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SCALE COVARIANT LAGRANGIANS AND SPACES RECIPROCAL TO STATIC EINSTEIN SPACES

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Abstract

The main aim of this paper is to derive a condition whose satisfaction ensures that the Euler–Lagrange equations of a non-linear scale covariant Lagrangian are satisfied by the metric of a space reciprocal to an arbitrary static Einstein space.

1. Introduction

In an earlier paper [1], hereafter referred to as N, the possibility of finding r-invariant non-linear Lagrangian densities $\mathfrak{L}$, that is, Lagrangian densities unaffected by the replacement of a given static metric by its reciprocal [2], was examined. Thus, let greek indices go over the range $1, 2, \ldots, n$, whilst roman indices go over the same range with one particular, fixed value excluded which is henceforth taken to be n without loss of generality. In appropriate coordinates the metric then satisfies the conditions $g_{jn} = 0$, $g_{\mu\nu,n} = 0$ and the metric reciprocal to it is $\bar{g}_{\mu\nu} = ((g_{nn})^q g_{ij}, (g_{nn})^{-1})$, $q := 2/(n-3)$. Evidently if the "field equations" $\delta\mathfrak{L}/\delta g_{\mu\nu} = 0$ are satisfied by the static metric $g_{\mu\nu}$ they will also be satisfied by the reciprocal metric $\bar{g}_{\mu\nu}$, granted that $\mathfrak{L}$ is r-invariant. In principle the situation would obtain also if $\mathfrak{L}$ were merely r-semi-invariant, if $\mathfrak{L}[\bar{g}_{\mu\nu}]$ differed from $\mathfrak{L}[g_{\mu\nu}]$ by an ordinary divergence. It is, however, not fruitful to contemplate this possibility in the present context.

In N it was shown that the Lagrangian density

$$\mathfrak{L}_1 = |g(R^2 - 2R_{\mu\nu}R^{\mu\nu})|^{\frac{1}{2}} \tag{1.1}$$

is r-invariant and it has every appearance of being unique except to the extent

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that the sum $\mathfrak{L}_2 = \mathfrak{R} - [n/(n-2)]^{\frac{1}{2}}\,\mathfrak{L}_1$ is r-semi-invariant. It was further shown that the equations $\delta\mathfrak{L}_2/\delta g_{\mu\nu} = 0$ are satisfied by the metric tensor reciprocal to that of an arbitrary static Einstein space. It is this latter result which is to be generalized in the present paper by inferring the existence of a wide class of Lagrangian densities $\mathfrak{L}$ such that the equation $\delta\mathfrak{L}/\delta g_{\mu\nu} = 0$ are satisfied by the metric reciprocal to that of an arbitrary static Einstein space. The latter itself will also satisfy these equations provided a further condition is satisfied by $\mathfrak{L}$.

2. Allowed Lagrangians. Scale weight

The only Lagrangians to be contemplated are those which are composed of the mixed Ricci tensor $R_\mu{}^\nu$ alone, that is, the uncontracted components of the Riemann tensor do not appear. The $R_\mu{}^\nu$ may be considered as the elements of an $n \times n$ matrix $\mathbf{R}$. Then L is a function of the invariants

$$I_s := \operatorname{tr}(\mathbf{R}^s), \tag{2.1}$$

where tr denotes the trace. Because of the operation of the Cayley–Hamilton theorem the "elementary invariants" $I_r\,(r > n)$ are functions of the $I_m\,(m \leqslant n)$. Accordingly

$$L = L(I_1, I_2, \ldots, I_n). \tag{2.2}$$

Under a scale transformation, that is, the multiplication of $g_{\mu\nu}$ by a constant (positive) factor α, $\mathbf{R}$ goes into $\alpha^{-1}\mathbf{R}$, I_s into $\alpha^{-s} I_s$ and so

$$L \to L(\alpha^{-1} I_1, \alpha^{-2} I_2, \ldots, \alpha^{-n} I_n). \tag{2.3}$$

At the same time

$$w := |g|^{\frac{1}{2}} := |\det g_{\mu\nu}|^{\frac{1}{2}} \to \alpha^{n/2} w.$$

