BULL. AUSTRAL. MATH. SOC. 58F10, 54H20
VOL. 45 (1992) [79–90]

STRUCTURALLY STABLE FLOWS

DONG PYO CHI, SUNG KYU CHOI AND JONG SUH PARK

We show that a C^1-flow on a compact Riemannian manifold is structurally stable and topologically stable if and only if it satisfies Axiom A and the strong transversality condition. This improves Smale's conjecture for flows.

1. INTRODUCTION

The concept of structural stability was first introduced by Andronov and Pontryagin in 1937. Roughly speaking, a given dynamical system is said to be structurally stable if a C^1-small change in the system does not change topologically the set of orbits in the system. That is, it is the type of stability referred to the permanence of orbits upon small perturbations of the parameters of the given system.

The well-known Smale's conjecture on structural stability [8] is the following:

> A dynamical system on a compact Riemannian manifold is structurally stable if and only if it satisfies Axiom A and the strong transversality condition.

Robbin [9] and Robinson [11] proved the sufficient condition of the above conjecture. For the necessary condition, Manẽ [5] obtained a partial answer for discrete dynamical systems by using his conjecture. However, by using Manẽ's result, Hurley [4] obtained a negative answer to Smale's conjecture for discrete dynamical systems.

In this paper, we prove Hurley's result for flows by using Manẽ's conjecture. Thus our main theorem is the following:

> A C^1-flow on a compact Riemannian manifold is structurally stable and topologically stable if and only if it satisfies Axiom A and the strong transversality condition.

Received 31 January 1991
The present studies were supported by the Basic Research Institute Program, Ministry of Education, Korea, 1990, Project No. BSR-90-110.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/92 $A2.00+0.00.

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

2. Basic terminologies

Let M be a compact Riemannian manifold and $X(M)$ be the set of all C^1-flows on M. For any $f, g \in X(M)$, we define

$$d(f, g) = \sup_{x\in M}\left\|\frac{\partial f}{\partial t}(x, 0) - \frac{\partial g}{\partial t}(x, 0)\right\|.$$

Let $f \in X(M)$. For a point $x \in M$, is said to be *nonwandering* if for every neighbourhood U of x and every $t \in \mathbb{R}$, there exist a point $y \in U$ and a number $s \geqslant t$ such that $f(y, s) \in U$. We denote by $\Omega(f)$ the set of all nonwandering points.

For points $x, y \in M$ and numbers $\varepsilon,\ a > 0$, a finite sequence $\{(x_i, t_i)\}_{i=1}^{n}$ in $M \times \mathbb{R}$ is called an (ε, a)-*chain* from x to y if $x_1 = x$, $t_i \geqslant a$ for $i = 1, \ldots, n$, $d(f(x_i, t_i), x_{i+1}) < \varepsilon$ for $i = 1, \ldots, n-1$ and $d(f(x_n, t_n), y) < \varepsilon$.

A point x is *related* to a point y (written $x \sim y$) if for any $\varepsilon > 0$ and any $a > 0$ there are (ε, a)-chains from x to y and from y to x, respectively. In particular, a point x is *chain recurrent* if $x \sim x$, and $CR(f)$ is denoted by the set of all chain recurrent points of f, that is, $CR(f) = \{x \in M : x \sim x\}$. Clearly, the relation "$\sim$" is an equivalence relation on the set $CR(f)$. The equivalence classes are called *chain components* of f.

A C^1-flow g in $X(M)$ is *semiconjugate* to $f \in X(M)$ if there exist a continuous surjection $h\colon M \to M$ and a continuous map $\lambda\colon M \times \mathbb{R} \to \mathbb{R}$ satisfying

(i) for all $x \in M$, $\lambda(x, 0) = 0$ and $\lambda(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism,

(ii) for all $x \in M$ and all $t \in \mathbb{R}$, $h(g(x, t)) = f(h(x), \lambda(x, t))$. We call (h, λ) a *semiconjugacy* from g to f.

A C^1-flow f is called *topologically stable* if for any $\varepsilon > 0$, there exists a neighbourhood U of f in $X(M)$ such that for every $g \in U$, there is a semiconjugacy (h, λ) from g to f with $d(h, 1) < \varepsilon$, where $d(h, 1) = \sup\limits_{x\in M} d(h(x), x)$.

A C^1-flow g is *conjugate* to f in $X(M)$ if there exist a homeomorphism $h\colon M \to M$ and a continuous map $\lambda\colon M \times \mathbb{R} \to \mathbb{R}$ satisfying the above conditions (i) and (ii). In such case, (h, λ) is called a *conjugacy* from g to f.

