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The adjunction morphism for regular differential forms and relative duality

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Abstract. Let $f : X \to Y$ be a morphism of noetherian schemes, generically smooth and equidimensional of dimension d, $\iota : X' \to X$ a closed embedding such that $f \circ \iota : X' \to Y$ is generically smooth and equidimensional of dimension d', and X', X and Y are excellent schemes without embedded components. We exhibit a concrete morphism

$$\mathrm{Res}_{X'/X} : \det \mathcal{N}_{X'/X} \otimes_{\mathcal{O}_{X'}} \iota^* \omega^d_{X/Y} \to \omega^{d'}_{X'/Y},$$

which transforms the integral of X/Y into the integral of X'/Y. Here $\mathcal{N}_{X'/X}$ denotes the normal sheaf of X'/X and $\omega^d_{X/Y}$ resp. $\omega^{d'}_{X'/Y}$ denotes the sheaf of regular differential forms of X/Y resp. X'/Y. Using generalized fractions we provide a canonical description of residual complexes and residue pairs of Cohen–Macaulay varieties, and obtain a very explicit description of fundamental classes and their traces.

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Relative duality theory for morphisms of noetherian schemes was studied and described by A. Grothendieck and R. Hartshorne (cf. [RD]). It settles the problem of duality theory for quasi-coherent sheaves from a theoretical point of view. This approach heavily relies on the theory of derived categories and constructs for a morphism $f : X \to Y$ of noetherian schemes a right adjoint $\mathbf{f}^!$ to $\mathbb{R}f_*$. In a given situation it is often hard to apply this theory since the objects and maps involved are rather complicated to determine. Better suited for explicit applications is Kleiman's fairly elementary approach to construct a right adjoint $f^!$ to the functor $R^d f_*$ (cf. [Kl]). Kleiman has studied $f^!(\mathcal{O}_Y)$ in various special situations.

In [HK$_2$] resp. [HS] the first author and E. Kunz resp. P. Sastry have been able to identify the sheaf $f^!(\mathcal{O}_Y)$ with the sheaf $\omega^d_{X/Y}$ of regular differential forms in case X/Y is generically smooth and equidimensional of relative dimension d. Further, the trace morphism tr_f was described in terms of residues and local and global integrals. In this paper we will continue this project. If $f : X \to Y$ is generically smooth and equidimensional of dimension d, $\iota : X' \to X$ is a closed embedding such that $f \circ \iota$ is generically smooth and equidimensional of dimension d' and X',

X and Y are excellent noetherian schemes without embedded components, we will relate the duality theory of $f \circ \iota$ to that of f and ι explicitly. To be more precise, we will exhibit a concrete morphism

$$\mathrm{Res}_{X'/X} : \det \mathcal{N}_{X'/X} \otimes_{\mathcal{O}_{X'}} \iota^* \omega^d_{X/Y} \to \omega^{d'}_{X'/Y}$$

using the second fundamental exact sequence for universal differential modules (cf. [KD], (4.17)). For varieties over perfect fields, the existence of this morphism was proved by Lipman [Li_1], Section 13. It is an isomorphism if X/Y is Cohen–Macaulay. Here $\mathcal{N}_{X/X'}$ denotes the normal sheaf of X'/X. It will be shown that $\mathrm{Res}_{X'/X}$ transforms the integral of X/Y into the integral of X'/Y. We will proceed by first studying the local situation and by proving an adjunction formula for the residue symbol and the local integral of [HK_1]. In particular we will show that the local integral satisfies the residue axiom (R3) of [RD], p. 197. From the local case we will derive the global adjunction theorem via the residue theorem.

The adjunction morphism and its explicit description in terms of regular differential forms plays a prominent role in Arakelov theory. S. Lang has used it (implicitly) in his version of the residue theorem $\big($[La], IV (4.1)$\big)$ which is an essential ingredient in his proof of the Arakelov adjunction formula $\big($[La], IV (5.3)$\big)$. However it is also of great interest in case of varieties over a field k of characteristic 0. El Zein ([EZ_1], [EZ_2]) has applied Grothendieck duality theory and the residual complex of a variety to the construction of the fundamental class of a cycle. Using generalized fractions we will provide a canonical description of residual complexes and residue pairs of Cohen–Macaulay varieties. Combining this with the concrete realization of the adjunction morphism in terms of generalized fractions we obtain a very explicit description of fundamental classes and their traces.

In fact, we deal with a more general situation considering differential algebras on Y admissible for X/Y and X'/Y.

1. Residues and Cousin complexes

In this section we provide the basic facts about residues of regular differential forms and the construction of Cousin complexes via generalized fractions needed in the later sections.

First, we recall some definitions (cf. [KD], app.B and EGA IV, Sect. 13) and give two lemmas about equidimensional morphisms. Then we formulate a situation when regular differential forms are defined and give a short description of them using traces.

(1.1) DEFINITION. Let $f: X \to Y$ be a morphism of schemes and $x \in X$.

A *quasi-normalization of f at x* is an equidimensional Y-morphism $g: U \to \mathbb{A}^d_Y$ of dimension 0. Here, U is an open neighborhood of x in X and $d \in \mathbb{N}$.

A *quasi-normalization of X/Y* is an equidimensional Y-morphism $g: X \to \mathbb{A}^d_Y$ of dimension 0.

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A *system of parameters of* X/Y *at* x is a sequence of elements $t_1, \dots, t_d$ of $\mathcal{O}_{X,x}$ whose residue classes $\bar{t}_1, \dots, \bar{t}_d$ in $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ are a system of parameters of this ring.

(1.2) LEMMA. *Let* $f : X \to Y$ *be an equidimensional morphism of schemes of dimension* d, *and* $x \in X$ *a closed point. Then:*

(a) $\dim \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = d$.
(b) *If* $(t_i)_{i=1}^d$ *is a system of parameters of* X/Y *at* x, *then there is an open neighborhood* U *of* $x \in X$ *such that there are sections* $s_i \in \mathcal{O}_X(U)$ *with* $(s_i)_x = t_i$ $(1 \leqslant i \leqslant d)$ *and the* Y*-morphism* $g : U \to \mathbb{A}^d_Y$ *induced by the sections* $(s_i)_{i=1}^d$ *is a quasi-normalization of* X/Y *at* x.

Proof. (a) is obvious, and (b) is a reformulation of [KW], (5.18) in the language of schemes. □

(1.3) LEMMA. *Let* $f : X \to Y$ *be a morphism of schemes, equidimensional of dimension* d, *and let* $g : X \to Q := \mathbb{A}^d_Y$ *be a quasi-normalization of* X/Y, *and* $x \in X$ *with* $\dim \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \dim \mathcal{O}_{Q,g(x)}/\mathfrak{m}_{f(x)}\mathcal{O}_{Q,g(x)}$. *Further, let* $z \in X$.

Then the following holds true:

(a) *If* X/Y *is Cohen–Macaulay at* x, *then* X/Q *is Cohen–Macaulay at* x. *If this is the case, then* $\widehat{\mathcal{O}}_{X,x}$ *is a finite free* $\widehat{\mathcal{O}}_{Q,g(x)}$*-module.*
(b) *If* X/Y *is a locally complete intersection at* x, *then* X/Q *is a locally complete intersection at* x.
(c) *If* X/Y *is locally a complete intersection at* x, *then the algebra* $\widehat{\mathcal{O}}_{X,z}/\widehat{\mathcal{O}}_{Q,g(z)}$ *is locally a complete intersection at all* $\mathfrak{P} \in \operatorname{Spec} \widehat{\mathcal{O}}_{X,z}$ *whose image under the canonical morphism* $\operatorname{Spec} \widehat{\mathcal{O}}_{X,z} \to X$ *of schemes is* x.
(d) *If* X/Y *is generically a complete intersection, then* X/Q *is generically a complete intersection and the algebra* $\widehat{\mathcal{O}}_{X,z}/\widehat{\mathcal{O}}_{Q,g(z)}$ *is finite, equidimensional of dimension* 0 *and generically a complete intersection.*

Proof. We may assume that X and Y are affine. Then (a) resp. (b) are special cases of [KD], (B.27), resp. [KD], (C.23). If X/Y is locally a complete intersection at x, then by (b) X/Q is locally a complete intersection at x. Set $R := \mathcal{O}_Q(Q)$, $S := \mathcal{O}_X(X)$, $\mathfrak{m}$ resp. $\mathfrak{p} \in \operatorname{Spec} S$ the prime ideals corresponding to z resp. x, $\widehat{R} := \widehat{R_{\mathfrak{m}\cap R}}$ and $\widehat{S} := \widehat{S_{\mathfrak{m}}}$. By Zariski's main theorem (cf. [KD], (B.16)), there is a finite subalgebra T/R of S/R such that $\operatorname{Spec} S \to \operatorname{Spec} T$ is an open immersion. The algebra T/R is locally a complete intersection at $\mathfrak{p} \cap T$. Let $\mathfrak{P} \in \operatorname{Spec} \widehat{S}$ with $\mathfrak{P} \cap S = \mathfrak{p}$. We have a canonical ring morphism $\widehat{R} \otimes_R T \to \widehat{S}$, and by [KD], (C.18) $\widehat{R} \otimes_R T/\widehat{R}$ is locally a complete intersection at $\mathfrak{P} \cap (\widehat{R} \otimes_R T)$. By [M$_2$], (8.15) $\widehat{S}$ is a localization of $\widehat{R} \otimes_R T$ so that $\widehat{S}/\widehat{R}$ is locally a complete intersection at $\mathfrak{P}$. This shows (c). If X/Y is generically a complete intersection, then (b) applied to the

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maximal points of X shows that X/Q is generically a complete intersection. The algebra $\widehat{\mathcal{O}}_{X,z}/\widehat{\mathcal{O}}_{Q,g(z)}$ is finite. By (c), it is generically a complete intersection. □

(1.4) LEMMA. *Let R be a universally japanese ring, and S an R-algebra which is essentially of finite type, where R and S have no embedded primary components and S is local. Then the completion $\widehat{S}$ of S has no embedded primary components.*

Proof. We may assume w.l.o.g. $S = R$. Then the assertion follows from [Mat], 9.B and [Mat], Thm. 70. □

(1.5) LEMMA. *Let $f: X \to Y$ be a morphism of schemes. Then the set*

$$\{x \in X;\ X/Y \text{ is a locally complete intersection at } x\}$$

is open in X.

Proof. By [EGA IV, 11.1.1], the set of all points $x \in X$ such that f is flat at x is open in X. We may assume X and Y to be affine and X/Y to be flat. Now the assertion follows from the criterion [KD, C.5] for locally complete intersections. □

(1.6) *Situation.* Let Y be a scheme which is universally japanese, and X a Y-scheme with structure morphism $f: X \to Y$, which is equidimensional of dimension d and generically a complete intersection. The schemes X and Y shall not have embedded primary components. Further, let Ω be a differential algebra on Y which is *admissible* for X/Y, i.e., Ω is an exterior coherent differential algebra on Y such that Ω^1 is a locally free $\mathcal{O}_Y$-module of constant rank, say r, and $\mathcal{M}_X(\Omega^1_X)$ is a locally free $\mathcal{M}_X$-module of rank $r + d$, where Ω^1_X denotes the universal X-extension of Ω^1 in the sense of [KD], Section 4.

(1.7) *Remark.* Consider Situation (1.6). Then by [KW], (3.8) the sheaf of regular differential forms $\omega^{\Omega}_{X/Y}$ for X/Y with respect to Ω is defined. We write $\omega_{X/Y}$ instead of $\omega^{\Omega}_{X/Y}$ if no confusion is likely.

Let $x \in X$ be a closed point and $(t_i)_{i=1}^d$ a system of parameters of X/Y at x. By (1.2) this induces a quasi-normalization $g: U \to Q := \mathbb{A}^d_Y$ of X/Y at x.

Set $R := \mathcal{O}_{Y,f(x)}$, $P := \mathcal{O}_{Q,g(x)}$ and $S := \mathcal{O}_{X,x}$.

By (1.4), $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ have no embedded primary components, and by (1.3) the algebra $\widehat{S}/\widehat{P}$ is finite, equidimensional of dimension 0 and generically a complete intersection.

We have $\widehat{P} = \widehat{R}[[T_1, \ldots, T_d]]$, and by [KD], (12.4) the universally finite extension $\widetilde{\Omega}$ of the differential algebra $\Omega_{f(x)}$ of R to $\widehat{P}$ exists, and we have $\widetilde{\Omega} = \widehat{P} \otimes_P (\Omega_{f(x)})_P$. In particular, $\widetilde{\Omega}^1$ is a free $\widehat{P}$-module of rank $r + d$, and if ω_0 is a basis of $\Omega^r_{f(x)}$ as an R-module, then $\omega_0\, \mathrm{d}T_1 \ldots \mathrm{d}T_n$ is a basis of $\widetilde{\Omega}^{r+d}$ as a $\widehat{P}$-module where ω_0 is an element of $\widetilde{\Omega}$ via the canonical map $\Omega_{f(x)} \to \widetilde{\Omega}$. By [KD], (11.9) and [KD], (12.4) $\widetilde{\Omega}_{\widehat{S}} = \widehat{S} \otimes_S \Omega_S$.

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Let K, L, K_1 and L_1 denote the full quotient rings of P, S, $\widehat{P}$ and $\widehat{S}$. Then $K_1 \otimes_{\widehat{P}} \widehat{S} = L_1$, $L_1 \otimes_L \Omega_L = \widetilde{\Omega}_{L_1}$ and $K_1 \otimes_K \Omega_K = \widetilde{\Omega}_{K_1}$. In particular, $\widetilde{\Omega}^1_{L_1}$ is a free L_1-module of rank $r+d$. So, $\widehat{S}/\widehat{P}$ together with the differential algebra $\widetilde{\Omega}$ of $\widehat{P}$ satisfies the conditions of [KW], (3.8) and [KW], (4.1). The module of regular differential forms and the complementary module in the sense of [KW], (Sections 3, 4) are defined for $\widehat{S}/\widehat{P}$ with respect to $\widetilde{\Omega}$.