A Lagrangian or Lagrangian density will be said to be scale covariant and to have scale weight σ if under a scale transformation it merely takes the factor $\alpha^{-\sigma}$ It is scale invariant when $\sigma = 0$. Note that if L has the definite scale weight σ

$$\sum_{s=1}^{n} s I_s \frac{\partial L}{\partial I_s} = \sigma L. \tag{2.4}$$

3. Einstein spaces and scale invariant Lagrangians

Let the metric $g_{\mu\nu}$ be specifically that of a (non-special) Einstein space so that

$$R_{\mu\nu} = \lambda g_{\mu\nu}. (\lambda = \text{constant} \neq 0). \tag{3.1}$$

(3.1) may be looked upon as the Euler–Lagrange equations of the variational principle

$$\delta \int [R-(n-2)\lambda]\, w\, dx = 0. \tag{3.2}$$

For further use set

$$\delta \int w\, dx =: \Delta, \tag{3.3}$$

so that (3.2) implies that

$$\int w \delta R\, dx \stackrel{*}{=} -2\lambda\Delta. \tag{3.4}$$

Throughout, the sign $\stackrel{*}{=}$ indicates that equality holds when (3.1) obtains and likewise quantities distinguished by an asterisk are to be thought of as evaluated by using (3.1). For example,

$$I_s^* = n\lambda^s. \tag{3.5}$$

Bearing (2.2) in mind and writing $L_s := \partial L/\partial I_s$,

$$\delta \int \mathfrak{L}\, dx = L^*\Delta + \sum_{s=1}^{n} L_s^* \int w \delta I_s\, dx.$$

Now

$$\delta I_s = \delta \operatorname{tr}(\mathbf{R}^s) = s\mathbf{R}^{s-1}\delta\mathbf{R} = s\lambda^{s-1}\mathbf{1}\delta\mathbf{R} \stackrel{*}{=} s\lambda^{s-1}\delta R.$$

In view of (3.4) it follows that

$$\delta \int \mathfrak{L}\, dx \stackrel{*}{=} \left(L^* - 2\sum_{s=1}^{n} s\lambda^s L_s^*\right)\Delta. \tag{3.6}$$

If $\mathfrak{L}$ is scale invariant (2.3) shows that one must have

$$\alpha^{n/2} L(\alpha^{-1} I_1, \ldots, \alpha^{-n} I_n) = L(I_1, \ldots, I_n), \tag{3.7}$$

which implies that

$$\sum s I_s L_s = \tfrac{1}{2} n L \tag{3.8}$$

and, in particular, from (3.5), that

$$\sum s\lambda^s L_s^* = \tfrac{1}{2} L^*. \tag{3.9}$$

Inspection of (3.6) thus leads to the conclusion that the Euler–Lagrange equations of a scale-invariant Lagrangian density are satisfied by the metric of an arbitrary Einstein space. This known result [3] will be referred to again in Section 7.

4. The reciprocal Lagrangian

Suppose now that $g_{\mu\nu}$ is static (with respect to x^n) and contemplate the metric $\bar{g}_{\mu\nu}$ reciprocal to it. From $N(4.1)$

$$\bar{R}_i^j = f^{-2q}(R_i^j + q\delta_i^j T), \quad \bar{R}_n^n = -f^{-2q} T, \quad \bar{R} = f^{-2q}(R + 2qT). \tag{4.1}$$

Here $f^2 := e^{2\tau} := g_{nn}$ and $T = f^{-1}\Box f$, where $\Box f$ is the contracted second covariant derivative of f, defined in the V_{n-1} whose metric tensor is g_{ij}. If $\bar{I}_s := \operatorname{tr}(\bar{\mathbf{R}}^s)$, then

$$f^{2sq}\bar{I}_s = \operatorname{tr}(\mathbf{R}' + qT\mathbf{1}) + (-1)^s T^s, \tag{4.2}$$

where $\mathbf{R}'$ is the matrix whose elements are R_i^j. The first term on the right of (4.2) is, with $I_0 := n$,

$$\begin{aligned}\sum_{r=0}^{s}\binom{s}{r}(qT)^r \operatorname{tr}(\mathbf{R}')^{s-r} &= \sum_{r=0}^{s}\binom{s}{r}(qT)^{s-r}(I_r - T^r)\\ &= \sum_{r=0}^{s}\binom{s}{r}(qT)^{s-r} I_r - (q+1)^s T^s.\end{aligned}$$