Finally, a C^1-flow f is called *structurally stable* if there exists a neighbourhood U of f in $X(M)$ such that for all $g \in U$, g is conjugate to f.

We let $B(x, \varepsilon)$ denote $\{y \in M : d(x, y) < \varepsilon\}$.

3. Chain components of topologically stable flows

First, we can obtain the following properties about (ε, a)-chains.

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

Lemma 3.1. *For any number $a > 0$, there exists a map $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following:*

(i) *for any $\varepsilon > 0$, $d(x, y) < h(\varepsilon)$ for $a \leqslant t \leqslant 2a$ implies $d(f(x, t), f(y, t)) < \varepsilon/4$,*

(ii) *$h^i < h^{i-1}/2$ for all $i \in \mathbb{Z}^+$,*

(iii) *if $\{(x_i, t_i)\}_{i=1}^n$ is an $(h^n(\varepsilon), a)$-chain and $t_i \leqslant 2a$ for all i, then*

$$d\left(f\left(x_1, \sum_{i=1}^n t_i\right), f(x_n, t_n)\right) < \varepsilon/2.$$

Proof: For any $\varepsilon > 0$, we can choose $\delta > 0$ with $\delta < \varepsilon/2$ such that $d(f(x, t), f(y, t)) < \varepsilon/4$ if $d(x, y) < \delta$ and $a \leqslant t \leqslant 2a$. Define $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ by $h(\varepsilon) = \delta$. Then, clearly, (i) and (ii) hold.

We prove (iii) by induction on n. Obviously, (iii) holds when $n = 1, 2$. Suppose that (iii) is true for $n > 2$. Let $\{(x_i, t_i)\}_{i=1}^{n+1}$ be an $(h^{n+1}(\varepsilon), a)$-chain with $t_i \leqslant 2a$ for all i. Since $\{(x_i, t_i)\}_{i=1}^n$ is an $(h^n h(\varepsilon), a)$-chain, we have

$$d\left(f\left(x_1, \sum_{i=1}^n t_i\right), f(x_n, t_n)\right) < h(\varepsilon)/2 < h(\varepsilon).$$

In view of (i),

$$d\left(f\left(x_1, \sum_{i=1}^{n+1} t_i\right), f(x_n, t_n + t_{n+1})\right) < \varepsilon/4.$$

It follows that

$$d(f(x_n, t_n + t_{n+1}), f(x_{n+1}, t_{n+1})) < \varepsilon/4$$

because

$$d(f(x_n, t_n), x_{n+1}) < h^{n+1}(\varepsilon) < h(\varepsilon).$$

Therefore we have

$$d\left(f\left(x_1, \sum_{i=1}^{n+1} t_i\right), f(x_{n+1}, t_{n+1})\right) < \varepsilon/2.$$

$\square$

Now, we prove that the relation "$\sim$" is equivalent to the property of (ε, a)-chains.

Proposition 3.2. *Let $a > 0$ and $x, y \in M$. Then $x \sim y$ if and only if for any $\varepsilon > 0$ there exist (ε, a)-chains from x to y and from y to x.*

Proof: $(\Rightarrow)$ It is straightforward.

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

($\Leftarrow$) Let $b > 0$. We may assume that $a < b$. Then $b \leqslant pa$ for some $p \in \mathbb{Z}^+$. Then for any $\varepsilon > 0$, there are $(h^{2p}(\varepsilon), a)$-chains $\{(x_i, t_i)\}_{i=1}^n$ from x to y and $\{(y_j, s_j)\}_{j=1}^m$ from y to x. Also, we may assume that $t_i, s_j \leqslant 2a$ for all i, j.

For $1 \leqslant k \leqslant (p+1)n + pm$, we define $z_k \in M$ and $a \leqslant r_k \leqslant 2a$ by

$$z_k = \begin{cases} x_{k-i(n+m)} & \text{if } i(n+m) < k \leqslant (i+1)n + im,\ i = 0, 1, \dots, p, \\ y_{k-(j+1)n-jm} & \text{if } (j+1)n + jm < k \leqslant (j+1)(n+m),\ j = 0, \dots, p-1, \end{cases}$$

and

$$r_k = \begin{cases} t_{k-i(n+m)} & \text{if } i(n+m) < k \leqslant (i+1)n + im,\ i = 0, \dots, p, \\ s_{k-(j+1)n-jm} & \text{if } (j+i)n + jm < k \leqslant (j+1)(n+m),\ j = 0, \dots, p-1. \end{cases}$$

