu-1}Dx^{2\mu+1}D$. In Section 3 we establish that the kernel K_m of the Hankel multiplier T_m is a Calderón–Zygmund kernel in the region $\Omega = \{(x,y) : 0 < x/b < y < bx\}$ for all $b > 1$. Moreover $|K_m(x,y)| \le C/x$ when $0 < y < x/2$, and $|K_m(x,y)| \le C/y$ when $2x < y$. Calderón–Zygmund theory of singular integrals and the boundedness properties of Hardy operators give the strong type (p,p), $1 < p < \infty$, and the weak type $(1,1)$, as stated in the following theorem, which is the main result of this paper. Here $A_p(0,\infty)$, $1 \le p < \infty$, represents the Muckenhoupt class of weights on $((0,\infty),\ dx)$.

Theorem 1.2 *Let $1 \le p < \infty$ and $w \in A_p(0,\infty)$. Suppose that m is a function of Laplace transform type. Then the Hankel multiplier operator T_m is bounded from $L^p(w)$ into itself, $1 < p < \infty$, and from $L^1(w)$ into $L^{1,\infty}(w)$.*

Similar results for $1 < p < \infty$ were proved in quite general settings [22]. To use [22, p. 58], we need the compactness of the space and that the derivative operator commutes with Δ_μ. It is required that the Poisson semigroup be markovian, that is, that it map constants into constants [22, p. 121]. None of these conditions are satisfied in our case. Indeed, in the Δ_μ-setting the Poisson semigroup P_μ is defined [21] by

$$P_\mu(f)(t,x) = \int_0^\infty P_\mu(t,x,y)f(y)\,dy, \quad t,x \in (0,\infty).$$

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According to [5, Remark 2.5], if $f \in L^\infty(0,\infty)$, then the function $u = P_\mu(f)$ is a solution of the Laplace type equation $\partial^2/\partial t^2 u - \Delta_\mu u = 0$. In particular this happens when $f = \chi_{(0,\infty)}$, the indicator function of $(0,\infty)$. However, it is clear that constant functions do not verify the last partial differential equation. Hence the Poisson semigroup P_μ is not markovian.

On the other hand, as was mentioned by Nowak and Stempak [20], by using transplantation theorems for Hankel transforms [14] we are able to derive L^p-boundedness results for Hankel multipliers by applying known results for Fourier multipliers (adapted to the cosine Fourier transform, for instance). However, using this transplantation procedure, the weak type results established in Theorem 1.2 cannot be obtained. Moreover, weighted L^p-boundedness for Hankel multipliers cannot be established by transference from boundedness results for multipliers with respect to other orthogonal systems (Jacobi [15], Laguerre [24] or Bessel [6] functions) for general A_p-weights.

A remarkable particular case of Hankel multipliers of Laplace transform type are the imaginary powers $\Delta_\mu^{i\lambda}$, $\lambda \in \mathbf{R}$, defined by $\Delta_\mu^{i\lambda} f = h_\mu(y^{2i\lambda} h_\mu(f))$. Note that $y^{2i\lambda}\Gamma(1-2i\lambda) = y\int_0^\infty e^{-yt} t^{-2i\lambda} dt$, $y \in (0,\infty)$. A straightforward corollary from Theorem 1.2 is the following.

Corollary 1.1 *Let $1 \le p < \infty$, $\lambda \in \mathbf{R}$ and $w \in A_p(0,\infty)$. Then the imaginary power $\Delta_\mu^{i\lambda}$ of Δ_μ is bounded from $L^p(w)$ into itself when $1 < p < \infty$, and from $L^1(w)$ into $L^{1,\infty}(w)$ when $p = 1$.*