The algebra L_1/K_1 is a finite locally complete intersection. So by [KD], (Sect. 16), we have a trace map

$$\sigma_{L_1/K_1}\colon \widetilde{\Omega}_{L_1} \to \widetilde{\Omega}_{K_1}$$

which induces an isomorphism of L_1-modules

$$\begin{aligned} \Phi_{t_1,\ldots,t_d}\colon L_1 \otimes_L \mathcal{M}_X(\Omega_X^{r+d})_x = \widetilde{\Omega}^{r+d}_{L_1} &\xrightarrow{\sim} \mathrm{Hom}_{K_1}(L_1, \widetilde{\Omega}^{r+d}_{K_1}) \\ \omega &\longmapsto (s \mapsto \sigma_{L_1/K_1}(s\omega)) \end{aligned} \tag{1.7.1}$$

by [KD], (16.8). Via (1.7.1) the isomorphism $\widetilde{\Omega}^{r+d}_{K_1} \to K_1$, $\omega_0\, \mathrm{d}T_1 \ldots \mathrm{d}T_d \mapsto 1$ of K_1-modules induces an isomorphism

$$\Phi^{\omega_0}_{t_1,\ldots,t_d}\colon L_1 \otimes_L \mathcal{M}_X(\Omega_X^{r+d})_x = \widetilde{\Omega}^{r+d}_{L_1} \xrightarrow{\sim} \mathrm{Hom}_{K_1}(L_1, K_1) \tag{1.7.2}$$

By [KW], (Sect. 4), we have a commutative diagram of $\widehat{S}$-linear maps

$$\begin{array}{ccc} \widehat{S} \otimes_S \omega^{r+d}_{X/Y,x} & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\widehat{P}}(\widehat{S}, \widehat{P}) \\ \Big\downarrow{\scriptstyle \text{can.}} & & \Big\downarrow{\scriptstyle \text{can.}} \\ L_1 \otimes_L \mathcal{M}_X(\Omega_X^{r+d})_x & \xrightarrow[\Phi^{\omega_0}_{t_1,\ldots,t_d}]{\sim} & \mathrm{Hom}_{K_1}(L_1, K_1) \end{array} \tag{1.7.3}$$

with bijective vertical maps.

The composition of the R-linear maps

$$\begin{array}{l} \omega^{r+d}_{X/Y,x} \xrightarrow{\text{can.}} \widehat{S} \otimes_S \omega^{r+d}_{X/Y,x} \xrightarrow{\varphi^{\omega_0}_{t_1,\ldots,t_d}} \mathrm{Hom}_{\widehat{P}}(\widehat{S}, \widehat{P}) \longrightarrow \widehat{R} \otimes_R \Omega^r_{f(x)} = \widehat{\Omega}^r_{f(x)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad f \longrightarrow f(1)(0)\omega_0 \end{array}$$

is independent on the choice of ω_0. It will be denoted by

$$\mathrm{Res}_{X/Y,x}\begin{bmatrix} \\ t_1, \ldots, t_d \end{bmatrix}\colon \omega^{r+d}_{X/Y,x} \to \widehat{\Omega}^r_{f(x)} \tag{1.7.4}$$

and is called the *residue map of X/Y with respect to Ω at x for the system of parameters* $(t_1, \ldots, t_d)$.

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Now we repeat the definition of Cousin complexes and the notion of relative M-active sequences according to [Ke] as they are relevant for the following.

(1.8) *Remark.*

(a) Let S be a ring, $\mathcal{D}^{\bullet} \subset \bigcup_{i\in\mathbb{N}} S^i$ a system of denominators of S, and M an S-module. Fix $p \in \mathbb{N}$. For a sequence $f = (f_1, \dots, f_p) \in \mathcal{D}^p := \mathcal{D}^{\bullet} \cap S^p$ the set $S(f) := \{g \in S : (f_1, \dots, f_p, g) \in \mathcal{D}^{p+1}\} \subset S$ is multiplicatively closed. The family $\big(M_{S(f)}/(f)M_{S(f)}\big)_{f\in\mathcal{D}^p}$ is a directed system of S-modules where the morphisms are given by the transition determinants, and its colimit, denoted by $C^p_{\mathcal{D}}(M)$, is called the p^{th} *Cousin module* of M with respect to $\mathcal{D}^{\bullet}$ (cf. [Ke], Sect. 2). There is a complex, the augmented Cousin complex,

$$0 \to M \xrightarrow{\varepsilon} C^0_{\mathcal{D}}(M) \xrightarrow{d} C^1_{\mathcal{D}}(M) \xrightarrow{d} \cdots$$

of S-modules where $\epsilon(m) = \frac{m}{1}$ and $\left(\begin{bmatrix} m/g \\ f_1, \dots, f_p \end{bmatrix}\right) = \begin{bmatrix} m/1 \\ f_1, \dots, f_p, g \end{bmatrix}$, where $m \in M$ and $(f_1, \dots, f_p, g) \in \mathcal{D}^{p+1}$.

(b) Let S be a ring, and M a finitely generated S-module. A sequence $f_1, \dots, f_p \in S^p$ ($p \in \mathbb{N}$) is called M*-active* if $\dim_{S_\mathfrak{p}} M_\mathfrak{p} \geqslant j$ for all $\mathfrak{p} \in \mathrm{Supp}_S M$ with $(f_1, \dots, f_j) \subset \mathfrak{p}$ and for all $0 \leqslant j \leqslant p$. A sequence $(f_1, \dots, f_p) \in S^p$ is M-active if and only if f_i is $M/(f_1, \dots, f_{i-1})M$-active for all $1 \leqslant i \leqslant p$. By $A^p_S(M)$ we denote the set of all M-active sequences of length p. Then $A^{\bullet}_S(M) := \bigcup_{p\in\mathbb{N}} A^p_S(M)$ is a system of denominators of S in the sense of [Ke], Section 1. Now, let S be local with maximal ideal $\mathfrak{m}$ and $M \neq 0$. If we have $d := \dim_S M = \dim_S(M/\mathfrak{p}M) + \dim_{S_\mathfrak{p}}(M_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Supp}_S M$, then a sequence $(f_1, \dots, f_p)$ in $\mathfrak{m}$ is M-active if and only if $p \leqslant d$ and there exist $f_{p+1}, \dots, f_d \in \mathfrak{m}$ such that $M/(f_1, \dots, f_d)M$ has finite length. If M is a Cohen–Macaulay module over S, then a sequence in $\mathfrak{m}$ is M-active if and only if it is M-regular.

(c) Let $(R, \mathfrak{m})$ be a local ring, S an R-algebra, and M a finitely generated S-module. Then $A^{\bullet}_{S/R}(M) := A^{\bullet}_S(M/\mathfrak{m}M)$ is a system of denominators of S in the sense of [Ke], Section 1 and consists of all sequences that are relatively M-active with respect to S/R (cf. [Ke], (2.10)).

(1.9) LEMMA.

(a) *Let S be a ring, $N \subset S$ a multiplicatively closed subset, and M a finite S-module. Then there is a canonical isomorphism*

$$C^{\bullet}_{A_S(M)}(M)_N \longrightarrow C^{\bullet}_{A_{S_N}(M_N)}(M_N), \qquad \frac{\begin{bmatrix} m/g \\ f \end{bmatrix}}{t} \longmapsto \begin{bmatrix} \frac{m}{t}/\frac{g}{1} \\ \frac{f}{1} \end{bmatrix}$$

of complexes of S_N-modules.

(b) *Let $(R, \mathfrak{m})$ be a local ring, S an R-algebra such that* $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ *is closed. Further, let M be a finite S-module which is a flat R-module and*

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such that $M/\mathfrak{m}M$ is a Cohen–Macaulay module over S. Then the augmented Cousin complex

$$0 \to M \xrightarrow{\varepsilon} C^0_{A_{S/R}(M)}(M) \xrightarrow{d} C^1_{A_{S/R}(M)}(M) \xrightarrow{d} \cdots$$

is exact.

Proof. (a) is clear as the image of $A_S(M)$ in $A_{S_N}(M_N)$ is a cofinal subsystem of denominators by the lemma about the avoidance of prime ideals.

(b) Let $f \in A^{p+1}_{S/R}(M)$ ($p \in \mathbb{N}$) and $\mathfrak{p} \in \operatorname{Max} S \cap \operatorname{Supp}_S M$ with $(f)S \subset \mathfrak{p}$. Then $\mathfrak{p} \cap R = \mathfrak{m}$ and by (1.8) f is an $M_\mathfrak{p}/\mathfrak{m}M_\mathfrak{p}$-regular sequence. As $M_\mathfrak{p}$ is a flat R-module, f is $M_\mathfrak{p}$-regular by [Mat], (20.F), and the assumption follows by [Ke], (2.8). □

(1.10) COROLLARY. *Let X be a scheme, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

(a) *By* (1.8) *for any open and affine $U \subset X$ we have complexes*

$$0 \to \mathcal{F}(U)^\sim \xrightarrow{\varepsilon} C^0_{A_{\mathcal{O}_X(U)}(\mathcal{F}(U))}(\mathcal{F}(U))^\sim \xrightarrow{d} C^1_{A_{\mathcal{O}_X(U)}(\mathcal{F}(U))}(\mathcal{F}(U))^\sim \xrightarrow{d} \cdots$$

of $\mathcal{O}_U$-modules which by (1.9) *glue to a complex*

$$0 \to \mathcal{F} \xrightarrow{\varepsilon} C^0(X, \mathcal{A}^\bullet(\mathcal{F}), \mathcal{F}) \xrightarrow{d} C^1(X, \mathcal{A}^\bullet(\mathcal{F}), \mathcal{F}) \xrightarrow{d} \cdots \qquad (*)$$

of quasicoherent $\mathcal{O}_X$-modules.

(b) *If $\mathcal{F}$ is a Cohen–Macaulay (resp. Gorenstein) sheaf, i.e., $\mathcal{F}_x$ is a Cohen–Macaulay (resp. Gorenstein) $\mathcal{O}_{X,x}$-module for all $x \in X$, then the above complex $(*)$ is a resolution (resp. a minimal injective resolution) of the $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof.

(a) is obvious. (b) The Cousin complex defines a resolution (resp. injective resolution) of $\mathcal{F}$ by [Ke], (2.9) at all $x \in X$. The minimality of the Cousin complex follows by the definition of essential extensions. □

(1.11) *Remark.* The complex $(*)$ in (1.10) is the Cousin complex of $\mathcal{F}$ with respect to the sheaf $\mathcal{A}(\mathcal{F})$ of systems of denominators of locally $\mathcal{F}$-active sequences, i.e.,

$$\mathcal{A}^p(\mathcal{F}(U)) = \{(f_1, \ldots, f_p) \in \mathcal{O}_X(U)^p;\ (f_{1,x}, \ldots, f_{p,x}) \in A^p_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \text{ for all } x \in U\}$$

($p \in \mathbb{N}$, $U \subset X$ open) in the sense of [Ke], Section 4.

We are going to apply this mainly to the sheaf of regular differential forms. Given situation (1.6) and a closed point $x \in X$, set $y = f(x)$. Let $t, s \in A^d_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}}(\omega^{r+d}_{X/Y,x})$ with $(s) \subset (t) \subset \mathfrak{m}_x$, and Δ a transition determinant from s to t., i.e., $\Delta = \det(\lambda_{ij})$ with $\lambda_{ij} \in \mathcal{O}_{X,x}$ such that $s_i = \sum_{j=1}^d \lambda_{ji} f_j$, ($1 \leqslant i \leqslant d$). As X/Y is equidimensional of dimension d and as we have $\operatorname{Supp} \omega^{r+d}_{X/Y} = X$, we

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conclude from (1.8) that t and s are relative systems of parameters for X/Y at x, hence the residue maps $\mathrm{Res}_{X/Y,x}\left[\,t\,\right]$ and $\mathrm{Res}_{X/Y,x}\left[\,s\,\right]$ are defined, and a minor modification of [HK$_1$], (2.4) $\big($see also [Hü$_2$], (2.11)$\big)$ shows that for $\omega \in \omega^{r+d}_{X/Y,x}$

$$\mathrm{Res}_{X/Y,x}\begin{bmatrix}\omega\\ t\end{bmatrix} = \mathrm{Res}_{X/Y,x}\begin{bmatrix}\Delta\omega\\ s\end{bmatrix}.$$

From this we conclude

(1.12) THEOREM. *In Situation* (1.6) *for a closed point* $x \in X$, *the residue maps induce a morphism of* $\mathcal{O}_{Y,f(x)}$*-modules*

$$\mathrm{Res}_{X/Y,x}\colon \mathcal{C}^d_{A_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}(\omega^{r+d}_{X/Y,x})}(\omega^{r+d}_{X/Y,x}) \to \widehat{\Omega}^r_{f(x)}$$

called the residue of X/Y *at* x.

(1.13) *Remark.* We consider Situation (1.6) with $Y = \mathrm{Spec}\, R$ where R is a local and complete ring, and X is affine. Furthermore, let $Z \subset X$ be a closed subscheme which is finite over Y and whose ideal $\mathcal{I}_Z$ can locally be generated by d elements up to radical. Then the relative $\omega^{r+d}_{X/Y}$-active sequences contained in $\mathcal{I}_Z$ induce in a canonical way a system of denominators, and the above calculations show that the local residue symbol passes to the limit to define

$$\int_{X/Y,Z}\colon H^d_Z(X, \omega^{r+d}_{X/Y}) \to \Omega^r$$

called the local integral of X/Y with respect to Z (cf. also [HK$_1$], Sect. 4).

If in Situation (1.6) $Y = \mathrm{Spec}\, k$ for some field k, then by an easy calculation we get

$$f_*\mathcal{C}^q(X, \mathcal{A}(\omega^{r+d}_{X/Y}), \omega^{r+d}_{X/Y}) = \bigoplus_{\substack{x\in X\\ \dim \mathcal{O}_{X,x}=q}} \mathcal{C}^q_{A_{\mathcal{O}_{X,x}}(\omega^{r+d}_{X/Y,x})}(\omega^{r+d}_{X/Y,x}),$$

for all $q \in \mathbb{N}$. Thus we conclude

(1.14) THEOREM. *Given Situation* (1.6) *with* $Y = \mathrm{Spec}\, k$ *for some field* k, *the local residue symbols induce a* k*-morphism*

$$\int_{X/Y}\colon f_*\mathcal{C}^d(X, \mathcal{A}(\omega^{r+d}_{X/Y}), \omega^{r+d}_{X/Y}) \to \Omega^r$$

called the (global) integral of X/Y.

2. The adjunction morphism for the sheaf of regular differential forms

In this section we will define the adjunction morphism for the sheaf of regular differential forms. For this we fix the following situation:

(2.1) *Situation.* Let $f: X \to Y$ be a morphism of noetherian schemes which is equidimensional of dimension d and generically a complete intersection. Further, let $\iota : X' \to X$ be a closed immersion such that $f' := f \circ \iota : X' \to Y$ is equidimensional of dimension d' and generically a complete intersection. Assume that Y is universally japanese, the schemes X, X' and Y have no embedded primary components, and the conormal sheaf $\mathcal{C}_{X'/X} := \mathcal{I}/\mathcal{I}^2$ of X'/X is locally generated by $n := d - d'$ elements where $\mathcal{I}$ is the ideal sheaf defining the closed subscheme X' of X. Let Ω be an differential algebra on Y such that Ω is admissible for X/Y and X'/Y. Let r be the rank of Ω^1. Further, suppose that X/Y is a locally complete intersection at x and $\Omega^1_{X,x}$ is a free $\mathcal{O}_{X,x}$-module for all maximal points x of X'.