Then $\{(z_k, r_k)\}_{k=1}^{(p+1)n+pm}$ is an $(h^{2p}(\varepsilon), a)$-chain from x to y. Let $(p+1)n + pm = sp + q$ for $0 \leqslant q < p$. Let

$$\begin{aligned} w_i &= z_{(i-1)p+1}, & i &= 1, \dots, s, \\ u_i &= \sum_{j=1}^{p} r_{(i-1)p+j}, & i &= 1, \dots, s-1, \end{aligned}$$

and

$$u_s = \sum_{j=1}^{p+q} r_{(s-1)p+j}.$$

For $1 \leqslant i < s$, since $\{z_{(i-1)p+j}, r_{(i-1)p+j}\}_{j=1}^p$ is an $(h^p(\varepsilon), a)$-chain, we have

$$\begin{aligned} d(f(w_i, u_i), w_{i+1}) &\leqslant d\left(f\left(z_{(i-1)p+1}, \sum_{j=1}^{p} r_{(i-1)p+j}, f(z_{ip}, r_{ip}) \right)\right) \\ &\quad + d(f(z_{ip}, r_{ip}), z_{ip+1}) \\ &< \varepsilon. \end{aligned}$$

Furthermore

$$\begin{aligned} d(f(w_s, u_s), y) &\leqslant d\left(f\left(z_{(s-1)p+1}, \sum_{j=1}^{p+q} r_{(s-1)p+j} \right)\right), f(z_{sp+q}, r_{sp+q}) \\ &\quad + d(f(z_{sp+q}, r_{sp+q}), y) \\ &< \varepsilon. \end{aligned}$$

since $\{(z_{(s-1)p+j}, r_{(s-1)p+j})\}_{j=1}^{p+q}$ is an $(h^{p+q}(\varepsilon), a)$-chain. Noting that $u_i \geqslant pa \geqslant b$ for $i = 1, \dots, s-1$ and $u_s \geqslant (p+q)a \geqslant b$, $\{(w_i, u_i)\}_{i=1}^s$ is an (ε, b)-chain from x to y.

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

An (ε, b)-chain from y to x can be constructed by a similar method. Consequently, we have $x \sim y$. $\square$

We can obtain a useful characterisation of chain components of f.

Proposition 3.3. *A subset X of $CR(f)$ is a chain component of f if and only if it is a connected component of $CR(f)$.*

Proof: ($\Rightarrow$) Let U and V be disjoint open subsets with the property that $U \cup V \subset X$. For a point $x \in X$, let $U(x) = \{t \in \mathbb{R} : f(x, t) \in U\}$ and $V(x) = \{t \in \mathbb{R} : f(x, t) \in V\}$. Then $U(x)$ and $V(x)$ are disjoint open and $U(x) \cup V(x) = \mathbb{R}$. Moreover, $U(x) = \emptyset$ or $V(x) = \emptyset$ since $\mathbb{R}$ is connected. Thus $U \cap X$ and $V \cap X$ are invariant, both open and closed sets in X. Since $U \cap X$ is compact, $B(U \cap X, \delta) \cap (V \cap X) = \emptyset$ for some $\delta > 0$. Assume that $U \cap X \neq \emptyset$ and $V \cap X \neq \emptyset$ and let $x \in U \cap X$ and $y \in V \cap X$. Then there is a $(\delta, 1)$-chain $\{(x_i, t_i)\}_{i=1}^{n}$ from x to y. From $f(x_1, t_1) \in U \cap X$ and $d(f(x_1, t_1), x_2) < \delta$, it follows that $x_2 \in U \cap X$. Also, $x_3 \in U \cap X$ follows from $f(x_2, t_2) \in U \cap X$ and $d(f(x_2, t_2), x_3) < \delta$. Continuing this process, we obtain $y \in U \cup X$ from $f(x_n, t_n) \in U \cap X$ and $d(f(x_n, t_n), y) < \delta$. This implies that $(U \cap X) \cap (V \cap X) \neq \emptyset$, which is a contradiction. Hence X is connected.