The situation is different when we consider the negative powers $\Delta_\mu^{-\alpha/2}$, $0 < \alpha < 1$, defined through

$$\Delta_\mu^{-\alpha/2} f = h_\mu(y^{-\alpha} h_\mu(f)). \tag{1.4}$$

This is not a Laplace transform type multiplier for Hankel transforms, since $y^{-\alpha} = y\int_0^\infty e^{-xy} x^\alpha\, dx$, $y \in (0,\infty)$ and x^α is not bounded on $(0,\infty)$. The right-hand side of (1.4) makes sense when, for instance, f belongs to the image by Hankel transform of the space $C_c^\infty(0,\infty)$ of the infinitely differentiable functions on $(0,\infty)$ with compact support (which is a dense subspace of $L^p(0,\infty)$, $1 \le p < \infty$, see [25, Corollary 4.8]). According to [23, Lemma 2, p. 23], we get $h_\mu(P_\mu(t,x,\cdot))(z) = e^{-tz}\sqrt{xz}J_\mu(xz)$ for $f \in h_\mu(C_c^\infty(0,\infty))$. Plancherel equality for Hankel transforms [28, Theorem 5.1-2] allows us to write

$$\Delta_\mu^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \int_0^\infty P_\mu(t,x,y) f(y)\, dy dt, \quad x \in (0,\infty).$$

Straightforward manipulations using homogeneity lead us to see that if $\Delta_\mu^{-\alpha/2}$ can be extended as a bounded operator from $L^p(0,\infty)$ into $L^{q,\infty}(0,\infty)$, then $p < q$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Hence it is clear that the multiplier operator $\Delta_\mu^{-\alpha/2}$ is bounded neither from $L^p(0,\infty)$ into itself nor from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$, in contrast with the Laplace transform type multipliers for Hankel transforms. According to [19, p. 86], $0 \le P_\mu(t,x,y) \le C\frac{t}{(x-y)^2+t^2}$ for $t, x, y \in (0,\infty)$, and we can obtain the

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L^p-boundedness properties for the negative powers $\Delta_\mu^{-\alpha/2}$ from the corresponding ones for the negative powers of the Laplacian operator in one dimension.

Theorem 1.3 *Let $0 < \alpha < 1$ and $1 \leq p < q < \infty$. Then the operator $\Delta_\mu^{-\alpha/2}$ is a bounded operator from $L^p(0,\infty)$ into $L^q(0,\infty)$, provided that $1 < p < \infty$ and $\frac{1}{p} - \frac{1}{q} = \alpha$, and from $L^1(0,\infty)$ into $L^{1/(1-\alpha),\infty}(0,\infty)$.*

Throughout this paper, C represents a suitable positive constant that can change from one line to another.

2 Proof of Theorem 1.1

The following lemma will be useful in the remainder of the paper.

Lemma 2.1 *For every $t, x, y \in (0,\infty)$, we have $|\frac{d}{dt}P_\mu(t,x,y)| \leq \frac{C}{(x-y)^2+t^2}$.*

Proof Let $t, x, y \in (0,\infty)$. By considering the formula (1.3) for the Poisson kernel P_μ, we can write

$$\begin{aligned}\Big|\frac{d}{dt}P_\mu(t,x,y)\Big| &\leq C(xy)^{\mu+1/2}\Big(\int_0^\pi \frac{\sin^{2\mu}\theta\, d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2}}\\ &\qquad + \int_0^\pi \frac{\sin^{2\mu}\theta t^2\, d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+5/2}}\Big)\\ &\leq C(xy)^{\mu+1/2}\Big(\int_0^{\pi/2} + \int_{\pi/2}^\pi\Big)\frac{\sin^{2\mu}\theta\, d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2}}\\ &= C(xy)^{\mu+1/2}(I_1(t,x,y)+I_2(t,x,y)).\end{aligned}$$

Taking into account that $\sin\theta \sim \theta$ and $\theta^2/2 \sim 1-\cos\theta$ when $\theta \in [0,\pi/2]$, and then applying the change of variables $z^2 = \frac{xy}{(x-y)^2+t^2}\theta^2$, we get

$$\begin{aligned}I_1(t,x,y) &\leq C\int_0^{\pi/2}\frac{\theta^{2\mu}\, d\theta}{((x-y)^2+t^2+xy\theta^2)^{\mu+3/2}}\\ &\leq \frac{C(xy)^{-\mu-1/2}}{(x-y)^2+t^2}\int_0^{\frac{\pi}{2}\frac{\sqrt{xy}}{\sqrt{(x-y)^2+t^2}}}\frac{z^{2\mu}dz}{(1+z^2)^{\mu+3/2}} \leq \frac{C(xy)^{-\mu-1/2}}{(x-y)^2+t^2}.\end{aligned}$$