(2.2) LEMMA. *In Situation* (2.1) *the following holds true:*

(i) *The conormal sheaf $\mathcal{C}_{X'/X}$ and thus the normal sheaf $\mathcal{N}_{X'/X} := \mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{C}_{X'/X}, \mathcal{O}_{X'})$ of X'/X are locally free $\mathcal{O}_{X'}$-modules of rank n.*

(ii) *The canonical sequence of $\mathcal{O}_{X'}$-modules (cf. [KD],* (4.17)*)*

$$0 \to \mathcal{C}_{X'/X} \to \iota^*\Omega^1_X \to \Omega^1_{X'} \to 0$$

is exact and induces an exact sequence of locally free $\mathcal{M}_{X'}$-modules

$$0 \to \mathcal{M}_{X'}(\mathcal{C}_{X'/X}) \to \mathcal{M}_{X'}(\iota^*\Omega^1_X) \to \mathcal{M}_{X'}(\Omega^1_{X'}) \to 0 \tag{2.2.1}$$

of ranks n, $r + d$ resp. $r + d'$.

(iii) *The exact sequence* (2.2.1) *induces a canonical isomorphism*

$$\alpha_{X'/X} : \mathcal{M}_{X'}(\det \mathcal{N}_{X'/X}) \otimes \mathcal{M}_{X'}(\iota^*\Omega^{r+d}_X) \xrightarrow{\sim} \mathcal{M}_{X'}(\Omega^{r+d'}_{X'})$$

of $\mathcal{M}_{X'}$-modules.

(iv) *There is a canonical isomorphism*

$$\mathcal{M}_{X'}(\iota^*\Omega_X) \cong \mathcal{M}_{X'}(\iota^*\omega_{X/Y})$$

of $\mathcal{M}_{X'}$-modules.

Proof. The statements are local in X and Y. Therefore, we may assume that X and Y are affine such that $\mathcal{C}_{X'/X}$ is generated as an $\mathcal{O}_{X'}$-module by n global sections.

Set $R := \mathcal{O}_Y(Y)$, $S := \mathcal{O}_X(X)$, $S' := \mathcal{O}_{X'}(X')$, $I := \mathcal{I}(X)$ and $\Omega := \Omega(Y)$. Then $S' = S/I$ and I/I^2 is generated as an S'-module by n elements. By assumption, for all minimal prime divisor $\mathfrak{p} \in \operatorname{Spec} S$ of I we have S/R is a locally complete intersection at $\mathfrak{p}$ and $(\Omega^1_S)_\mathfrak{p}$ is a free $S_\mathfrak{p}$-module of rank $r + d$. By [KW], (3.9) and (3.6) we get $(\Omega_S)_\mathfrak{p} = (\omega_{S/R})_\mathfrak{p}$. By assumption, S' has no embedded primary components. Therefore the full quotient ring $L' := Q(S')$ of S' has the

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form $L' = \bigoplus_{\mathfrak{q}} S_{\mathfrak{q}}/IS_{\mathfrak{q}}$ where $\mathfrak{q} \in \operatorname{Spec} S$ runs over all minimal prime divisors of I, and hence $L' \otimes_S \Omega_S = L' \otimes_S \omega_{S/R}$ and $L' \otimes_S \Omega^1_S$ is a free L'-module of rank $r+d$. In particular, there is a canonical isomorphism

$$\mathcal{M}_{X'}(\iota^*\Omega_X) = (L' \otimes_S \Omega_S)^{\sim} \cong (L' \otimes_S \omega_{S/R})^{\sim} = \mathcal{M}_{X'}(\iota^*\omega_{X/Y})$$

of $\mathcal{M}_{X'}$-modules, and $\mathcal{M}_{X'}(\iota^*\Omega^1_X)$ is a locally free $\mathcal{M}_{X'}$-module of rank $r+d$. By assumption, $L' \otimes_{S'} \Omega^1_{S'}$ is a free L'-module of rank $r+d'$ and by [KD], (4.17) we have a canonical exact sequence of S'-modules

$$I/I^2 \xrightarrow{\alpha} S' \otimes_S \Omega^1_S \xrightarrow{\beta} \Omega^1_{S'} \to 0. \tag{2.2.2}$$

Let M denote the kernel of the canonical map β. Then there is a canonical surjection $I/I^2 \twoheadrightarrow M$. Thus, M is generated as an S'-modules by n elements, and there is an exact sequence of L'-modules

$$0 \to L' \otimes_{S'} M \to L' \otimes_S \Omega^1_S \to L' \otimes_{S'} \Omega^1_{S'} \to 0,$$

where $L' \otimes_S \Omega^1_S$ resp. $L' \otimes_{S'} \Omega^1_{S'}$ are free L'-modules of rank $r+d$ resp. $r+d'$. Consequently, $L' \otimes_{S'} M$ is a free L'-module of rank $n = r+d-(r+d')$, and as M is generated by n elements, M is a free S'-module of rank n. Since I/I^2 is generated by n elements, the canonical surjection $I/I^2 \twoheadrightarrow M$ is bijective, and hence α is injective and (2.2.2) induces an exact sequence

$$0 \to L' \otimes_{S'} I/I^2 \xrightarrow{\alpha_{L'}} L' \otimes_S \Omega^1_S \xrightarrow{\beta_{L'}} L' \otimes_{S'} \Omega^1_{S'} \to 0$$

of free L'-modules of ranks n, $r+d$ and $r+d'$. Let $f_1, \dots, f_n$ be a basis of I/I^2 as an S'-module, and $f_1^*, \dots, f_n^* \in \operatorname{Hom}_{S'}(I/I^2, S')$ its dual basis. Further, choose $\omega_1, \dots, \omega_{r+d'} \in L' \otimes_S \Omega^1_S$ such that $\beta_{L'}(\omega_1), \dots, \beta_{L'}(\omega_{r+d'})$ is a basis of $L' \otimes_S \Omega^1_{S'}$ as an L'-module. Then $\omega_1 \wedge \cdots \wedge \omega_{r+d'} \wedge (1 \otimes df_1) \wedge \cdots \wedge (1 \otimes df_n)$ is an L'-basis of $L' \otimes_S \Omega^{r+d}_S$, and there is a unique isomorphism of L'-modules

$$(L' \otimes_{S'} \det(\operatorname{Hom}_{S'}(I/I^2, S'))) \otimes_{S'} (L' \otimes_S \Omega^{r+d}_S) \xrightarrow{\sim} L' \otimes_{S'} \Omega^{r+d'}_{S'} \tag{2.2.3}$$

which maps $(1 \otimes (f_1^* \wedge \cdots \wedge f_n^*)) \otimes (\omega_1 \wedge \cdots \wedge \omega_{r+d'}) \wedge (1 \otimes df_1) \wedge \cdots \wedge (1 \otimes df_n)$ to $\beta_{L'}(\omega_1) \wedge \cdots \wedge \beta_{L'}(\omega_{r+d'})$. This map is independent of the choice of $f_1, \dots, f_n \in I/I^2$ and $\omega_1, \dots, \omega_{r+d'} \in L' \otimes_{S'} \Omega^1_{S'}$. The L'-linear map (2.2.3) corresponds to an isomorphism

$$\alpha_{X'/X} \colon \mathcal{M}_{X'}(\det \mathcal{N}_{X'/X}) \otimes \mathcal{M}_{X'}(\iota^*\Omega^{r+d}_X) \xrightarrow{\sim} \mathcal{M}_{X'}(\Omega^{r+d'}_{X'})$$

of $\mathcal{M}_{X'}$-modules. □

Now, we formulate and prove our first main result.

(2.3) THEOREM (Adjunction). *In Situation* (2.1) *there is a unique morphism*

$$\operatorname{Res}_{X'/X} \colon \det \mathcal{N}_{X'/X} \otimes_{\mathcal{O}_{X'}} \iota^*\omega^{r+d}_{X/Y} \longrightarrow \omega^{r+d'}_{X'/Y}$$

of $\mathcal{O}_{X'}$-modules such that the diagram

$$\begin{tikzcd}
\det \mathcal{N}_{X'/X} \otimes_{\mathcal{O}_{X'}} \iota^*\omega^{r+d}_{X/Y} \arrow[rr, "\mathrm{Res}_{X'/X}"] \arrow[d, "\mathrm{can.}"] & & \omega^{r+d'}_{X'/Y} \arrow[dd, "\mathrm{can.}"'] \\
\mathcal{M}_{X'}(\det \mathcal{N}_{X'/X}) \otimes_{\mathcal{O}_{X'}} \mathcal{M}_{X'}(\iota^*\omega^{r+d}_{X/Y}) \arrow[d, "\mathrm{id}\otimes\mathrm{can.}", "\wr"'] & & \\
\mathcal{M}_{X'}(\det \mathcal{N}_{X'/X}) \otimes_{\mathcal{O}_{X'}} \mathcal{M}_{X'}(\iota^*\Omega^{r+d}_{X}) \arrow[rr, "\alpha_{X'/X}", "\sim"'] & & \mathcal{M}_{X'}(\Omega^{r+d'}_{X'})
\end{tikzcd}$$

commutes. It is called the adjunction morphism for the sheaf of regular differential forms of X'/X with respect to Ω.

Proof. The uniqueness of $\mathrm{Res}_{X'/X}$ is obvious. Let a be defined by the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{X'}(\det \mathcal{N}_{X'/X}) \otimes \mathcal{M}_{X'}(\iota^*\omega^{r+d}_{X/Y}) \arrow[d, "\mathrm{id}\otimes\mathrm{can.}", "\wr"'] & \det \mathcal{N}_{X'/X} \otimes \iota^*\Omega^{r+d}_{X} \arrow[l, "\mathrm{can.}"'] \arrow[d, "a"] \\
\mathcal{M}_{X'}(\det \mathcal{N}_{X'/X}) \otimes \mathcal{M}_{X'}(\iota^*\Omega^{r+d}_{X}) \arrow[r, "\alpha_{X'/X}", "\sim"'] & \mathcal{M}_{X'}(\Omega^{r+d'}_{X'})
\end{tikzcd} \tag{2.3.1}$$

For the existence of $\mathrm{Res}_{X'/X}$ it suffices to show $\mathrm{Im}(a) \subset \omega^{r+d'}_{X'/Y}$. The statement is local in X and Y, and we may assume X and Y to be affine. Because of the local-global principle it is enough to show this for the stalks in the closed points of X'.

Let $x \in X'$ be a closed point. We have $\mathcal{O}_{X',x} = \mathcal{O}_{X,x}/\mathcal{I}_x$. Since $\mathcal{C}_{X'/X,x}$ is generated as an $\mathcal{O}_{X',x}$-module by n elements, Nakayama's lemma implies that the ideal $\mathcal{I}_x$ is generated by $n = d - d'$ elements. We see that there is a system of parameters $(t_i)_{i=1}^{d}$ of X/Y at x such that $t_{d'+1}, \dots, t_d$ generate the ideal $\mathcal{I}_x$, and the residue classes $\bar{t}_i := t_i + \mathcal{I}_x \in \mathcal{O}_{X',x}$, $(1 \leqslant i \leqslant d')$ are a system of parameters of X'/Y at x. Possibly after shrinking X, we may assume by (1.2) that the systems of parameters $(t_i)_{i=1}^{d}$ resp. $(\bar{t}_i)_{i=1}^{d'}$ of X/Y resp. X'/Y at x define quasi-normalizations

$$g\colon X \to Q := \mathbb{A}^d_Y \quad \text{resp.}\; g'\colon X' \to Q' := \mathbb{A}^{d'}_Y$$

of X/Y resp. X'/Y.

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Set $R := \mathcal{O}_{Y,f(x)}$, $P := \mathcal{O}_{Q,g(x)}$, $P' := \mathcal{O}_{Q',g'(x)}$, $S := \mathcal{O}_{X,x}$, $S' := \mathcal{O}_{X',x}$ and $I := \mathcal{I}_x$. Let K_1, K_1', L_1 and L_1' the full quotient rings of $\widehat{P}$, $\widehat{P}'$, $\widehat{S}$ and $\widehat{S}'$. Further, let $\omega_0 \in \Omega^r_{f(x)}$ be an R-basis of $\Omega^r_{f(x)}$.

Then by (1.7), we have $\widehat{P} = \widehat{R}[[T_1, \dots, T_d]]$ and $\widehat{P}' = \widehat{R}'[[T_1, \dots, T_{d'}]]$ and isomorphisms

$$\varphi := \varphi^{\omega_0}_{t_1,\dots,t_d} \colon \widehat{S} \otimes_S \omega^{r+d}_{X/Y,x} \xrightarrow{\sim} \mathrm{Hom}_{\widehat{P}}(\widehat{S}, \widehat{P}),$$

$$\varphi' := \varphi^{\omega_0}_{\bar{t}_1,\dots,\bar{t}_{d'}} \colon \widehat{S}' \otimes_{S'} \omega^{r+d'}_{X'/Y,x} \xrightarrow{\sim} \mathrm{Hom}_{\widehat{P}'}(\widehat{S}', \widehat{P}')$$

and

$$\Phi := \Phi^{\omega_0}_{\bar{t}_1,\dots,\bar{t}_{d'}} \colon L_1' \otimes_{L'} \mathcal{M}_{X'}(\Omega^{r+d'}_{X'})_x \xrightarrow{\sim} \mathrm{Hom}_{K_1'}(L_1', K_1')$$

of $\widehat{S}$-, resp. $\widehat{S}'$- resp. L_1'-modules.