($\Leftarrow$) Let X be a connected component of $CR(f)$ and $x, y \in X$. For any $\varepsilon > 0$, there exists a δ with $0 < \delta < \varepsilon/2$ and $d(f(a, t), f(b, t)) < \varepsilon/2$ when $d(a, b) < \delta$ and $1 \leqslant t \leqslant 2$. It is clear that $\{B(x, \delta/2) : x \in X\}$ forms an open cover of X. Thus there are finitely many points $x_i \in X$, $i \in I$ for some index set I, such that $X \subset \bigcup_{i \in I} B(x_i, \delta/2)$. Since X is also connected, for a subset $I' \subset I$, we have $\left(\bigcup_{i \in I'} B(x_i, \delta/2)\right) \cap \left(\bigcup_{i \in I - I'} B(x_i, \delta/2)\right) \neq \emptyset$. Therefore we have a finite sequence $\{x_i\}_{i=1}^{n}$ in X such that

(i) $x_n = x_1$,

(ii) $d(x_i, x_{i+1}) < \delta$ for $i = 1, \ldots, n-1$,

(iii) $X \subset \bigcup_{i=1}^{n} B(x_i, \delta/2)$.

Hence for $i = 1, \ldots, n$, we have a $(\delta, 1)$-chain $C_i = \left\{\left(x_{i_j}, t_{i_j}\right)\right\}_{j=1}^{m_i}$ from x_i to x_i.

Now, we may assume that $t_{i_j} \leqslant 2$ for all i, j. Let $d(x_i, x) < \delta/2$ and $d(x_k, y) < \delta/2$. Then, for $i = k$, we obtain an $(\varepsilon, 1)$-chain from x to y:

$$(x, t_{i_1}), (x_{i_2}, t_{i_2}), \ldots, \left(x_{i_{m_i}}, t_{i_{m_i}}\right)$$

and an $(\varepsilon, 1)$-chain from y to x:

$$(y, t_{i_1}), (x_{i_2}, t_{i_2}), \ldots, \left(x_{i_{m_i}}, t_{i_{m_i}}\right).$$

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

For $i < k$, we obtain an $(\varepsilon, 1)$-chain from x to y:

$$(x, t_{i_1}), (x_{i_2}, t_{i_2}), \ldots, \left(x_{i m_i}, t_{i m_i}\right), C_{i+1}, \ldots, C_k$$

and an $(\varepsilon, 1)$-chain from y to x:

$$(y, t_{k_1}), (x_{k_2}, t_{k_2}), \ldots, \left(x_{k m_k}, t_{k m_k}\right), C_{k+1}, \ldots, C_n, C_1, \ldots, C_i.$$

This implies that $x \sim y$. It completes the proof. ∎

When $f \in X(M)$ is topologically stable, Hurley [5] obtained the following:

Proposition 3.4. *If $f \in X(M)$ is topologically stable, then*

(i) *there are only finitely many chain components of f,*
(ii) *for any chain component X of f, $\omega_f(x) = X$ for some x in X, where $\omega_f(x)$ is the ω-limit set of f at x.*

We can obtain a property of chain components of topologically stable flows.

Proposition 3.5. *Let $f \in X(M)$ be topologically stable and X be a chain component of f. For any $g \in X(M)$, if (h, λ) is a semiconjugacy from g to f, then*

(i) $h(CR(g)) = CR(f)$,
(ii) *if Y is a chain component of g and $Y \cap h^{-1}(X) \neq \emptyset$, then $Y \subset h^{-1}(X)$,*
(iii) $h(X) = X$ *for some chain component Y of g.*

Proof: (i) Since $\lambda(\cdot, 1)\colon M \to \mathbb{R}$ is continuous on the compact manifold M, there exists $K = \min \lambda(M, 1)$. Let $K = \lambda(z, 1)$. Then $K > 0$ since $\lambda(z, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and $\lambda(z, 0) = 0$.

Let $y \in h(CR(g))$. Then $y = h(x)$ for some x in $CR(g)$. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(a, b) < \delta$ implies $d(h(a), h(b)) < \varepsilon$. Also, there is a $(\delta, 1)$-chain $\{(x_i, t_i)\}_{i=1}^n$ from x to x. Note that $h(x_1) = h(x) = y$ and $\lambda(x_i, t_i) \geqslant \lambda(x_i, 1) \geqslant K$. Since $d(g(x_i, t_i), x_{i+1}) < \delta$, we have

$$d(h(g(x_i, t_i)), h(x_{i+1})) = d(f(h(x_i), \lambda(x_i, t_i)), h(x_{i+1})) < \varepsilon.$$

Also, we have

$$d(h(g(x_n, t_n)), h(x)) = d(f(h(x_n), \lambda(x_n, t_n)), y) < \varepsilon$$

since $d(g(x_n, t_n), x) < \delta$. Thus $\{(h(x_i), \lambda(x_i, t_i))\}_{i=1}^n$ is an (ε, K)-chain from y to y. By Proposition 3.2, $y \in CR(f)$. Therefore $h(CR(g)) \subset CR(f)$. Consequently, we have $h(CR(g)) = CR(f)$ by (iii) (we prove (iii) later).