On the other hand, by considering the change of variable $\sigma = \pi - \theta$ and using again that $\sin\sigma \sim \sigma$ and $\sigma^2/2 \sim 1-\cos\sigma$, $\sigma \in [0,\pi/2]$, we obtain

$$\begin{aligned}I_2(t,x,y) &= \int_0^{\pi/2}\frac{\sin^{2\mu}\sigma\, d\sigma}{((x-y)^2+t^2+2xy(1+\cos\sigma))^{\mu+3/2}}\\ &\leq C\int_0^{\pi/2}\frac{\sigma^{2\mu}\, d\sigma}{((x-y)^2+t^2+xy\sigma^2)^{\mu+3/2}} \leq C\frac{(xy)^{-\mu-1/2}}{(x-y)^2+t^2}. \qquad \blacksquare\end{aligned}$$

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Proof of Theorem 1.1 Let $f \in L^2(0,\infty)$. Our objective is to establish that

$$\langle T_m f, g\rangle = \Big\langle \int_0^\infty K_m(x,y) f(y)\,dy\,,\, g(x)\Big\rangle,$$

for every $g \in L^2(0,\infty)$ with compact support outside of the support of function f. Using Plancherel equality for Hankel transforms [28, Theorem 5.1-2] we get

$$\begin{aligned}\langle T_m f, g\rangle &= \int_0^\infty h_\mu(m h_\mu f)(x) g(x)\,dx = \int_0^\infty m(u) h_\mu(f)(u) h_\mu(g)(u)\,du \\ &= \int_0^\infty u \int_0^\infty e^{-tu} k(t)\,dt h_\mu(f)(u) h_\mu(g)(u)\,du \\ &= \int_0^\infty k(t) \int_0^\infty \Big(-\frac{d}{dt}\Big) e^{-tu} h_\mu(f)(u) h_\mu g(u)\,du\,dt,\end{aligned}$$

where, to justify the interchange of integrals in the last equality, we have used Hölder's inequality, the fact that h_μ is an isometry in $L^2(0,\infty)$, and the boundedness of the function k, in order to see that

$$\int_0^\infty |h_\mu(f)(u) h_\mu(g)(u)| \int_0^\infty u e^{-tu} |k(t)|\,dt du \le C\|h_\mu f\|_2 \|h_\mu g\|_2 \le C\|f\|_2\|g\|_2.$$

Since for every $t > 0$ there exists $C > 0$ such that $\int_0^\infty |u e^{-tu} h_\mu(f)(u) h_\mu(g)(u)|\,du \le C\|f\|_2\|g\|_2$, we can write

$$\langle T_m f, g\rangle = \int_0^\infty k(t)\Big(-\frac{d}{dt}\Big) \int_0^\infty e^{-tu} h_\mu(f)(u) h_\mu(g)(u)\,dudt.$$

On the other hand, we note that the functions $h_1(u) = e^{-tu} h_\mu(f)(u)$ and $h_2(u) = \sqrt{xu} J_\mu(xu) e^{-tu}$, $u \in (0,\infty)$, belong to $L^2(0,\infty)$ for every $t, x \in (0,\infty)$. Then the Plancherel formula for h_μ and $h_\mu(P_\mu(t,x,\cdot))(z) = e^{-tz}\sqrt{xz} J_\mu(xz)$ lead to

$$\begin{aligned}\langle T_m f, g\rangle &= \int_0^\infty k(t)\Big(-\frac{d}{dt}\Big) \int_0^\infty h_\mu(e^{-tu} h_\mu f)(x) g(x)\,dxdt \\ &= \int_0^\infty k(t)\Big(-\frac{d}{dt}\Big) \int_0^\infty \int_0^\infty h_\mu(\sqrt{xu} J_\mu(xu) e^{-tu})(y) f(y)\,dy g(x) dxdt \\ &= \int_0^\infty k(t)\Big(-\frac{d}{dt}\Big) \int_0^\infty g(x) \int_0^\infty P_\mu(t,x,y) f(y)\,dydxdt.\end{aligned}$$