By (2.2), $\mathcal{C}_{X'/X,x}$ is free with basis $\bar{t}_{d'+1}, \dots, \bar{t}_d$ where $\bar{t}_i := t_i + I^2 \in I/I^2$, ($d' < i \leqslant d$). Therefore we have an isomorphism

$$\alpha_{\bar{t}_{d'+1},\dots,\bar{t}_d} \colon \widehat{S}' \otimes_{S'} \det \mathcal{N}_{X'/X,x} \xrightarrow{\sim} \widehat{S}', \qquad 1 \otimes \bar{t}^*_{d'+1} \wedge \cdots \wedge \bar{t}^*_d \mapsto 1$$

of $\widehat{S}'$-modules where $\bar{t}^*_{d'+1}, \dots, \bar{t}^*_d$ denotes the dual basis of the basis $\bar{t}_{d'+1}, \dots, \bar{t}_d$ of I/I^2. Let $J := (T_{d'+1}, \dots, T_d)\widehat{P}$. Then $\widehat{P}' = \widehat{P}/J$ and $\widehat{S}' = \widehat{S}/J\widehat{S}$, and by (1.7) $L_1' = K_1' \otimes_{\widehat{P}'} \widehat{S}'$. The maps $\alpha_{\bar{t}_{d'+1},\dots,\bar{t}_d}$ and $\varphi^{\omega_0}_{t_1,\dots,t_d}$ induce an isomorphism

$$\gamma := \gamma^{\omega_0}_{t_1,\dots,t_d} \colon \widehat{S}' \otimes_{S'} (\det \mathcal{N}_{X'/X} \otimes_{\mathcal{O}_{X'}} \iota^* \omega^{r+d}_{X/Y})_x \xrightarrow{\sim} \mathrm{Hom}_{\widehat{P}}(\widehat{S}, \widehat{P})/J\mathrm{Hom}_{\widehat{P}}(\widehat{S}, \widehat{P})$$

of $\widehat{S}'$-modules.

(2.4) LEMMA. *The diagram of $\widehat{S}'$-linear maps*

$$\begin{tikzcd}
\widehat{S}' \otimes_{S'} (\det \mathcal{N}_{X'/X} \otimes_{\mathcal{O}_{X'}} \iota^* \omega^{r+d}_{X/Y})_x \arrow[r, "\sim", "\gamma^{\omega_0}_{t_1,\dots,t_d}"'] \arrow[d, "b_x := \mathrm{incl.} \otimes a_x"] & \mathrm{Hom}_{\widehat{P}}(\widehat{S}, \widehat{P})/J\mathrm{Hom}_{\widehat{P}}(\widehat{S}, \widehat{P}) \arrow[d, "c"', "\mathrm{can.}"] \\
L_1' \otimes_{L'} \mathcal{M}_{X'}(\Omega^{r+d'}_{X'})_x \arrow[r, "\sim", "\Phi^{\omega_0}_{\bar{t}_1,\dots,\bar{t}_{d'}}"'] & \mathrm{Hom}_{K_1'}(L_1', K_1')
\end{tikzcd}$$

is commutative.

We suppose, that (2.4) is proved, and finish the proof of (2.3).

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As the canonical map c from (2.4) factors through $\mathrm{Hom}_{\widehat{P}'}(\widehat{S}', \widehat{P}')$, it follows with (1.7) that $\mathrm{Im}(b_x) \subset \widehat{S}' \otimes \omega^{r+d'}_{X'/Y,x}$ and as $\widehat{S}'/S'$ is faithfully flat, this implies the assertion. $\square$

Proof of (2.4). The proof will be done in several steps.

Step 1. Consequences of (1.7).

The algebras $\widehat{S}/\widehat{P}$ and $\widehat{S}'/\widehat{P}'$ are finite, equidimensional of dimension 0 and generically locally complete intersections. The rings $\widehat{R}$, $\widehat{P}$, $\widehat{P}'$, $\widehat{S}$ and $\widehat{S}'$ have no embedded primary components and are universally japanese. We have $\widehat{P}' = \widehat{P}/J$, $J\widehat{S} = I\widehat{S}$ and $\widehat{S}' = \widehat{S}/J\widehat{S}$. The algebras L_1/K_1 and L_1'/K_1' are finite locally complete intersections, and we have

$$L_1 = K_1 \otimes_{\widehat{P}} \widehat{S} \quad \text{and} \quad L_1' = K_1' \otimes_{\widehat{P}'} \widehat{S}'.$$

If $N := \widehat{P} \setminus \bigcup_{\mathfrak{p}} \mathfrak{p}$ where $\mathfrak{p}$ runs over all minimal prime divisors of J, then

$$K_1' = \widehat{P}_N/J\widehat{P}_N \quad \text{and} \quad L_1' = \widehat{S}_N/I\widehat{S}_N = \widehat{S}_N/J\widehat{S}_N. \tag{2.4.1}$$

The universally finite extensions $\widetilde{\Omega}$, resp. $\widetilde{\Omega}'$ of $\Omega_{f(x)}$ to $\widehat{P}$, resp. $\widehat{P}'$ exist, and we have $\widetilde{\Omega}' = \widetilde{\Omega}_{\widehat{P}'}$. The $\widehat{P}$- resp. $\widehat{P}'$-modules $\widetilde{\Omega}^1$ resp. $\widetilde{\Omega}'^1$ are free of rank $r+d$ resp. $r+d'$. We have

$$\widetilde{\Omega}_{\widehat{S}} = \widehat{S} \otimes_S \Omega_S \quad \text{and} \quad \widetilde{\Omega}_{L_1} = L_1 \otimes_L \Omega_L \ \text{ resp.,}$$

$$\widetilde{\Omega}'_{\widehat{S}'} = \widehat{S}' \otimes_{S'} \Omega_{S'} \quad \text{and} \quad \widetilde{\Omega}'_{L_1'} = L_1' \otimes_{L'} \Omega_{L'},$$

where $L := Q(S)$ and $L' := Q(S')$ are the full quotient rings of S and S'. Then $\widetilde{\Omega}_{L_1}$ resp. $\widetilde{\Omega}'_{L_1'}$ are free L_1- resp. L_1'-modules of rank $r+d$ resp. $r+d'$. The module of regular differential forms of $\widehat{S}/\widehat{P}$ with respect to $\widetilde{\Omega}$ resp. of $\widehat{S}'/\widehat{P}'$ with respect to $\widetilde{\Omega}'$ in the sense of [KW], (Sect. 3) are defined. By [KW], (3.14) and [KW] (3.6) in connection with [M$_1$], (24.C) we have:

$$\widehat{S} \otimes_S \omega^{r+d}_{X/Y,x} = (\omega^{\widetilde{\Omega}}_{\widehat{S}/\widehat{P}})^{r+d} \quad \text{and} \quad \widehat{S}' \otimes_{S'} \omega^{r+d'}_{X'/Y,x} = (\omega^{\widetilde{\Omega}'}_{\widehat{S}'/\widehat{P}'})^{r+d'}.$$

Step 2. We prove that $\widehat{S}_N/\widehat{P}_N$ is a finite locally complete intersection, and $\widetilde{\Omega}^1_{\widehat{S}_N}$ is a projective $\widehat{S}_N$-module of rank $r+d$.

The algebra $\widehat{S}_N/\widehat{P}_N$ is finite. Let $\mathfrak{n} \in \mathrm{Max}(\widehat{S}_N)$. Then $\mathfrak{n} \cap \widehat{P}_N \in \mathrm{Max}(\widehat{P}_N)$ and therefore contains $J\widehat{P}_N$. In particular, $I\widehat{S}_N = J\widehat{S}_N \subset \mathfrak{n}$. Hence $\mathfrak{n} \cap \widehat{S}$ is a minimal prime divisor of $I\widehat{S}$ and consequently $\mathfrak{m} := \mathfrak{n} \cap S$ is a minimal prime divisor of I. Set $\mathfrak{P} := \mathfrak{n} \cap P$ and $\mathfrak{p} := \mathfrak{n} \cap R$. As $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ resp. $S'_{\mathfrak{p}}/\mathfrak{p}S'_{\mathfrak{p}}$ are equidimensional of dimension d resp. d', and $\mathfrak{m}$ is a minimal prime divisor of I, we get

$$\dim S_{\mathfrak{m}}/\mathfrak{p}S_{\mathfrak{m}} = \dim(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}) - \dim(S_{\mathfrak{p}}/\mathfrak{m}S_{\mathfrak{p}}) = d - d' = n.$$

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Analogously, we see that $\dim(P_{\mathfrak{P}}/\mathfrak{p}P_{\mathfrak{P}}) = n$. By assumption and (1.3), $\widehat{S}/\widehat{P}$ is a locally complete intersection at $\mathfrak{n} \cap \widehat{S}$ and therefore $\widehat{S}_N/\widehat{P}_N$ is locally a complete intersection at $\mathfrak{n}$. Further, $(\widetilde{\Omega}^1_{\widehat{S}_N})_{\mathfrak{n}} = (\widehat{S}_N)_{\mathfrak{n}} \otimes_{S_{\mathfrak{m}}} \Omega^1_{S_{\mathfrak{m}}}$. By assumption, $\Omega^1_{S_{\mathfrak{m}}}$ is a locally free $S_{\mathfrak{m}}$-module, necessarily of rank $r+d$, and hence $(\widetilde{\Omega}^1_{\widehat{S}_N})_{\mathfrak{n}}$ is a locally free $(\widehat{S}_N)_{\mathfrak{n}}$-module of rank $r+d$. The assertion follows by (1.5).

Step 3. Now we show a commutative diagram of traces of differential algebras. We have shown that there is a commutative diagram of ring morphisms

$$\begin{tikzcd} K_1 \arrow[r, "\alpha_1"] \arrow[d, "\text{can.}"] & (K_1)_N \arrow[d, "\text{can.}"] & \widehat{P}_N \arrow[l, "\alpha_2"'] \arrow[r, "\alpha_3"] \arrow[d, "\text{can.}"] & K_1' \arrow[d, "\text{can.}"] \\ L_1 \arrow[r, "\beta_1"] & (L_1)_N & \widehat{S}_N \arrow[l, "\beta_2"'] \arrow[r, "\beta_3"] & L_1' \end{tikzcd} \tag{2.4.3}$$

where the vertical maps are finite locally complete intersections which are induced by the canonical map $\widehat{P} \to \widehat{S}$ by base change. By [KD], (Sect. 16, Tr.3), there is an associated diagram of differential algebras

$$\begin{tikzcd} \widetilde{\Omega}_{K_1} \arrow[r, "\alpha_1"] & \widetilde{\Omega}_{(K_1)_N} & \widetilde{\Omega}_{\widehat{P}_N} \arrow[l, "\alpha_2"'] \arrow[r, "\alpha_3"] & \widetilde{\Omega}_{K_1'} = \widetilde{\Omega}'_{K_1'} \\ \widetilde{\Omega}_{L_1} \arrow[r, "\beta_1"] \arrow[u, "\sigma_{L_1/K_1}"] & \widetilde{\Omega}_{(L_1)_N} \arrow[u, "\sigma_{(L_1)_N/(K_1)_N}"] & \widetilde{\Omega}_{\widehat{S}_N} \arrow[l, "\beta_2"'] \arrow[r, "\beta_3"] \arrow[u, "\sigma_{\widehat{S}_N/\widehat{P}_N}"] & \widetilde{\Omega}_{L_1'} = \widetilde{\Omega}'_{L_1'} \arrow[u, "\sigma_{L_1'/K_1'}"] \end{tikzcd} \tag{2.4.4}$$

where the horizontal maps are the morphisms of differential algebras induced by the horizontal maps in (2.4.3). The vertical maps are the traces corresponding to the ring morphisms in (2.4.3) in the sense of [KD], (Sect. 16).

Step 4. Proof of (2.4).
Let $\omega \in (\omega^{\widetilde{\Omega}}_{\widehat{S}/\widehat{P}})^{r+d}$. By [KW], (3.10) Step 2 implies

$$\widetilde{\Omega}_{\widehat{S}_N} = \omega^{\widetilde{\Omega}_{\widehat{P}_N}}_{\widehat{S}_N/\widehat{P}_N} = (\omega^{\widetilde{\Omega}}_{\widehat{S}/\widehat{P}})_N \tag{2.4.5}$$

and for the image of ω under the canonical map $\omega^{\widetilde{\Omega}}_{\widehat{S}/\widehat{P}} \to \widetilde{\Omega}_{\widehat{S}_N}$ we also write ω. By Step 2 using rank arguments, we see that the canonical sequence (cf. [KD], (4.17))

$$0 \to I\widehat{S}_N/I^2\widehat{S}_N \to \widetilde{\Omega}^1_{\widehat{S}_N}/I\widetilde{\Omega}^1_{\widehat{S}_N} \to \widetilde{\Omega}^1_{L_1'} \to 0$$

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is an exact sequence of free L'_1-modules of ranks n, $r+d$ and $r+d'$. Hence there is some $\omega_1 \in \widetilde{\Omega}^{r+d'}_{\widehat{S}_N}$ such that in $\widetilde{\Omega}^{r+d}_{\widehat{S}_N}$

$$\omega \equiv \omega_1 \wedge \mathrm{d}t_{d'+1} \wedge \ldots \wedge \mathrm{d}t_d \quad \operatorname{mod} I\widetilde{\Omega}^{r+d}_{\widehat{S}_N} = J\widetilde{\Omega}^{r+d}_{\widehat{S}_N}. \tag{2.4.6}$$

We see from the construction of a $\big($cf. (2.4.2)$\big)$ that

$$b_x(\bar{t}^*_{d'+1} \wedge \cdots \wedge \bar{t}^*_d \otimes \bar{\omega}) = \beta_3(\omega_1), \tag{2.4.7}$$

where $\bar{\omega}$ denotes the residue class of ω in $\omega^{r+d}_{\widehat{S}/\widehat{P}}/I\,\omega^{r+d}_{\widehat{S}/\widehat{P}}$.

Let $s \in \widehat{S}$, $z := \varphi(\omega)(s) \in \widehat{P}$, and let $\bar{s}$ resp. $\bar{z}$ be the residue class of s resp. z in $\widehat{S}' = \widehat{S}/I\widehat{S}$ resp. $\widehat{P}' = \widehat{P}/J$.