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

(ii) Let $x \in Y \cap h^{-1}(X)$. Then $h(X) \in h(Y) \cap X$. If $Y \subset CR(g)$ is connected, then $h(Y) \subset CR(f)$ is also connected. Since X is a connected component of $CR(f)$, we have $h(Y) \subset X$. Therefore $Y \subset h^{-1}(h(Y)) \subset h^{-1}(X)$.

(iii) By Proposition 3.4, there is a point x in X such that $\omega_f(x) = X$. Let $y \in h^{-1}(x)$. Since $\omega_g(y) \subset CR(g)$ is connected, $\omega_g(y) \subset Y$ for some chain component Y of g. We claim that $X \subset h(\omega_g(y)) \subset h(Y)$. Let $z \in X$. Then $f(x, t_i) \to z$ for some $t_i \to \infty$. Let $\psi = \lambda(y, \cdot): \mathbb{R} \to \mathbb{R}$. Then the sequence $\{g(y, \psi^{-1}(t_i))\}$ has a convergent subsequence. So we assume $g(y, \psi^{-1}(t_i)) \to a$. Since $\psi^{-1}(t_i) \to \infty$, $a \in \omega_g(y)$. Notice that

$$\begin{aligned} h\big(g\big(y, \psi^{-1}(t_i)\big)\big) &= f\big(h(y), \lambda\big(y, \psi^{-1}(t_i)\big)\big) = f\big(x, \psi\big(\psi^{-1}(t_i)\big)\big) \\ &= f(x, t_i) \to z. \end{aligned}$$

Thus $z = h(a) \in h(\omega_g(y))$. Since $Y \cap h^{-1}(X) \neq \emptyset$, we have $Y \subset h^{-1}(X)$. Hence $h(Y) \subset h\big(h^{-1}(X)\big) = X$. Therefore $h(Y) = X$. $\square$

4. Closed orbits of topologically stable flows

In this section we obtain some properties of closed orbits of topologically stable flows. $O(x)$ denotes the orbit of a point x in M.

Let $f \in X(M)$. For a point x in M, we define

$$W^s(x) = \{y \in M : d(f(x, t), f(y, t)) \to 0 \text{ as } t \to \infty\}$$

and

$$W^u(x) = \{y \in M : d(f(x, t), f(y, t)) \to 0 \text{ as } t \to -\infty\}.$$

We call $W^s(x)$ and $W^u(x)$ a *stable set* and an *unstable set* of x, respectively.

For a subset $\Lambda \subset M$, let $W^\sigma(\Lambda) = \bigcup_{x \in \Lambda} W^\sigma(x)$, where $\sigma = s, u$. If x is a hyperbolic fixed point of f, then $W^s(x)$ and $W^u(x)$ are immersed submanifolds of M whose dimensions sum to the dimension of M [7].

For a periodic orbit γ of f, we define

$$W^s(\gamma) = \{x \in M : d(\gamma, f(x, t)) \to 0 \text{ as } t \to \infty\}$$

and

$$W^u(\gamma) = \{x \in M : d(\gamma, f(x, t)) \to 0 \text{ as } t \to -\infty\}.$$

We call $W^s(\gamma)$ and $W^u(\gamma)$ a *stable set* and an *unstable set* of γ, respectively. If γ is a hyperbolic orbit of f, then $W^\sigma(\gamma) = \bigcup_{x \in \gamma} W^\sigma(x)$, where $\sigma = s, u$, and $W^s(\gamma)$, $W^u(\gamma)$ are immersed submanifolds of M whose dimensions sum to one plus the dimension of M [7]. When we need to emphasise the dependence of any these sets on the flow f, we write them as $W^s(x; f) = W^s(x)$, *et cetera*.