From here, we can write

$$\begin{aligned}\langle T_m f, g\rangle &= \int_0^\infty k(t) \int_0^\infty g(x)\Big(-\frac{d}{dt}\Big) \int_0^\infty P_\mu(t,x,y) f(y)\,dydxdt \\ &= \int_0^\infty k(t) \int_0^\infty g(x) \int_0^\infty \Big(-\frac{d}{dt}\Big) P_\mu(t,x,y) f(y)\,dydxdt \\ &= \int_0^\infty g(x) \int_0^\infty f(y) K_m(x,y)\,dydx.\end{aligned}$$

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Thus, to finish the proof we have to justify the interchange of the integrals and the differentiation under the integral sign in the above equalities. Since $x \in \operatorname{supp} g$, $y \in \operatorname{supp} f$, $\int_0^\infty |f(y)\frac{d}{dt}P_\mu(t,x,y)|\,dy < \infty$ by Hölder's inequality and Lemma 2.1. Since $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$, Hölder's inequality, and Lemma 2.1 give us

$$\int_0^\infty \int_0^\infty |g(x)f(y)\frac{d}{dt}P_\mu(t,x,y)|\,dxdy \leq \frac{C}{\varepsilon^2}\|g\|_2\|f\|_2,$$

where $\varepsilon \leq |x-y|$, $x \in \operatorname{supp} g$ and $y \in \operatorname{supp} f$. Finally, by proceeding as above we get that

$$\int_0^\infty \int_0^\infty \int_0^\infty |g(x)k(t)f(y)\frac{d}{dt}P_\mu(t,x,y)|\,dydtdx$$
$$\leq C\int_0^\infty \int_0^\infty |g(x)| \int_0^\infty |f(y)| \int_0^\infty \frac{dt}{(x-y)^2+t^2}\,dydx \leq \frac{C}{\varepsilon}\|g\|_2\|f\|_2. \quad \blacksquare$$

3 Proof of Theorem 1.2.

The following result shows that the kernel K_m is locally a Calderón–Zygmund kernel.

Proposition 3.1 *Let $b > 1$. There exists $C > 0$ such that for every $x, y \in (0,\infty)$, $x \neq y$*

(i) $|K_m(x,y)| \leq \frac{C}{|x-y|}$.

(ii) $|\partial_x K_m(x,y)| + |\partial_y K_m(x,y)| \leq \frac{C}{|x-y|^2}$, *provided that* $\frac{1}{b} \leq y \leq bx$.

Proof Since k is a bounded function on $(0,\infty)$, by using Lemma 2.1 we get

$$|K_m(x,y)| = \Big|\int_0^\infty k(t)\Big(-\frac{d}{dt}\Big)P_\mu(t,x,y)\,dt\Big|$$
$$\leq C\int_0^\infty \frac{1}{(x-y)^2+t^2}\,dt \leq \frac{C}{|x-y|}.$$

Let $b > 1$. To analyze the estimate in (ii) we consider the expression (1.3) for Poisson kernel P_μ and write K_m in the following form:

$$K_m(x,y) = -\frac{2\mu+1}{\pi}(xy)^{\mu+1/2}\int_0^\infty k(t)$$
$$\Big(\int_0^\pi \frac{\sin^{2\mu}\theta d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+3/2}}$$
$$-(2\mu+3)t^2\int_0^\pi \frac{\sin^{2\mu}\theta d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+5/2}}\Big)\,dt.$$