By the definition of φ (cf. (1.7)) we have in $\widetilde{\Omega}^{r+d} \subset \widetilde{\Omega}^{r+d}_{K_1}$:

$$z\omega_0 \wedge \mathrm{d}T_1 \wedge \ldots \wedge \mathrm{d}T_d = \sigma_{L_1/K_1}(s\omega)$$

and hence in $\widetilde{\Omega}^{r+d}_{\widehat{P}_N} \subset \widetilde{\Omega}^{r+d}_{(K_1)_N}$ using (2.4.4), (2.4.5) and (2.4.6) we have

$$\begin{aligned}
z\omega_0 \wedge \mathrm{d}T_1 \wedge \ldots \wedge \mathrm{d}T_d &= \sigma_{(L_1)_N/(K_1)_N}(s\,\omega) \\
&= \sigma_{\widehat{S}_N/\widehat{P}_N}(s\,\omega) \\
&\equiv \sigma_{\widehat{S}_N/\widehat{P}_N}(s\,\omega_1 \wedge \mathrm{d}t_{d'+1} \wedge \cdots \wedge \mathrm{d}t_d) \quad \operatorname{mod} J\widetilde{\Omega}^{r+d}_{\widehat{P}_N} \\
&= \sigma_{\widehat{S}_N/\widehat{P}_N}(s\,\omega_1) \wedge \mathrm{d}T_{d'+1} \wedge \cdots \wedge \mathrm{d}T_d.
\end{aligned}$$

Consequently, in $\widetilde{\Omega}^{r+d'}_{\widehat{P}_N}$ we have the congruence

$$z\omega_0 \wedge \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_{d'} \equiv \sigma_{\widehat{S}_N/\widehat{P}_N}(s\,\omega_1)$$

modulo $J\widetilde{\Omega}^{r+d'}_{\widehat{P}_N} + \mathrm{d}T_{d'+1}\widetilde{\Omega}^{r+d'-1}_{\widehat{P}_N} + \cdots + \mathrm{d}T_d\widetilde{\Omega}^{r+d'-1}_{\widehat{P}_N}$. By (2.4.4) and the definition of Φ (cf. (1.7)), this implies the equation in $\widetilde{\Omega}'_{K'_1}$

$$\begin{aligned}
\bar{z}\omega_0 \wedge \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_{d'} &= \alpha_3(\sigma_{\widehat{S}_N/\widehat{P}_N}(s\,\omega_1)) \\
&= \sigma_{L'_1/K'_1}(s\beta_3(\omega_1)) \\
&= \Phi(\beta_3(\omega_1))(\bar{s})\,\omega_0 \wedge \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_{d'}.
\end{aligned}$$

Therefore, using the definition of γ, we get

$$(c \circ \gamma)(\bar{t}^*_{d'+1} \wedge \cdots \wedge \bar{t}^*_d \otimes \bar{\omega})(\bar{s}) = \bar{z} = \Phi(\beta_3(\omega_1))(\bar{s}).$$

Now (2.4) follows from (2.4.7), the fact that all maps in (2.4) are $\widehat{S}$-linear, and (2.4.1). $\square$

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(2.5) *Remark.* (Flat base change).

Given Situation (2.1) and a flat morphism $g : Y_1 \to Y$, there is commutative diagram of morphisms of schemes

$$\begin{tikzcd} X_1' \arrow[r, "\iota_1"] \arrow[d, "g_1'"] & X_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & Y_1 \arrow[d, "g"] \\ X' \arrow[r, "\iota"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

with cartesian squares. Assume that $\Omega_1 := g^*\Omega$ has the structure of a differential algebra on Y_1 such that for open affine sets $V \subset Y$ and $V_1 \subset g^{-1}(V)$ the canonical map $\Omega(V) \to \mathcal{O}_{Y_1}(V_1) \otimes_{\mathcal{O}_Y(V)} \Omega(V) = \Omega_1(V_1)$ is a morphism of differential algebras. If X_1', X_1 and Y_1 have no embedded primary components and Y_1 is universally japanese, then the morphism f_1, the closed immersion ι_1 and the differential algebra Ω_1 on Y_1 satisfy the conditions of situation (2.1) and there are canonical isomorphisms

$$\begin{aligned} \omega^{\Omega_1}_{X_1/Y_1} &\cong g_1^*\omega_{X/Y}, \\ \omega^{\Omega_1}_{X_1'/Y_1} &\cong {g_1'}^*\omega_{X'/Y}, \qquad \mathcal{C}_{X_1'/X_1} \cong {g_1'}^*\mathcal{C}_{X'/X} \end{aligned} \tag{2.5.1}$$

of $\mathcal{O}_{X_1}$- resp. $\mathcal{O}_{X_1'}$-modules (cf. [KW], (3.13)). The diagram of morphisms of $\mathcal{O}_{X_1'}$-modules

$$\begin{tikzcd} {g_1'}^*(\det \mathcal{N}_{X'/X} \otimes_{\mathcal{O}_{X'}} \iota^*(\omega^{\Omega}_{X/Y})^{r+d}) \arrow[r, "{g_1'}^* \mathrm{Res}_{X'/X}"] \arrow[d, "\wr", "\text{can.}"'] & {g_1'}^*(\omega^{\Omega}_{X'/Y})^{r+d'} \arrow[d, "\wr", "\text{can.}"'] \\ \det \mathcal{N}_{X_1'/X_1} \otimes_{\mathcal{O}_{X_1'}} \iota_1^*(\omega^{\Omega_1}_{X_1/Y_1})^{r+d} \arrow[r, "\mathrm{Res}_{X_1'/X_1}"] & (\omega^{\Omega_1}_{X_1'/Y_1})^{r+d'} \end{tikzcd}$$

commutes, where the vertical morphisms are induced by the isomorphisms from (2.5.1).

The following proposition is sometimes useful to gain results on the adjunction morphism as is illustrated by the next corollary and examples. It will also be applied to get the second main result about the adjunction morphism and the local residue maps.

(2.6) PROPOSITION. *In Situation* (2.1), *let* $x \in X'$ *be a closed point and* $(t_i)_{i=1}^d$ *a system of parameters of* X/Y *at* x *such that* $t_{d'+1}, \ldots, t_d$ *generate the ideal* $\mathcal{I}_x \subset \mathcal{O}_{X,x}$. *Via the* $\widehat{\mathcal{O}}_{Y,f(x)}$*-algebra morphism*

$$P := \widehat{\mathcal{O}}_{Y,f(x)}[[T_1, \ldots, T_d]] \to \widehat{\mathcal{O}}_{X,x} =: S, \qquad T_i \mapsto t_i.$$

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S is a finite P-algebra and the residue classes $\bar{t}_i$ of t_i in $\mathcal{O}_{X',x} = \mathcal{O}_{X,x}/\mathcal{I}_x$, $(1 \leqslant i \leqslant d')$ are a system of parameters for X'/Y at x. If J denotes the ideal of P generated by $T_{d'+1}, \dots, T_d$, then $\widehat{\mathcal{O}}_{X',x} = S/JS$ and we have the commutative diagram of $\widehat{\mathcal{O}}_{X',x}$-linear maps

$$\begin{tikzcd}
\widehat{\mathcal{O}}_{X',x} \otimes (\det \mathcal{N}_{X'/X} \otimes \iota^*\omega^{r+d}_{X/Y})_x \arrow[r, "\sim", "\gamma^{\omega_0}_{t_1,\dots,t_d}"'] \arrow[d, "\mathrm{id}\otimes(\mathrm{Res}_{X'/X})_x"] & \mathrm{Hom}_P(S,P)/J\mathrm{Hom}_P(S,P) \arrow[d, "c"', "\text{can.}"] \\
\widehat{\mathcal{O}}_{X',x} \otimes \omega^{r+d'}_{X'/Y} \arrow[r, "\sim", "\varphi^{\omega_0}_{\bar{t}_1,\dots,\bar{t}_{d'}}"'] & \mathrm{Hom}_{P/J}(S/JS, P/J)
\end{tikzcd}$$

with bijective horizontal maps which are defined in the proof of (2.3).

(2.7) COROLLARY.

(i) *Given Situation* (2.1) *with $n = 1$, then the adjunction morphism* $\mathrm{Res}_{X'/X}$ *of X'/X with respect to Ω is injective.*

(ii) *If $x \in X'$ in Situation* (2.1) *is a closed point such that $f: X \to Y$ at x is Cohen–Macaulay, then the adjunction morphism* $\mathrm{Res}_{X'/X}$ *is bijective at x. In particular,* $\mathrm{Res}_{X'/X}$ *is an isomorphism if X/Y is Cohen–Macaulay at all points of X'.*

Proof. Without loss of generality we may assume X and Y to be affine. Let $x \in X'$ be a closed point, and $t_1, \dots, t_d \in \mathcal{O}_{X,x}$ a system of parameters of X/Y at x satisfying the assumptions of (2.6). With the notions of (2.6), we have

$(\mathrm{Res}_{X'/X})_x$ is injective, resp. surjective, resp. bijective if and only if the canonical map

$$c\colon \mathrm{Hom}_P(S,P)/J\mathrm{Hom}_P(S,P) \to \mathrm{Hom}_{P/J}(S/JS, P/J)$$

is injective, resp. surjective, resp. bijective, since $\widehat{\mathcal{O}}_{X',x}/\mathcal{O}_{X',x}$ is faithfully flat. If $n = 1$, then c is injective as T_d is a non-zerodivisor of P. If X/Y is Cohen–Macaulay at x, then by (1.3) S is a finite free P-module and hence c bijective. $\square$

The statements of Corollary 1 become wrong if we omit the assumption '$n = 1$' in (i) and if we don't assume X/Y to be Cohen–Macaulay at x in (ii), as we can see from the following examples.

(2.8) EXAMPLES. The following examples, which R. Waldi helped us to find, show that in Situation (2.1), the adjunction morphism is neither surjective nor injective in general:

Let k be a perfect field of characteristic $\mathrm{char}(k) \neq 5$, and denote by A the Segre product of $k[X_0, X_1, X_2, X_3, X_4]/(X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5)$ with $k[Y_0, Y_1]$

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over k. Then A is an integral affine k-algebra of dimension 5 with one and only one singular point $\mathfrak{m} \in \operatorname{Spec} A$, and we have depth $A_{\mathfrak{m}} = 4$ (cf. [SV], Chap. I, (4.14)). There are $f, g \in \mathfrak{m}$ such that f, g form an $A_{\mathfrak{m}}$-regular sequence and the rings $A_{\mathfrak{m}}/fA_{\mathfrak{m}}$ and $A_{\mathfrak{m}}/(f,g)A_{\mathfrak{m}}$ are normal (cf. [F], (3.3)). By [EGA IV], (6.13) there is an $h \in A\backslash\mathfrak{m}$ such that the rings A_h/fA_h and $A_h/(f,g)A_h$ are normal.

Then the k-scheme $X := \operatorname{Spec} A_h$, the closed subscheme $X' := \operatorname{Spec} A_h/fA_h$ resp. $X'' := \operatorname{Spec} A_h/(f,g)A_h$ of X and the trivial differential algebra $\Omega = \mathcal{O}_Y$ on $Y := \operatorname{Spec} k$ satisfy the assumptions of Situation (2.1). By the smoothness criterion [KD], (8.1), Ω is admissible for X/Y, X'/Y and X''/Y. Using (2.6), we will show that the adjunction morphism $\operatorname{Res}_{X'/X}$ resp. $\operatorname{Res}_{X''/X}$ at $x := \mathfrak{m}A_h/(f,g)A_h \in X'' \subset X'$ is not surjective resp. not injective.

Let $S := \widehat{A}_{\mathfrak{m}}$, and $t_1, \ldots, t_5 \in S$ a system of parameters of S with $t_4 = g$ and $t_5 = f$. Then S is an isolated singularity (cf. [Mat], (21.E)), and S and $P := k[[T_1, \ldots, T_5]]$ are complete, equidimensional rings of dimension 5, and S is a finite P-algebra via the k-algebra morphism $P \to S$, $T_i \mapsto t_i$. If $\mathfrak{p} \in \operatorname{Spec} S$ is different from the maximal ideal and $\mathfrak{q} := \mathfrak{p} \cap P$, then $\widehat{S}_{\mathfrak{p}}/\widehat{P}_{\mathfrak{q}}$ is a finite algebra of regular rings of the same dimensions. The Auslander–Buchsbaum formula shows that $\widehat{S}_{\mathfrak{p}}/\widehat{P}_{\mathfrak{q}}$ and hence $S_{\mathfrak{p}}/P_{\mathfrak{q}}$ are flat algebras. Therefore, S_{T_5} resp. S_{T_4} are a finite flat and hence projective P_{T_5}- resp. P_{T_4}-modules. Consequently, $\operatorname{Ext}^1_P(S, P)_{T_5} = 0$.

As $H^4_{\mathfrak{m}_P}(S) = H^4_{\mathfrak{m}_S}(S) \neq 0$, local duality implies $\operatorname{Ext}^1_P(S, P) \neq 0$ (cf. [HK$_1$], (3.5)) and so T_5 is a zerodivisor on $\operatorname{Ext}^1_P(S, P)$. The long exact Ext-sequence of P-modules

$$\begin{aligned} \cdots \to \operatorname{Hom}_P(S, P) &\xrightarrow[\text{can.}]{\alpha} \operatorname{Hom}_P(S, P/T_5P) \\ \to \operatorname{Ext}^1_P(S, P) &\xrightarrow{T_5} \operatorname{Ext}^1_P(S, P) \to \cdots \end{aligned}$$

shows that α is not surjective, and consequently, (2.6) implies that $(\operatorname{Res}_{X'/X})_x$ is not surjective.

Set $J := (T_4, T_5)\,P$. We have the canonical exact sequence

$$0 \to \operatorname{Hom}_P(S, P)/T_5\operatorname{Hom}_P(S, P) \xrightarrow{\text{can.}} \operatorname{Hom}_{P/T_5}(S/T_5, P/T_5) \to C \to 0$$

of P/T_5P-modules. Since $C \neq 0$, $C_{T_4} = 0$ and T_4 is a non-zerodivisor on P/T_5P, we have $\operatorname{Tor}_1^{P/T_5P}(P/J, C) \neq 0$. Similarly, we see $\operatorname{Tor}_1^{P/T_5P}(P/J, \operatorname{Hom}_{P/T_5P}(S/T_5S, P/T_5P)) = 0$. With (2.6) we get that $(\operatorname{Res}_{X''/X})_x$ is not injective.

Now, we show the compatibility of the adjunction morphism and the residue map. To make precise what we mean, we need the following lemma about Cousin complexes:

(2.9) LEMMA. *Let R be a local ring, S an R-algebra, M a finitely generated S-module, and $I \subsetneq S$ an ideal that is generated by a relative M-active sequence with respect to S/R of length $n \in \mathbb{N}$. Set $S' := S/I$ and $M' := M/IM$.*

(a) *There is a unique morphism*

$$\delta^{\bullet}_{S'/S/R}\colon C^{\bullet}_{A_{S'/R}(M')}\left(\bigwedge^{n} \mathrm{Hom}_{S'}(I/I^2, S') \otimes_{S'} M'\right) \to C^{\bullet}_{A_{S/R}(M)}(M)[n]$$

of complexes of S-modules with the following property: For an element $(t_1, \dots, t_{n+p}, g) \in A^{n+p+1}_{S/R}(M)$ with $(t_1, \dots, t_n)S = I$, $m \in M$ and $\varphi_1, \dots, \varphi_n \in \mathrm{Hom}_{S'}(I/I^2, M')$ we have

$$\delta^{p}_{S'/S/R}\begin{bmatrix} \varphi_1 \wedge \cdots \wedge \varphi_n \otimes \bar{m}/\bar{g} \\ \bar{t}_{n+1}, \dots, \bar{t}_{n+p} \end{bmatrix} = (-1)^{pn}\begin{bmatrix} sm/g \\ t_1, \dots, t_{n+p} \end{bmatrix},$$

where $s \in S$ is a representative of $\det\left(\varphi_i(\bar{t}_j)\right)_{1 \leqslant i,j \leqslant n} \in S'$. Here $\bar{m}$, resp. $\bar{t}_i$ are the residue classes of m, resp. t_i in M/IM, resp. S/IS for $n+1 \leqslant i \leqslant n+p$ and in I/I^2 for $1 \leqslant i \leqslant n$. (If $(X^{\bullet}, d^{\bullet})$ is a complex, then $X^{\bullet}[n]$ denotes the complex with $X[n]^p = X^{n+p}$ and $d[n]^p = (-1)^n d^{n+p}$ for $n, p \in \mathbb{Z}$.)