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

Lemma 4.1. *Let $f, g \in X(M)$. If (h, λ) is a semiconjugacy from g to f and γ is any closed orbit of f, then for any point x in $h^{-1}(\gamma)$, we have $h(W^\sigma(O(x); g)) \subset W^\sigma(\gamma; f)$ for $\sigma = s, u$.*

Proof: Let $y \in W^s(x; g)$. Then $d(g(x, t), g(y, t)) \to 0$ as $t \to \infty$. Also, we have

$$d(h(g(x, t)), h(g(y, t))) = d(f(h(x), \lambda(x, t)), f(h(y), \lambda(y, t))) \to 0$$

as $t \to \infty$. Since $f(h(x), \lambda(x, t)) \in \gamma$ and $\lambda(y, t) \to \infty$ as $t \to \infty$, $d(\gamma, f(h(y), t)) \to 0$ as $t \to \infty$. This means that $h(y) \in W^s(\gamma; f)$. Thus $h(W^s(x; g)) \subset W^s(\gamma; f)$. Since $h^{-1}(\gamma)$ is g-invariant, $O(x) \subset h^{-1}(\gamma)$. Therefore we have

$$\begin{aligned}\bigcup_{\gamma \in O(x)} h(W^s(y; g)) &= h\left(\bigcup_{y \in O(x)} W^s(y; g)\right) \\ &= h(W^s(O(x); g)) \subset W^s(\gamma; f).\end{aligned}$$

For the case $\sigma = u$, the proof is essentially the same. $\square$

A C^1-flow f satisfies the *strong transversality condition* if for all x in M, $W^s(x)$ and $W^u(x)$ intersect transversally at x.

A C^1-flow f is called a *Smale flow* if

(i) $CR(f)$ has a hyperbolic structure,
(ii) each chain component of f is either one dimensional or is a single point,
(iii) f satisfies the strong transversality condition.

Proposition 4.2. [1] *A set of Smale flows on M is dense in $X(M)$.*

Proposition 4.3. [3] *Let X be a chain component of a Smale flow f. Then for all x, y in X, $W^s(O(x)) \cap W^u(O(y)) \cap X$ is dense in X.*

We can obtain the above result of Franks for the closed orbits of topologically stable flow f.

Proposition 4.4. *Let X be a chain component of a topologically stable flow f in $X(M)$. Then for all closed orbits γ_1, γ_2 of f, $W^s(\gamma_1; f) \cap W^u(\gamma_2; f) \cap X$ is dense in X.*

Proof: There exists a neighbourhood U of f in $X(M)$ such that for all g in U, there is a semiconjugacy (h, λ) from g to f with $d(h, 1) < 1$. Also, there is a Smale flow g in U. By Proposition 3.5, $h(Y) = X$ for some chain component Y of g. Since $\gamma_j \subset X$ for $j = 1, 2$, $h^{-1}(\gamma_j) \cap Y \neq \emptyset$. Let $y_j \in h^{-1}(\gamma_j) \cap Y$ for $j = 1, 2$. In view of Proposition 4.3, $W^s(O(y_1); g) \cap W^u(O(y_2); g) \cap Y$ is dense in Y and so

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

$h(W^s(O(y_1);g) \cap W^u(O(y_2);g) \cap Y)$ is dense in $h(Y) = X$. Therefore we have

$$\begin{aligned} &h(W^s(O(y_1);g) \cap W^u(O(y_2);g) \cap Y) \\ &\subset h(W^s(O(y_1);g)) \cap h(W^u(O(y_2);g)) \cap h(Y) \\ &\subset W^s(\gamma_1;f) \cap W^u(\gamma_2;f) \cap X. \end{aligned}$$

It follows that $W^s(\gamma_1;f) \cap W^u(\gamma_2;f) \cap X$ is dense in X. ∎

A C^1-flow f is called *Kupka-Smale* if

(i) every closed orbit of f is hyperbolic,

(ii) any intersection of stable and unstable manifolds of closed orbits is transverse.

PROPOSITION 4.5. [7] *A set of Kupka-Smale flows on M is dense in $X(M)$.*

Let γ be a hyperbolic closed orbit of f. An index of γ (written index γ) is defined to be a pair of integers (S, U), where $S = \dim W^s(\gamma)$ and $U = \dim W^u(\gamma)$.

Now, we can obtain the following useful property about the index of the hyperbolic closed orbit.

PROPOSITION 4.6. *Let X be a chain component of a topologically stable flow f in $X(M)$. If γ_1 and γ_2 are distinct hyperbolic closed orbits of f, then they have the same index.*

PROOF: By the Hartman-Grobman theorem and the local stability of hyperbolic orbits [**7, 12**], there exist disjoint neighbourhoods U_1 and U_2 of γ_1 and γ_2, respectively, and a neighbourhood G' of f in $X(M)$ such that

(i) for any $g \in G'$, there exist hyperbolic closed orbits $\lambda_1(g) \subset U_1$, $\lambda_2(g) \subset U_2$ of g such that index $\lambda_1(g) =$ index γ_1 and index $\lambda_2(g) =$ index γ_2;

(ii) $\lambda_1(g)$ and $\lambda_2(g)$ are unique g-invariant sets in U_1 and U_2, respectively.