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Note that by symmetry it is sufficient to study the term $\partial_x K_m$. Assume that $x, y \in (0, \infty)$, $x \neq y$, and $\frac{1}{b}x \leq y \leq bx$. Differentiating under the integral sign, we obtain

$$\begin{aligned}\partial_x K_m(x,y) &= (\mu+1/2)\frac{K_m(x,y)}{x} + \frac{(2\mu+1)(2\mu+3)}{\pi}(xy)^{\mu+1/2}\times \\ &\quad\times \int_0^\infty k(t)\Big[\int_0^\pi \frac{(x-y\cos\theta)\sin^{2\mu}\theta\,d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+5/2}} \\ &\qquad - \int_0^\pi \frac{(2\mu+5)t^2(x-y\cos\theta)\sin^{2\mu}\theta d\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+7/2}}\Big]\,dt \\ &= I_1(x,y)+I_2(x,y)+I_3(x,y).\end{aligned}$$

When $\frac{1}{b}x \leq y \leq bx$, we have $|x-y| \leq Cx$. Then the estimate in (i) gives that $|I_1(x,y)| \leq \frac{C}{x|x-y|} \leq \frac{C}{|x-y|^2}$. To analyze $I_j(x,y)$, $j = 2,3$, we observe first that $|I_j(x,y)| \leq CI(x,y)$, $j = 2,3$, where

$$I(x,y) = (xy)^{\mu+1/2}\int_0^\pi |x-y\cos\theta|\sin^{2\mu}\theta \times \int_0^\infty \frac{dtd\theta}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+5/2}}.$$

If $\theta \in [0,\pi]$, by making the change of variables $t^2 = ((x-y)^2+2xy(1-\cos\theta))u^2$, it is easy to see that

$$\int_0^\infty \frac{1}{((x-y)^2+t^2+2xy(1-\cos\theta))^{\mu+5/2}}\,dt = \frac{C_\mu}{((x-y)^2+2xy(1-\cos\theta))^{\mu+2}},$$

being $C_\mu = \int_0^\infty \frac{1}{(1+u^2)^{\mu+5/2}}\,du$. Then since $|x-y| \leq Cy$,

$$|I(x,y)| \leq C(xy)^{\mu+1/2}\int_0^\pi \frac{\sin^{2\mu}\theta|x-y\cos\theta|\,d\theta}{((x-y)^2+2xy(1-\cos\theta))^{\mu+2}} \leq C(J_1(x,y)+J_2(x,y)),$$

where by the change of variables $\sigma = \pi-\theta$, $\theta \in [\pi/2,\pi]$. Then by using $\sin z \sim z$, $1-\cos z \sim z^2/2$, $z \in [0,\pi/2]$, we have

$$\begin{aligned}J_1(x,y) &= (xy)^{\mu+1/2}|x-y|\int_0^\pi \frac{\sin^{2\mu}\theta}{((x-y)^2+2xy(1-\cos\theta))^{\mu+2}}\,d\theta \\ &\leq C(xy)^{\mu+1/2}|x-y| \\ &\quad\times\int_0^{\pi/2}\sin^{2\mu}\theta\Big(\frac{1}{((x-y)^2+2xy(1-\cos\theta))^{\mu+2}} \\ &\qquad + \frac{1}{((x-y)^2+2xy(1+\cos\theta))^{\mu+2}}\Big)\,d\theta \\ &\leq C(xy)^{\mu+1/2}|x-y|\int_0^{\pi/2}\frac{\theta^{2\mu}}{((x-y)^2+xy\theta^2)^{\mu+2}}\,d\theta.\end{aligned}$$

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The change of variables $\sigma^2 = \frac{xy}{|x-y|^2}\theta^2$ leads to $J_1(x,y) \le C|x-y|^{-2}$. Also, by proceeding in a similar way and taking into account that $|x-y| \le Cx$, we get

$$J_2(x,y) = (xy)^{\mu+1/2}\frac{1}{x}\int_0^\pi \frac{\sin^{2\mu}\theta}{((x-y)^2+2xy(1-\cos\theta))^{\mu+1}}\,d\theta \le \frac{C}{|x-y|^2}. \qquad \blacksquare$$