(b) *If R is a field, I/I^2 a free S/I-module of rank n and M a Cohen–Macaulay S-module, then $\delta^{\bullet}_{S'/S/R}$ is injective and its image is the submodule $\mathrm{Ann}_I(C^{\bullet}_{A_S(M)}(M))$ of $C^{\bullet}_{A_S(M)}(M)$.*

Proof. (a) is a straightforward computation using the universal properties of exterior algebras, tensor products and colimits. In (b) (with the notation introduced in the lemma) $(\bar{t}_i)_{i=1}^{n}$ is a basis of I/I^2. We can assume that $(\varphi_i)_{i=1}^{n}$ is its dual basis and $s = 1$. Then $\left[\begin{smallmatrix} m/g \\ t_1, \dots, t_{n+p} \end{smallmatrix}\right] = 0$ implies $m \in (t_1, \dots, t_{n+p})M$, hence

$$\varphi_1 \wedge \cdots \wedge \varphi_n \otimes \bar{m} \in (\bar{t}_{n+1}, \dots, \bar{t}_{n+p}) \bigwedge^{n} \mathrm{Hom}_{S'}(I/I^2, S') \otimes_{S'} M'$$

implying the injectivity of $\delta^{\bullet}_{S'/S/R}$.

Clearly $\mathrm{Im}(\delta^{\bullet}_{S'/S/R}) \subseteq \mathrm{Ann}_I(C^{\bullet}_{A_S(M)}(M))$. Assume now conversely that we have an element $\left[\begin{smallmatrix} m/g \\ f_1, \dots, f_{n+p} \end{smallmatrix}\right] \in \mathrm{Ann}_I(C^{\bullet}_{A_S(M)}(M))$. As

$$C^{n+p}_{A_S(M)}(M) = \bigoplus_{ht_M(\mathfrak{p}) = n+p} C^{n+p}_{A_{S_{\mathfrak{p}}}(M_{\mathfrak{p}})}(M_{\mathfrak{p}})$$

and as we may replace S by $S/\mathrm{Ann}_S(M)$, we reduce to the case that $(S, \mathfrak{m})$ is local and $f_1, \dots, f_{n+p}$ is a system of parameters of S. We may assume that $t_1, \dots, t_n \in \mathfrak{m}$ as well. Choose $t_{n+1}, \dots, t_{n+p} \in \mathfrak{m}$ in such a way that $t_1, \dots, t_{n+p}$

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is an active sequence, hence a system of parameters of S as well. Choose $\rho \in \mathbb{N}$ such that $(t_1^\rho, \dots, t_{n+p}^\rho) \subseteq (f_1, \dots, f_{n+p})$. Then we may write

$$\begin{bmatrix} m/g \\ f_1, \dots, f_{n+p} \end{bmatrix} = \begin{bmatrix} m'/g \\ t_1^\rho, \dots, t_{n+p}^\rho \end{bmatrix}$$

for a suitable $m' \in M$. By the assumption we have that $t_i \cdot m' \in (t_1^\rho, \dots, t_{n+p}^\rho)M$ for all $i \in \{1, \dots, n\}$. As $t_1, \dots, t_{n+p}$ and any permutation thereof is an M-regular sequence, this implies that there exists an $m_1 \in M$ such that

$$\begin{bmatrix} m'/g \\ t_1^\rho, \dots, t_{n+p}^\rho \end{bmatrix} = \begin{bmatrix} m_1/g \\ t_1, t_2^\rho, \dots, t_{n+p}^\rho \end{bmatrix}$$

and inductively we proceed to see that there exists an element $\widetilde{m} \in M$ with

$$\begin{bmatrix} m/g \\ f_1, \dots, f_{n+p} \end{bmatrix} = \begin{bmatrix} \widetilde{m}/g \\ t_1, \dots, t_n, t_{n+1}^\rho, \dots, t_{n+p}^\rho \end{bmatrix}$$

and the latter generalized fraction is in the image of $\delta^p_{S'/S/R}$. □

Now we are in a position to relate the residues on X' with the residues on X.

(2.10) THEOREM. *Given Situation.* (2.1), *let $x \in X'$ be a closed point. Set $R := \mathcal{O}_{Y,f(x)}$, $S := \mathcal{O}_{X,x}$, and $S' := \mathcal{O}_{X',x}$. Then the diagram of R-linear maps*

$$\begin{tikzcd}
C^{d'}(\det \mathcal{N}_{X'/X,x} \otimes \iota^*\omega^{r+d}_{X/Y,x}; S'/R) \arrow[rr, "C^{d'}((\mathrm{Res}_{X'/X})_x; S'/R)"] \arrow[d, "\delta_{S'/S/R}"] & & C^{d'}(\omega^{r+d'}_{X'/Y,x}; S'/R) \arrow[dd, "\mathrm{Res}_{X'/Y,x}"] \\
C^d(\omega^{r+d}_{X/Y,x}; S/R) \arrow[d, "\mathrm{Res}_{X/Y,x}"] & & \\
\widehat{\Omega}^r_{f(x)} \arrow[rr, equal] & & \widehat{\Omega}^r_{f(x)}
\end{tikzcd}$$

is commutative where in the first resp. middle row, the Cousin modules are formed with repect to the system of denominators $A^\bullet_{S'/R}(\iota^*\omega^{r+d}_{X/Y,x}) = A^\bullet_{S'/R}(S') = A^\bullet_{S'/R}(\omega^{r+d'}_{X'/Y,x})$, resp. $A^\bullet_{S/R}(S) = A^\bullet_{S/R}(\omega^{r+d}_{X/Y,x})$ of S' resp. S

Proof. We use the notations of the proof to (2.4). Set $\tau := \bar{t}^*_{d'+1} \wedge \dots \wedge \bar{t}^*_d \otimes \bar{\omega}$. Using (2.6) we get

$$(\mathrm{Res}_{X/Y,x} \circ \delta_{S'/S/R}) \left(\begin{bmatrix} \tau \\ \bar{t}_1, \dots, \bar{t}_{d'} \end{bmatrix} \right)$$

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$$
\begin{aligned}
&= \operatorname{Res}_{X/Y,x}\left[\begin{smallmatrix}\omega\\ t_1,\ldots,t_d\end{smallmatrix}\right]\\
&= \varphi^{\omega_0}_{t_1,\ldots,t_d}(\omega)(1)(0)\,\omega_0\\
&= \varphi^{\omega_0}_{\bar t_1,\ldots,\bar t_{d'}}((\operatorname{Res}_{X'/X})_x(\tau))(1)(0)\,\omega_0\\
&= (\operatorname{Res}_{X'/Y,x}\circ C^{d'}((\operatorname{Res}_{X'/X})_x;S'/R))\left(\left[\begin{smallmatrix}\tau\\ \bar t_1,\ldots,\bar t_{d'}\end{smallmatrix}\right]\right).
\end{aligned}
$$

This shows the assertion. □

Our result (2.10) implies in particular residue formula (R3) of [RD], p. 197 for the local integrals of [HK_1]. In fact, using [HK_1], (2.7) we obtain

(2.11) COROLLARY. *In the Situation of* (2.1) *let* $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} S$ *and* $X' = \operatorname{Spec} S'$ *be affine and assume that* X/Y *and* X'/Y *are smooth. Furthermore assume that* $S' = S/(t_{d'+1},\ldots,t_d)$ *for some regular sequence* $t_{d'+1},\ldots,t_d$ *in* S*. Let* $t_1,\ldots,t_{d'}$ *be a regular sequence in S such that* $S'/(\bar t_1,\ldots,\bar t_{d'})$ *is finite as an* R*-module, and set* $Z = \mathfrak{V}(t_1,\ldots,t_{d'}) \subseteq X'$*. Then for each* $\omega \in \Omega^{d'}_{X/Y}$ *with image* $\bar\omega \in \Omega^{d'}_{X'/Y}$ *we have*

$$\int_{X'/Y,Z}\left[\begin{matrix}\bar\omega\\ \bar t_1,\ldots,\bar t_{d'}\end{matrix}\right] = \int_{X/Y,Z}\left[\begin{matrix}\omega\cdot \mathrm{d}t_{d'+1}\cdots\mathrm{d}t_d\\ t_1,\ldots,t_d\end{matrix}\right]$$

(2.12) *Remark.* Residue axiom (R3) is the only one of Hartshorne's residue axioms which has not been shown previously for the local integral of [HK_1]. Proving these axioms was one of Lipman's original goals in this area.

There are similar formulas to express (2.10) in the general Situation of (2.1), however they are not as explicit as in general the canonical morphism $\mathcal{M}_{X'}(\iota^*\Omega^{r+d'}_{X/Y}) \to \mathcal{M}_{X'}(\Omega^{r+d'}_{X'/Y})$ does not induce a surjection

$$\iota^*\omega^{r+d'}_{X/Y} \to \omega^{r+d'}_{X'/Y}.$$

3. Relative duality and adjunction

In this section we prove the theorem relating the relative duality theory of morphisms $f\colon X\to Y$ and $\iota\colon X'\to X$ with the duality theory of $f\circ\iota$ in the situation described in the introduction. More precisely, we fix the following

(3.1) *Situation.* Let X and Y be excellent noetherian schemes without embedded components, let Ω be an exterior differential algebra on Y such that Ω^1 is locally free of rank r and let $f\colon X\to Y$ be a morphism of finite type, Cohen-Macaulay, equidimensional of relative dimension d and generically a complete intersection, and such that $\mathcal{M}_X(\Omega^1_X)$ is locally free as an $\mathcal{M}_X$-module of rank $r+d$. Finally

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let $\iota\colon X'\to X$ be a closed immersion such that the composition $g := f\circ\iota : X'\to Y$ is flat and equidimensional of relative dimension d'. Set $n := d - d'$.

In the above situation, we will express the relation arising from abstract duality theory in terms of differential forms. In particular in case ι is a regular immersion such that no component of X' is completely contained in the non-smooth locus of f, it is given by the adjunction morphism constructed in (2.3). This will be very useful in our description of traces on the level of residual complexes (cf. Sect. 4).

(3.2) LEMMA. *Given Situation* (3.1), *we have* $\mathcal{E}xt^q_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y}) = 0$ *for* $q < n$.

Proof. As the assertion is local in X and Y we may assume that $Y = \operatorname{Spec} R$ for some local ring $(R,\mathfrak{m})$ and $X = \operatorname{Spec} S$. By [Mat], (3.E) we also may assume that R is complete. Let $X' = \operatorname{Spec} T$, $T = S/\mathfrak{A}$ and let $\mathfrak{M}\in\operatorname{Max} S$ with $\mathfrak{M}\cap R = \mathfrak{m}$ and $\mathfrak{A}\subseteq\mathfrak{M}$. By [HK$_1$], (4.10) we find (possibly after shrinking S as a neighborhood of $\mathfrak{M}$) elements $t_1,\dots,t_d\in\mathfrak{M}$ such that $I = (t_1,\dots,t_d)$ satisfies

(i) $\mathfrak{M}$ is the only maximal ideal of S containing I.
(ii) $t_{d'+1},\dots,t_d\in\mathfrak{A}$.
(iii) S/I is finite as an R-module.

Denoting by $\widehat{\ }$ I-adic completion and by $\overline{}$ residue classes mod $\mathfrak{m}$ we have a local integral

$$\int_{\widehat{S}/R,I} : H^d_I(\widetilde{\omega}^{d+r}_{\widehat{S}/R}) \to \Omega^r,$$

(where $\widetilde{\omega}^{\bullet}_{\widehat{S}/R}$ denotes the module of universally finite regular differential forms, cf. [HK$_1$], Sect. 1) which induces by compatibility of H^d_I with base change a map

$$\int_{\overline{\widehat{S}/R},\overline{I}} : H^d_{\overline{I}}(\overline{\widetilde{\omega}^{r+d}_{\widehat{S}/R}}) \to \bar{\Omega}^r.$$

The Cohen–Macaulay property of S/R implies that the isomorphism

$$\begin{array}{ccc} \widetilde{\omega}^{d'+r}_{\widehat{S}/R}/I\,\widetilde{\omega}^{d'+r}_{\widehat{S}/R} & \longrightarrow & \operatorname{Hom}_R(\widehat{S}/I,\Omega^r) \\ \overline{\omega} & \longmapsto & \left(\bar{s}\longmapsto \int_{\widehat{S}/R,I}\left[\begin{smallmatrix} s\cdot\omega \\ t\end{smallmatrix}\right]\right) \end{array}$$

of [HK$_1$], (3.3) commutes with passing to residue classes mod $\mathfrak{m}$, hence the local duality isomorphism commutes with base change to induce an isomorphism

$$\delta_M : \operatorname{Hom}_{\overline{\widehat{S}}}(M, \overline{\widetilde{\omega}^{r+d}_{\widehat{S}/R}}) \to \operatorname{Hom}_{\overline{R}}(H^d_{\overline{I}}(M), \overline{\Omega^r})$$

from which we conclude formally as in the proof of [HK$_1$], (3.5)(c) that the induced maps

$$\delta^i_M : \operatorname{Ext}^i_{\overline{\widehat{S}}}(M, \overline{\widetilde{\omega}^{r+d}_{\widehat{S}/R}}) \to \operatorname{Hom}_{\overline{R}}(H^{d-i}_{\overline{I}}(M), \overline{\Omega^r})$$