Thus

$$\bar{I}_s = f^{-2sq}\left\{\sum_{r=1}^{s}\binom{s}{r}(qT)^{s-r} I_r - [(q+1)^s - nq^s - (-1)^s]T^s\right\}. \tag{4.3}$$

When (3.1) obtains, $T^* = (R_n^n)^* = \lambda$. Using also (3.5) it follows from (4.3) after a little simplification that

$$\bar{I}_s^* = f^{-2sq}\lambda^s[(n-1)(q+1)^s + (-1)^s]. \tag{4.4}$$

It is convenient to write

$$\bar{I}_s = f^{-2sq}\hat{I}_s \tag{4.5}$$

and

$$\hat{I}_s^* = \lambda^s i_s, \quad i_s := 2q^{-1}(q+1)^{s+1} + (-1)^s. \tag{4.6}$$

The reciprocal Lagrangian is

$$\bar{L} := L(\bar{I}_1, \ldots, \bar{I}_n) = L(f^{-2q}\hat{I}_1, \ldots, f^{-2nq}\hat{I}_n).$$

Let it now be assumed that L has the definite scale weight p. Then

$$\bar{L} = f^{-2pq}L(\hat{I}_1, \ldots, I_n) =: f^{-2pq}\hat{L}.$$

Bearing in mind that $\bar{w} = f^{2q}w$ it follows that

$$\bar{\mathfrak{L}} = f^{2(1-p)q}\hat{\mathfrak{L}}.$$

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Accordingly, choose $p = 1$, so that

$$\bar{\mathfrak{L}} = \hat{\mathfrak{L}}. \tag{4.7}$$

5. The variation of $\int \bar{\mathfrak{L}}\, dx$

In view of (4.7),

$$\delta \int \bar{\mathfrak{L}}\, dx \overset{*}{=} \hat{L}^* \Delta + \sum_{s=1}^{n} \hat{L}_s^* \int w \delta \hat{I}_s\, dx. \tag{5.1}$$

As regards the second term on the right, one has from (4.3,5)

$$\int w \delta \hat{I}_s\, dx \overset{*}{=} \int w \left\{ \sum_{r=1}^{s} \binom{s}{r} q^{s-r} [\lambda^{s-r} (\delta I_r)^* + (s-r) \lambda^{s-r-1} I_r^* (\delta T)^*] \right.$$
$$\left. - s[(q+1)^s - nq^s - (-1)^s] \lambda^{s-1} (\delta T)^* \right\} dx.$$

Here one has to insert (3.5) and

$$\int w \delta I_r\, dx \overset{*}{=} -2r\lambda^r \Delta, \quad \int w \delta T\, dx \overset{*}{=} -\lambda \Delta. \tag{5.2}$$

This leads directly to

$$\int w \delta \hat{I}_s\, dx \overset{*}{=} \left\{ \sum_{r=1}^{s} \binom{s}{r} [(n-2)r - ns] q^{s-r} + s[(q+1)^s - nq^s - (-1)^s] \right\} \lambda^s \Delta$$
$$= -s[(2q+3)(q+1)^{s-1} + (-1)^s] \lambda^s \Delta =: -\gamma_s \lambda^s \Delta \tag{5.3}$$

say. (5.1) thus becomes

$$\delta \int \bar{\mathfrak{L}}\, dx \overset{*}{=} \left(\hat{L}^* - \sum_{s=1}^{n} \gamma_s \lambda^s \hat{L}_s^* \right) \Delta. \tag{5.4}$$