Proof of Theorem 1.2 Let φ be a smooth function on $(0,\infty)\times(0,\infty)$ with support in the region $\Omega = \{(x,y) \in (0,\infty)\times(0,\infty) : \frac{x}{3} \le y \le 3x\}$, such that $0 \le \varphi \le 1$, $\varphi(x,y) = 1$, when $x, y \in (0,\infty)$, $\frac{x}{2} < y < 2x$, and satisfying $|\partial_x\varphi(x,y)| + |\partial_y\varphi(x,y)| \le \frac{C}{|x-y|}$, $x \ne y$.

We define operators

$$T_m^{\rm glob} f(x) = \int_0^\infty K_m(x,y)(1-\varphi(x,y))f(y)\,dy, \qquad T_m^{\rm loc} f(x) = T_m f(x) - T_m^{\rm glob} f(x).$$

Let us analyze first the operator $T_m^{\rm glob}$. We can write, for $x \in (0,\infty)$,

$$T_m^{\rm glob} f(x) = \Big(\int_0^{x/2} + \int_{2x}^\infty\Big) K_m(x,y)(1-\varphi(x,y))f(y)\,dy = T_{m,1}^{\rm glob} f(x) + T_{m,2}^{\rm glob} f(x).$$

We observe that $|x-y| \sim x$ when $y \le x/2$, and $|x-y| \sim y$ when $y \ge 2x$. These estimates and Proposition 3.1(i) allow us to write

$$|T_{m,1}^{\rm glob} f(x)| \le CH_1(|f|)(x), \qquad |T_{m,2}^{\rm glob} f(x)| \le CH_2(|f|)(x),$$

H_1 and H_2 being the classical Hardy operators defined by

$$H_1(f)(x) = \frac{1}{x}\int_0^x f(x)\,dx \quad \text{and} \quad H_2(f)(x) = \int_x^\infty \frac{f(y)}{y}\,dy, \quad x \in (0,\infty).$$

It is well known that Hardy operators are bounded from $L^p(w)$ into itself for $1 < p < \infty$ and from $L^1(w)$ into $L^{1,\infty}(w)$, when w belongs to the Muckenhoupt class of weights $A_p(0,\infty)$ on $((0,\infty), dx)$ (see [1, 18]). Then the global part $T_m^{\rm glob}$ verifies the assertion of theorem.

Let us now study the operator $T_m^{\rm loc}$. We observe first that since h_μ is an isometry of $L^2(0,\infty)$, the operator T_m is a bounded operator from $L^2(0,\infty)$ into itself. Moreover, we have just seen that $T_m^{\rm glob}$ is a bounded operator from $L^2(0,\infty)$ into itself. Then also $T_m^{\rm loc}$ is a bounded operator from $L^2(0,\infty)$ into itself.

On the other hand, $T_m^{\rm loc}$ is a Calderón–Zygmund operator with kernel $K_m(x,y)\varphi(x,y)$. In fact, by taking into account Proposition 3.1 and the imposed conditions on function φ, we have for every $x, y \in (0,\infty)$, $x \ne y$,

$$|K_m(x,y)\varphi(x,y)| \le \frac{C}{|x-y|},$$

$$|\partial_x(K_m(x,y)\varphi(x,y))| + |\partial_y(K_m(x,y)\varphi(x,y))| \le \frac{C}{|x-y|^2}.$$

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Classical Calderón–Zygmund theory gives that T_m^{loc} is bounded from $L^p(w)$ into itself for $1 < p < \infty$, and from $L^1(w)$ into $L^{1,\infty}(w)$, provided that w belongs to the Muckenhoupt class of weights $A_p(0,\infty)$ on $((0,\infty), dx)$. This completes the proof of the theorem. ■

To finish, we would like to comment that after having proved Proposition 3.1, the boundedness of the local part T_m^{loc} also can be seen by using [20, Proposition 4.2].

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