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are bijective for all $\widehat{\overline{S}}$-modules M. Thus

$$\mathrm{Ext}^q_{\widehat{\overline{S}}}(\widehat{\overline{T}}, \overline{\widetilde{\omega}^{r+d}_{\widehat{S}/R}}) \cong \mathrm{Hom}_{\overline{R}}(H^{d-q}_{\overline{I}}(\widehat{\overline{T}}), \bar{\Omega}^r) \cong \mathrm{Hom}_{\overline{R}}(H^{d-q}_{(t_1,\ldots,t_{d'})}(\widehat{\overline{S}}), \bar{\Omega}^r) = 0$$

as $q < n$. By [AK$_2$], (1.8) the canonical map

$$\beta\colon \mathrm{Ext}^q_S(T, \omega^{r+d}_{S/R}) \otimes_R \overline{R} \to \mathrm{Ext}^q_{\overline{S}}(\overline{T}, \overline{\widetilde{\omega}^d_{\widehat{S}/R}})$$

is surjective, and from the R-flatness of $\omega^{r+d}_{S/R}$ and [AK$_2$], (1.9)(i) we conclude that β is bijective, hence the Nakayama lemma completes the proof of the lemma. □

In Situation (3.1) the isomorphism $f_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \_\_) = g_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \_\_)^{\sim}$ (where $\mathcal{F}^{\sim}$ denotes the $\mathcal{O}_{X'}$-module associated to an $\iota_*\mathcal{O}_{X'}$-module $\mathcal{F}$) of functors on the category $\mathrm{Coh}\, X$ of coherent $\mathcal{O}_X$-modules induces by [RD], I.(5.4) an isomorphism in the derived category

$$\mathbb{R}g_* \circ \mathbb{R}\,\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \_\_)^{\sim} \to \mathbb{R}(f_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \_\_)).$$

The associated spectral sequence degenerates sufficiently to yield an edge morphism

$$\varphi\colon R^{d'}g_*(\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^{\sim}) \to \mathcal{E}xt^d_f(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y}) \to R^d f_*\omega^{r+d}_{X/Y},$$

(where $\mathcal{E}xt_f$ denotes the right derived functors of $f_*\mathcal{H}om_{\mathcal{O}_X}$ cf. [Kl]). In case f is proper, this morphism, combined with the integral of [HK$_2$] and [HS] defines a local integral

$$\int_{X/Y,X'}\colon R^{d'}g_*(\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^{\sim}) \to \Omega^r$$

and the isomorphism

$$c_{\iota,f}\colon \mathbf{g}^! = (\mathbf{f} \circ \boldsymbol{\iota})^! \xrightarrow{\sim} \boldsymbol{\iota}^! \circ \mathbf{f}^! \tag{$*$}$$

can be expressed as follows:

(3.3) PROPOSITION. *In Situation* (3.1)

$$\left(\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^{\sim}, \int_{X/Y,X'}\right) \cong (g^!(\Omega^r), t_g(\Omega^r)).$$

Proof. First recall that by [RD], III.6 and III.8

$$\boldsymbol{\iota}^!(\mathcal{G}) = (\mathbb{R}\,\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}))^{\sim}.$$

For f and g we have by [Li$_2$], (4.5.4) $\omega^{r+d}_{X/Y} = f^!\mathcal{O}_Y = H^{-d}(\mathbf{f}^!\mathcal{O}_Y)$. Furthermore $H^n(\boldsymbol{\iota}^!\omega^{r+d}_{X/Y}) = \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^{\sim}$ and $H^q(\boldsymbol{\iota}^!\omega^{r+d}_{X/Y}) = 0$ for $q < n$ by (3.2).

As f is Cohen–Macaulay we know by [RD] V.(9.7) and [HS], main theorem

$$H^q(\mathbf{f}^!\Omega^r) = \begin{cases} \omega_{X/Y}^{r+d} & \text{if } q = d \\ 0 & \text{if } q \neq d \end{cases}.$$

Thus and by (3.2) the spectral sequence associated to the $c_{\iota,f}$ of $(*)$ degenerates sufficiently to yield an isomorphism

$$c_{\iota,f}\colon g^!\Omega^r \xrightarrow{\sim} \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega_{X/Y}^{r+d})^{\sim}.$$

Furthermore the trace Tr_ι induces a map

$$t_\iota\colon R^{d'}f_*(\iota_*\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega_{X/Y}^{r+d})^{\sim}) \longrightarrow R^d f_*\omega_{X/Y}^{r+d}$$

making the following diagram commute

$$\begin{tikzcd}
R^{d'}g_*(g^!\Omega^r) \arrow[r, "t_g(\Omega^r)"] \arrow[d, "R^{d'}g_*(c_{\iota,f})"] & \Omega^r \\
R^{d'}f_*(\iota_*\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega_{X/Y}^{r+d})^{\sim}) \arrow[r, "t_\iota"] & R^d f_*\omega_{X/Y}^{r+d} \arrow[u, "\int_{X/Y}"']
\end{tikzcd}$$

where t_g is induced by Tr_g as in [Li2], (4.5.5). To complete the proof of the proposition it remains to show that $t_\iota = \varphi$.

Let $\omega_{X/Y}^{r+d}[-d] \to \mathcal{I}^\bullet$ be a resolution by injective quasi-coherent $\mathcal{O}_X$-modules. Then $\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \mathcal{I}^\bullet)^{\sim}$ is a complex of injective, hence ι_*-acyclic $\mathcal{O}_{X'}$-modules. Thus

$$\mathbb{R}\iota_*\boldsymbol{\iota}^!(\omega_{X/Y}^{r+d}[-d]) \cong \iota_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \mathcal{I}^\bullet)^{\sim}$$

and Tr_ι is the canonical map

$$\iota_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \mathcal{I}^\bullet)^{\sim} \xrightarrow{\gamma^{-1}} \mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \mathcal{I}^\bullet) \xrightarrow{\mathrm{ev}} \mathcal{I}^\bullet,$$

where γ is the obvious isomorphism (cf. [Li2], (4.6.1.1)), and where 'ev' is evaluation at 1. Then the following diagram commutes

$$\begin{tikzcd}
\iota_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \mathcal{I}^\bullet)^{\sim} \arrow[r, "\gamma^{-1}"] \arrow[d, "\text{via}"', "\gamma^{-1}"] & \mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \mathcal{I}^\bullet) \arrow[d, "\mathrm{ev}"] \\
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}^\bullet) \arrow[r, "\mathrm{ev}"] & \mathcal{I}^\bullet
\end{tikzcd}$$

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By (3.2) we have in the derived category for the truncation functors of [Li2], Section 1

$$\tau_{\le -d'}(\iota_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\mathcal{I}^\bullet)^{\sim}) \cong \iota_*((\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\omega^{r+d}_{X/Y})[-d'])^{\sim})$$

and similarly

$$\tau_{\le -d'}(\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\mathcal{I}^\bullet)) \cong \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\omega^{r+d}_{X/Y})[-d'].$$

Applying this and $\mathbb{R}f_*$ to the above diagram we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{R}f_*(\iota_*\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\mathcal{I}^\bullet)[-d']^{\sim}) \arrow[r, "\mathbb{R}f_*\gamma^{-1}"] \arrow[d, "\text{via }\gamma^{-1}"] & \mathbb{R}f_*(\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\omega^{r+d}_{X/Y})[-d']) \arrow[dd, "\mathbb{R}f_*\mathrm{ev}"] \\
\mathbb{R}f_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\mathcal{I}^\bullet) \arrow[d, "\mathrm{can}"] & \\
\mathbb{R}f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{I}^\bullet) \arrow[r, "\mathbb{R}f_*\mathrm{ev}"] & \mathbb{R}f_*\mathcal{I}^\bullet
\end{tikzcd}$$

As $\mathbb{R}f_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\mathcal{I}^\bullet) = f_*\mathcal{H}om_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\mathcal{I}^\bullet)$ and as $\mathbb{R}f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{I}^\bullet) = f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{I}^\bullet)$, by taking H^0 of the above diagram we obtain a commutative diagram

$$\begin{tikzcd}
R^{d'}f_*(\iota_*\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\omega^{r+d}_{X/Y})^{\sim}) \arrow[r, "\text{via }\gamma^{-1}"] \arrow[d, "\text{via }\gamma^{-1}"] & R^{d'}f_*(\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'},\omega^{r+d}_{X/Y})) \arrow[dd, "\text{via ev}"] \\
\mathcal{E}xt^d_f(\mathcal{O}_{X'},\omega^{r+d}_{X/Y}) \arrow[d, "\mathrm{can}"] & \\
\mathcal{E}xt^d_f(\mathcal{O}_X,\omega^{r+d}_{X/Y}) \arrow[r, "\text{via ev}"] & R^d f_*\omega^{r+d}_{X/Y}
\end{tikzcd}$$

Going around the top of this diagram gives the map t_ι, whereas going around the bottom gives φ, thus completing the proof of the proposition. □

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(3.4) *Remark.* We consider Situation (3.1) where $\iota\colon X' \to X$ is a regular immersion. Then there is a natural 'fundamental local isomorphism'

$$\begin{aligned}\lambda\colon \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_X, \omega^{r+d}_{X/Y})^{\sim} &\to \mathcal{H}om_{\mathcal{O}_{X'}}(\det \mathcal{C}_{X'/X}, \iota^*\omega^{r+d}_{X/Y})\\ &= \det \mathcal{N}_{X'/X} \otimes \iota^*\omega^{r+d}_{X/Y}\end{aligned}$$

(cf. [AK$_1$], I.(4.5)). Assume in addition that $Y = \operatorname{Spec} R$ for some complete local ring R, and let $z \in X'$ be a closed point. Then there exists an open neighborhood $U = \operatorname{Spec} T$ of $z \in X'$ and $t_1, \ldots, t_{d'} \in T$ such that $P = \mathfrak{V}(t_1, \ldots, t_{d'}) \subseteq X'$ satisfies [HK$_1$], (4.8) (by [HK$_1$], (4.10)). After shrinking U as an open affine neighborhood of z we may assume $U = \operatorname{Spec} S \cap X'$ for some open affine $\operatorname{Spec} S \subseteq X$, and the ideal $I \subseteq S$ of U is generated by a quasi-regular sequence $t_{d'+1}, \ldots, t_d$. Given an element $\xi \in H^{d'}_P(X', \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y}))$ we may write

$$\xi = \begin{bmatrix} \alpha \\ t_1^{n_1}, \ldots, t_{d'}^{n_{d'}} \end{bmatrix}, \qquad n_i \in \mathbb{N}_+, \alpha \in \operatorname{Hom}_S(\det I/I^2, \omega^{r+d}_{S/R}/I\omega^{r+d}_{S/R}).$$

By abuse of notation we denote by $\alpha(\mathrm{d}t_{d'+1} \wedge \cdots \wedge \mathrm{d}t_d)$ a preimage of the differential form $\alpha(\mathrm{d}t_{d'+1} \wedge \cdots \wedge \mathrm{d}t_d) \in \omega^{r+d}_{S/R}/I\omega^{r+d}_{S/R}$ in $\omega^{r+d}_{S/R}$, and by $t_1, \ldots, t_{d'}$ preimages of $t_1, \ldots, t_{d'} \in T$ in S, and we define

$$\operatorname{Res}_P(\xi) := \int_{X/Y,P} \left(\begin{bmatrix} \alpha(\mathrm{d}t_{d'+1} \wedge \cdots \wedge \mathrm{d}t_d) \\ t_1^{n_1}, \ldots, t_{d'}^{n_{d'}}, t_{d'+1}, \ldots, t_d \end{bmatrix} \right),$$

i.e., Res_P is the composition

$$\begin{aligned} H^{d'}_P(X', \operatorname{Ext}^n_S(T, \omega^{r+d}_{S/R})) &\xrightarrow{\text{can}} H^{d'}_P(H^n_I(\omega^{r+d}_{S/R})) \\ &\xrightarrow{\varepsilon} H^d_P(X, \omega^{r+d}_{X/Y}) \xrightarrow{\int_{X/Y,P}} \Omega^r, \end{aligned}$$

where ε is the isomorphism arising from the Leray spectral sequence ([LSy], Sect. 3). Then the following diagram commutes

$$\begin{tikzcd} H^{d'}_P(\operatorname{Ext}^n_S(T, \omega^{r+d}_{S/R})) \arrow[r] \arrow[d, "\text{can}"'] & H^{d'}_P(H^n_I(\omega^{r+d}_{S/R})) \arrow[r, "\varepsilon"] & H^d_P(X, \omega^{r+d}_{X/Y}) \arrow[d, "\text{can}"] \\ H^{d'}(X', \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^{\sim}) \arrow[rr, "\varphi"] & & H^d(X, \omega^{r+d}_{X/Y}) \end{tikzcd}$$

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hence we conclude from [HS], residue theorem, that also the following diagram commutes ('residue theorem')

$$\begin{tikzcd} H^{d'}_P(X', \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^\sim) \arrow[rr, "\text{can}"] \arrow[dr, "\mathrm{Res}_P"'] & & H^{d'}(X', \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})) \arrow[dl, "\int_{X/Y,X'}"] \\ & \Omega^r & \end{tikzcd}$$

Let $f : X \to Y$ and $\iota : X' \to X$ satisfy the assumptions (3.1), and assume in addition that f is proper, that X' has no embedded components and that the composition $g = f \circ \iota : X' \to Y$ satisfies: $\mathcal{M}_{X'}(\Omega_{X'})$ is free of rank $r + d'$. Then by [HS], main theorem, and the above there is a canonical isomorphism

$$\psi : \omega^{r+d'}_{X'/Y} \to \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^\sim,$$

transforming $\int_{X'/Y}$ into $\int_{X/Y,X'}$. Assume now that ι is a regular immersion and that, whenever $z \in X$ is the generic point of an irreducible component of X', then X/Y is smooth at $\iota(z)$, i.e., no irreducible component of X' is completely contained in the singular locus of f. Then we also have an obvious morphism

$$\varphi := \lambda^{-1} \circ (\mathrm{Res}^{\Omega}_{X'/X})^{-1} : \omega^{r+d'}_{X'/Y} \to \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^\sim$$

by (2.7) and the fundamental local isomorphism (cf. (3.4)).