There exists an $\varepsilon > 0$ such that $B(\gamma_1, \varepsilon) \subset U_1$ and $B(\gamma_2, \varepsilon) \subset U_2$. Also, there is a neighbourhood $G \subset G'$ of f in $X(M)$ with the property that for all g in G, there is a semiconjugacy (h, λ) from g to f such that $d(h, 1) < \varepsilon$. We claim that $h^{-1}(\gamma_1) \subset U_1$ and $h^{-1}(\gamma_2) \subset U_2$.

Let $x \in h^{-1}(\gamma_j)$ for $j = 1, 2$. Then $h(x) \in \gamma_j$ for $j = 1, 2$. Thus $d(\gamma_j, x) \leqslant d(h(x), x) \leqslant d(h, 1) < \varepsilon$. It follows that $x \in B(\gamma_j, \varepsilon) \subset U_j$. Now, $h^{-1}(\gamma_j)$ is closed g-invariant for $j = 1, 2$, since each γ_j is closed f-invariant. Hence $h^{-1}(\gamma_1) = \lambda_1(g)$ and $h^{-1}(\gamma_2) = \lambda_2(g)$.

We show that $W^s(\lambda_1(g);g) \cap W^u(\lambda_2(g);g) \neq \emptyset$ for some Kupka-Smale flow g in G. Using Proposition 4.3, we choose a point x in $W^s(\gamma_1;f) \cap W^u(\gamma_2;f)$ and let $y \in h^{-1}(x)$. Since $h(\omega_g(y)) \subset \omega_f(x) = \gamma_1$, we have $\omega_g(y) \subset h^{-1}(\gamma_1) = \lambda_1(g)$. Thus $y \in W^s(\lambda_1(g);g)$. Similarly, $y \in W^u(\lambda_2(g);g)$ since $h(\alpha_g(y)) \subset \alpha_f(x) = \gamma_2$, where

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

$\alpha_f(x)$ is the α-limit set of f at x. Reversing the roles of $\lambda_1(g)$ and $\lambda_2(g)$, it follows that $W^s(\lambda_1(g);g)\cap W^u(\lambda_2(g);g) \neq \emptyset$. Since g is Kupka-Smale, $W^s(\lambda_1(g);g)\cap W^u(\lambda_2(g);g)$ and $W^u(\lambda_1(g);g)\cap W^s(\lambda_2(g);g)$ are submanifolds of dimension at least one. In what follows (S_j, U_j) represents the index of $\lambda(g)$ with respect to g and $m = \dim M$, $U_1 + S_2 \geqslant m+1$ and $U_2 + S_1 \geqslant m+1$. In addition, general facts about indices of hyperbolic closed orbits give a second pair of inequalities $U_1 + S_1 \leqslant m+1$ and $U_2 + S_2 \leqslant m+1$. Hence $S_1 = S_2$ and $U_1 = U_2$. Combining these equalities with (i) gives the conclusion of the proposition. □

5. Structurally stable flows

In this section, we prove the main result.

Proposition 5.1. [10] *If a C^1-flow is structurally stable, then it is Kupka-Smale.*

A C^1-flow f satisfies *Axiom A* if

(i) the set of the periodic orbits of f are dense in $\Omega(f)$,

(ii) $\Omega(f)$ has a hyperbolic structure.

Smale's conjecture [8] is the following:

> $f \in X(M)$ is structurally stable if and only if it satisfies Axiom A and the strong transversality condition.

Robinson [10] obtained the following sufficient condition of Smale's conjecture.

Proposition 5.2. *If $f \in X(M)$ satisfies Axiom A and the strong transversality condition, then it is structurally stable.*

Moreover, Nitecki [6] proved the following:

Proposition 5.3. *If $f \in X(M)$ satisfies Axiom A and the strong transversality condition, then it is topologically stable.*

Let

$$\mathcal{F}(M) = \{ f \in X(M) : \text{ there exists a neighbourhood } U \text{ of} f \text{ in } X(M) \text{ such that for all } g \text{ in } U, \text{ every closed orbit of } g \text{ is hyperbolic}\}.$$

Also, for f in $\mathcal{F}(M)$, we denote by $\Lambda_i(f)$ the closure of the set of hyperbolic closed orbits of f whose stable manifold has dimension i, where $0 \leqslant i \leqslant \dim M$.