6. The main result derived

According to (5.4), the variation of $\int \mathfrak{L}\, dx$ will vanish if the factor multiplying Δ on the right vanishes, that is to say,

$$\sum_{s=1}^{n} \gamma_s \frac{\partial L(i)}{\partial i_s} = L(i). \tag{6.1}$$

Here λ has been set equal to unity since it disappears from the equation because of the assumed scale weight unity of L; $L(i)$ stands for $L(i_1, i_2, \ldots, i_n)$; and the i_s are not to be regarded as standing for the numbers given by (4.6) until after the differentiations have been carried out. Equation (6.1) may be written

$$\sum_{s=1}^{n} s\{\tfrac{1}{2}q(q+1)^{-2}(2q+3)[i_s-(-1)^s]+(-1)^s\}\frac{\partial L(i)}{\partial i_s} = L(i). \tag{6.2}$$

However, since L has the scale weight $\sigma = 1$, one has from (2.4) that

$$\sum_{s=1}^{n} s i_s \frac{\partial L(i)}{\partial i_s} = L(i). \tag{6.3}$$

Upon using this equation in (6.2) the latter reduces to

$$\sum_{s=1}^{n} (-1)^s s \frac{\partial L(i)}{\partial i_s} = L(i). \tag{6.4}$$

One thus has the desired result that, if the Lagrangian $L(I_1, \ldots, I_n)$ of scale weight unity satisfies the condition (6.4), then the field equations $\delta L/\delta g_{\mu\nu} = 0$ are satisfied by the metric of a space reciprocal to an arbitrary static Einstein space.

7. Miscellaneous remarks

(a) As mentioned in Section 1, if

$$L = I_1+\alpha(I_1^2-2I_2)^{\frac{1}{2}}, \quad \alpha = -[n/(n-2)]^{\frac{1}{2}}, \tag{7.1}$$

then $\delta L/\delta g_{\mu\nu} = 0$ when $g_{\mu\nu}$ is the metric reciprocal to that of an arbitrary static Einstein space. Evidently L has the scale weight required for the applicability of the result of Section 6. Thus

$$L(i) = i_1+\alpha(i_1^2-2i_2)^{\frac{1}{2}},$$

whence (6.4) becomes in particular

$$(1+i_1)(i_1^2-2i_2)^{\frac{1}{2}}+\alpha(i_1^2-2i_2+i_1+2) = 0. \tag{7.2}$$

Here one now has to insert

$$i_1 = -1+2(q+1)^2/q, \quad i_2 = 1+2(q+1)^3/q.$$

After some manipulation it turns out that

$$\alpha = -[(3q+2)/(q+2)]^{\frac{1}{2}} = -[n/(n-2)]^{\frac{1}{2}},$$

in agreement with (7.1).

(b) According to N, in the case of the Lagrangian (7.1), the equations $\delta L/\delta g_{\mu\nu} = 0$ are satisfied also by the metric of an arbitrary Einstein space. Now

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the scale weight of L is 1, whereas that the the Lagrangians contemplated in Section 3 was $\frac{1}{2}n$. The conflict is only apparent: (3.8) is a sufficient but not a necessary condition. In fact it is only necessary that (3.9) should be satisfied. That L has $\sigma = 1$ is compatible with (3.9) if L satisfies the additional condition $L^* = 0$, that is

$$L(n, n, \ldots, n) = 0, \tag{7.3}$$

and this is indeed the case for the Lagrangian (7.1).

(c) In this subsection it will suffice to make the special choice $n = 4$. Then one may think of the Lagrangian

$$L = I_1 - 16I_2/5I_1 \tag{7.4}$$

as the "simplest" rational Lagrangian of the kind under consideration—rational in the sense that it is a rational function of the invariants I_s. It satisfies (6.3) and (6.4), but not (7.3). To accommodate the latter as well one has to add at least one further term on the right of (7.4). Indeed, an example of a rational Lagrangian which satisfies (7.3) as well is

$$L = I_1 - 6I_2/I_1 + 8I_3/I_1^2. \tag{7.5}$$

On the other hand,

$$L = I_1 - 16\alpha I_2/5I_1 + 64(\alpha - 1) I_3/7I_1^2,$$

where α is an arbitrary constant, is an example of a one-parameter family of rational Lagrangians all of which satisfy (6.3) and (6.4).

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