Manẽ [5] conjecutred the following:

> For f in $\mathcal{F}(M)$, if $\Lambda_i \cap \Lambda_j = \emptyset$ for all $0 \leqslant i < j \leqslant \dim M$, then f satisfies Axiom A.

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

He showed that his conjecture is true in the case of a discrete dynamical system.

Now, we prove the main theorem.

Theorem 5.4. *$f \in X(M)$ is structurally stable and topologically stable if and only if it satisfies Axiom A and the strong transversality condition.*

Proof: ($\Rightarrow$) Since f is structurally stable, it is Kupka-Smale by Proposition 5.1. Thus $f \in \mathcal{F}(M)$.

We claim that $\Lambda_i \cap \Lambda_j = \emptyset$ for all $0 \leqslant i < j \leqslant \dim M$. Suppose that $\Lambda_i \cap \Lambda_j \neq \emptyset$, say $x \in \Lambda_i \cap \Lambda_j$. If we denote by Per_i the set of periodic points whose stable manifold has dimension i, then there exist sequences $\{x_n\} \subset \mathrm{Per}_i$ and $\{y_n\} \subset \mathrm{Per}_j$ such that $x_n \to x$ and $y_n \to y$. In view of Propositions 3.4 and 4.6, we may assume that x_n and y_n are sequences in the chain components of X and Y, respectively. But $x \in \overline{X} = X$ and $x \in \overline{Y} = Y$. This contradicts $X \cap Y = \emptyset$.

Now f satisfies Axiom A by Mañé's conjecture. Since f is structurally stable, it satisfies the strong transversality condition [**10**].

($\Leftarrow$) It follows from Propositions 5.2 and 5.3. $\square$

Let $f \in X(M)$. A closed subset Λ of $\Omega(f)$ is called an *attractor* if

(i) Λ is f-invariant;

(ii) there is a neighbourhood U of Λ in M such that $\bigcap_{t \geqslant 0} f(U, t) = \Lambda$;

(iii) $f\,|_{\Lambda \times \mathbb{R}}$ is transitive.

Smale showed that if $f \in X(M)$ satisfies Axiom A, then it has only a finite number of hyperbolic attractors. Therefore we can obtain the following corollary.

Corollary 5.5. *If $f \in X(M)$ is structurally stable and topologically stable, then it has only a finite number of hyperbolic attractors.*

References

[1] R. Bowen, 'Periodic orbits for hyperbolic flows', *Amer. J. Math.* **94** (1972), 1–30.

[2] S.K. Choi and J.S. Park, 'Hyperbolic flows are topologically stable', *Bull. Austral. Math. Soc.* **43** (1991), 225–232.

[3] J. Franks, *Homology and dynamical systems*, C.B.M.S. Regional Conf. Ser. Math. No. 49 (Amer. Math. Soc. Providence, R.I., 1982).

[4] M. Hurley, 'Combined structural and topological stability are equivalent to Axiom A and the strong transversality condition', *Ergodic Theory Dynamical Systems* **4** (1984), 81–88.

[5] R. Mañé, 'A ergodic closing lemma', *Ann. Math.* **116** (1982), 503–540.

[6] Z. Nitecki, 'On semistability for diffeomorphisms', *Invent. Math.* **14** (1971), 83–122.

[7] J. Palis and W. de Melo, *Geometric theory of dynamical systems* (Springer-Verlag, Berlin, Heidelberg, New York, 1982).

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press

[8] J. Palis and S. Smale, 'Structural stability theorems', in *Proc. Sympos. Pure Math* . **14**, pp. 223–231 (Amer. Math. Soc., Providence, R.I., 1970).

[9] J. Robbin, 'A structural stability theorem', *Ann. of Math.* **94** (1971), 447–493.

[10] C. Robinson, 'C^r structural stability implies Kupka-Smale', in *Dynamical systems* (Academic Press, 1973).

[11] R.C. Robinson, 'Structural stability of vector fields', *Ann. of Math.* **99** (1974), 154–175.

[12] S. Smale, 'Differentiable dynamical systems', *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

Department of Mathematics
Chungnam National University
Taejon 305-764
Korea

https://doi.org/10.1017/S0004972700037035 Published online by Cambridge University Press