(3.5) THEOREM (Adjunction). *In the above situation we have $\varphi = \psi$.*

Proof. It remains to show that φ satisfies the property uniquely determining ψ, i.e., that the following diagram commutes

$$\begin{tikzcd} R^{d'}g_*\omega^{r+d'}_{X'/Y} \arrow[d, "R^{d'}g_*(\varphi)"'] \arrow[dr, "\int_{X'/Y}"] & \\ R^{d'}g_*(\mathcal{E}xt^n_{\mathcal{O}_X}(\iota_*\mathcal{O}_{X'}, \omega^{r+d}_{X/Y})^\sim) \arrow[d, "\text{can}"'] & \Omega^r \\ R^d f_*\omega^{r+d}_{X/Y} \arrow[ur, "\int_{X/Y}"'] & \end{tikzcd} \qquad (**)$$

To see this we may assume that $Y = \operatorname{Spec} R$ is affine, and after replacing R by $Q(R)$ we may also assume that $\dim Y = 0$ and consists of a single point only. Hence for each $z \in X'$ we can find an open affine neighborhood $U = \operatorname{Spec} S \subset X$

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of z and an ideal $\overline{J} = (\bar{t}_1, \dots, \bar{t}_{d'}) \subset T := \iota_* \mathcal{O}_{X'}(U)$ such that the assumptions of [HK$_1$] are satisfied. By $t_1, \dots, t_{d'}$ we denote preimages of $\bar{t}_1, \dots, \bar{t}_{d'}$ in S and set $J = (t_1, \dots, t_{d'})$. For each ring A, each ideal $\mathfrak{A} \subseteq A$ and each A-module M we have a canonical morphism $\mathrm{Ext}^p_A(A/\mathfrak{A}, M) \to H^p_{\mathfrak{A}}(M)$. Thus we have (with $K = I + J$) a canonical morphism from the spectral sequence

$$\mathrm{Ext}^p_T(T/\overline{J}, \mathrm{Ext}^q_S(S/I, \omega^{r+d}_{S/R})) \Longrightarrow \mathrm{Ext}^{p+q}_S(S/K, \omega^{r+d}_{S/R})$$

to the spectral sequence

$$H^p_J(H^q_I(\omega^{r+d}_{S/R})) \Longrightarrow H^{p+q}_{I+J}(\omega^{r+d}_{S/R})$$

implying that the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Ext}^{d'}_T(T/\overline{J}, \omega^{r+d'}_{T/R}) \arrow[r, "\mathrm{can}"] \arrow[d, "\text{via } \varphi"'] & H^{d'}(X', \omega^{r+d'}_{X'/Y}) \arrow[d, "\text{via } \varphi"'] \\
H^{d'}_J(H^n_I(\omega^{r+d}_{S/R})) \arrow[d, "\nu"] & \mathrm{Ext}^{d'}_T(T/\overline{J}, \mathrm{Ext}^n) \arrow[l, "\mathrm{can}"'] \arrow[r, "\mathrm{can}"] \arrow[d, "\mu"] & H^{d'}(X', \mathcal{E}xt^n) \arrow[d, "\lambda"] \\
H^d_{I+J}(\omega^{r+d}_{S/R}) \arrow[dr, "\mathrm{can}"'] & \mathrm{Ext}^d_S(S/K, \omega^{r+d}_{S/R}) \arrow[l, "\mathrm{can}"'] \arrow[r, "\mathrm{can}"] & \mathrm{Ext}^d \arrow[dl, "\mathrm{can}"] \\
 & H^d(X, \omega^{r+d}_{X/Y}) &
\end{tikzcd}$$

where we write for short $\mathrm{Ext}^n = \mathrm{Ext}^n_S(S/I, \omega^{r+d}_{S/R})$, $\mathcal{E}xt^n = \mathcal{E}xt^n_{\mathcal{O}_X}(\iota_* \mathcal{O}_{X'}, \omega^{r+d}_{X/Y})$ and $\mathrm{Ext}^d = \mathrm{Ext}^d_{\mathcal{O}_X}(\iota_* \mathcal{O}_{X'}, \omega^{r+d}_{X/Y})$, and where μ, ν and λ are the canonical morphisms arising from the various Leray spectral sequences.

Now let $\eta \in \mathrm{Ext}^{d'}_T(T/\overline{J}, \omega^{r+d'}_{T/R})$. By the fundamental local isomorphism (cf. [AK$_1$], I.(4.5)), η may be thought of as a $T/\overline{J}$-linear morphism

$$\bigwedge^{d'}(\overline{J}/\overline{J}^2) \to \omega^{r+d'}_{T/R}/\overline{J}\omega^{r+d'}_{T/R}, \quad \bar{t}_1 \wedge \cdots \wedge \bar{t}_{d'} \mapsto \overline{\omega} + \overline{J} \cdot \omega^{r+d'}_{T/R}$$

for some $\omega \in (\Omega^{d'}_S)_N$ with image $\overline{\omega}$ mod I (where $N \subseteq S$ is the preimage of the set of all nonzerodivisors of T). Thus we see that the image of η in $H^{d'}_J(\omega^{r+d'}_{T/R})$ can be described by the generalized fraction

$$\begin{bmatrix} \omega \\ t_1, \dots, t_{d'} \end{bmatrix} \in H^{d'}_J(\omega^{r+d'}_{T/R}).$$

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Using the fundamental local isomorphism twice, we see that the image of η by

$$\mathrm{Ext}_T^{d'}(T/\overline{J}, \omega_{T/R}^{r+d'}) \xrightarrow{\text{via } \varphi} \mathrm{Ext}_T^{d'}(T/\overline{J}, \mathrm{Ext}_S^n(S/I, \omega_{S/R}^{r+d})) \xrightarrow{\text{can}} H_J^{d'}(H_I^n(\omega_{S/R}^{r+d}))$$

is equal to the generalized fraction

$$\left[\begin{matrix} \left[\begin{matrix} \omega^* \\ t_{d'+1}, \ldots, t_d \end{matrix}\right] \\ t_1, \ldots, t_{d'} \end{matrix}\right],$$

where $\omega^* \in \omega_{S/R}^{r+d}$ is some regular differential form of S/R having the same image in $(\Omega_S^{r+d})_N / I \cdot (\Omega_S^{r+d})_N$ as $\omega \cdot \mathrm{d}t_{d'+1} \cdots \mathrm{d}t_d$. As

$$\nu \left[\begin{matrix} \left[\begin{matrix} \omega^* \\ t_{d'+1}, \ldots, t_d \end{matrix}\right] \\ t_1, \ldots, t_{d'} \end{matrix}\right] = \left[\begin{matrix} \omega* \\ t_1, \ldots, t_d \end{matrix}\right]$$

by [LSy], (3.3.1), theorem (2.10) implies that the following diagram commutes

$$\begin{tikzcd}
H_J^{d'}(\omega_{T/R}^{r+d'}) \arrow[dr, "\int_{T/R,\overline{J}}"] & \\
\mathrm{Ext}_T^{d'}(T/\overline{J}, \omega_{T/R}^{r+d'}) \arrow[u, "\text{can}"] \arrow[d, "\text{via}\ \varphi"] & \Omega^r \\
H_{I+J}^d(\omega_{S/R}^{r+d}) \arrow[ur, "\int_{S/R,I+J}"'] &
\end{tikzcd}$$

As the canonical map

$$\bigoplus_{z \in X' \text{ closed}} H_{\{z\}}^{d'}(\omega_{X'/Y}^{r+d'}) \to H^{d'}(X', \omega_{X'/Y}^{r+d'})$$

is surjective, we conclude from [HS], residue theorem, that $(**)$ commutes, completing the proof of (3.5). $\square$

(3.6) *Remark.* Suppose in Situation (3.1) that X/Y and X'/Y are smooth, and that $\Omega = \mathcal{O}_Y$ so that $\omega_{X/Y}^d = \Omega_{X/Y}^d$ and $\omega_{X'/Y}^{d'} = \Omega_{X'/Y}^{d'}$. Then the adjunction map φ may be identified with the obvious isomorphism

$$\omega_{X'/Y}^{d'} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{X'}}(\det \mathcal{C}_{X'/X}, \iota^* \omega_{X/Y}^d)$$

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arising from the canonical short exact sequence

$$0 \to \mathcal{C}_{X'/X} \to \iota^*\Omega^1_X \to \Omega^1_{X'} \to 0$$

of locally free $\mathcal{O}_{X'}$-modules.

4. Fundamental classes of subschemes

In this section we will apply the adjunction morphism to study morphisms of residual complexes and subschemes of a given smooth variety over a field of characteristic 0 and their fundamental classes. Most of the results in this section are already contained in [EZ$_2$] and to some extend in [Li$_1$], at least from a theoretical point of view. The presentation given here might however help to clarify some of the ideas in these papers. An alternative treatment of fundamental classes in terms of adelic residues is given by Yekutieli [Ye$_2$].

First recall that a residual complex on a noetherian scheme X is a complex $\mathcal{K}^\bullet_X$ of quasi-coherent injective $\mathcal{O}_X$-modules, bounded below with coherent cohomology sheaves, and such that there exists an isomorphism of $\mathcal{O}_X$-modules

$$\bigoplus_{n\in\mathbb{N}} \mathcal{K}^n_X \cong \bigoplus_{x\in X} J(x),$$

where $J(x)$ is a skyscraper sheaf on the closed set $\overline{\{x\}}$ whose underlying $\mathcal{O}_{X,x}$-module is an $\mathcal{O}_{X,x}$-injective hull of $k(x)$. They are of particular interest as they are representatives of (pointwise) dualizing complexes on X, but they are hard to describe in general as injective hulls are unique only up to non-canonical isomorphism.

Suppose now that k is a field and that X is a k-variety, i.e., a reduced k-scheme of finite type, which is Cohen-Macaulay and equidimensional of dimension d. Furthermore assume that Ω is an exterior differential algebra on $Y = \operatorname{Spec} k$ which is admissible for X/Y and let $r := \dim_k(\Omega^1)$.

(4.1) PROPOSITION. *In the above situation $\omega^{r+d}_{X/Y}$ is a Gorenstein sheaf, and for each $x \in X$ we have*

$$\dim_{k(x)} \operatorname{Ext}^i_{\mathcal{O}_{X,x}}(k(x), \omega^{r+d}_{X/Y,x}) = \begin{cases} 0 & \text{for } i \neq \dim \mathcal{O}_{X,x} \\ 1 & \text{for } i = \dim \mathcal{O}_{X,x} \end{cases}.$$

Proof. As the assertion is local in X we may assume that $X = \operatorname{Spec} R$ with some (reduced and equidimensional) Cohen–Macaulay algebra R/k. As $\omega^{r+d}_{X/Y,x}$ is a canonical module for $\mathcal{O}_{X,x}$ by [KW], (4.11) and [HeK], (5.12) for each $x \in X$, it is Gorenstein by [HeK], (6.10). Now let $x \in X$ be a point of X. If $x = \mathfrak{m}$ is a closed point of X, and if we set $S := R_\mathfrak{m}$ then by the local duality theorem [HK$_1$], (3.4) we have

$$\operatorname{Ext}^p_{\widehat{S}}(k(x), \widetilde{\omega}^{r+d}_{\widehat{S}/k}) = \operatorname{Hom}_k(H^{d-p}_\mathfrak{m}(k(x)), \Omega^r) = \begin{cases} 0 & \text{for } i \neq \dim \mathcal{O}_{X,x} \\ k(x) & \text{for } i = \dim \mathcal{O}_{X,x} \end{cases},$$

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implying the claim in this case. For an arbitrary $x = \mathfrak{p} \in \operatorname{Spec} R$ with height $i < d$ let

$$P = k[X_1, \dots, X_d] \to R$$

be a noetherian normalization of R/k such that we have $\mathfrak{p} \cap P = (X_1, \dots, X_i)$. Setting $K' = k(X_{i+1}, \dots, X_n)$ and $R' := R \otimes_{k[X_{i+1},\dots,X_n]} K'$ we can apply the above to

$$f' \colon X' := \operatorname{Spec} R' \to \operatorname{Spec} K' = Y'$$

and the point $x' \in X'$ corresponding to x. As $x' \in X'$ is closed we obtain as above

$$\operatorname{Ext}^i_{\mathcal{O}_{X',x'}}(k(x), (\omega^{\Omega_{K'}}_{X'/Y',x'})^{r+d})) = \begin{cases} 0 & \text{for } i \neq \dim \mathcal{O}_{X',x'} \\ k(x') & \text{for } i = \dim \mathcal{O}_{X',x'} \end{cases}$$

and the claim follows. □

Using the notation of Section 1 we write $\mathcal{C}^\bullet_X$ for the Cousin complex (with respect to the system of denominators of active sequences) $\mathcal{C}^\bullet(X, \mathcal{A}^\bullet(\omega^{r+d}_{X/Y}), \omega^{r+d}_{X/Y})$ of $\omega^{r+d}_{X/Y}$ if no confusion is likely.

(4.2) COROLLARY. *In the above situation $\mathcal{C}^\bullet_X[d]$ is a residual complex on X.*

Proof. By (1.10)(b) $\mathcal{C}^\bullet_X$ is a minimal injective resolution of $\omega^{r+d}_{X/Y}$, hence the claim follows from (4.1). □

From now on suppose that k is a perfect field and that $\Omega = \mathcal{O}_Y$. Then by (1.14) the global integral

$$\int_{X/Y} : \Gamma(X, \mathcal{C}^\bullet_X[d]^0) \to k$$

is defined and we obtain

(4.3) THEOREM. *Suppose that $Y = \operatorname{Spec} k$ for a perfect field k and that X/Y is a proper Cohen–Macaulay variety. Then $\int_{X/Y}$ induces for each complex $\mathcal{F}^\bullet$ of quasi-coherent sheaves on X, bounded above, an isomorphism*

$$\mathbb{R}\operatorname{Hom}^\bullet_X(\mathcal{F}^\bullet, \mathcal{C}^\bullet_X[d]) \to \operatorname{Hom}^\bullet_Y(\mathbb{R}f_*\mathcal{F}^\bullet, k)$$

in the derived category $\mathbb{D}^+(X)$, i.e., $(\mathcal{C}^\bullet_X[d], \int_{X/Y})$ is a residue pair in the sense of [Sas$_1$].

Proof. By local duality $\big([\mathrm{HK}_1], (3.4)\big)$, by (4.2) and by [Sas$_1$], thm. 2 it suffices to show that $\int_{X/Y}$ is a morphism of complexes. In [Ye$_1$], A. Yekutieli has used Beilinson's theory of higher-dimensional adeles and the residues of Parshin and

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Lomadze to construct a residue pair $(\mathcal{K}^\bullet_X, \mathrm{tr}_{X/Y})$, and in [Hü$_1$], Section 2 the second author has, for $U = \mathrm{Spec}\, R \subseteq X$ open and affine, constructed an isomorphism

$$\mathcal{C}^\bullet_X[d](U) \to \mathcal{K}^\bullet_X(U)$$

of complexes (see also [SY]). These maps glue to give a global isomorphism $\mathcal{C}^\bullet_X[d] \to \mathcal{K}^\bullet_X$, mapping – up to a sign – $\int_{X/Y}$ to $\mathrm{tr}_{X/Y}$, i.e., being compatible with the local residues, thus implying the theorem. □

(4.4) *Remark.* In case X/k is projective, E. Kunz [Ku] has given a direct proof that $(\mathcal{C}^\bullet_X[d], \int_{X/Y})$ is a residue pair for X.

(4.5) *Remark.* In the general situation, P. Sastry [Sas$_2$] has used Cousin-complexes $\mathcal{C}^\bullet_X$ on smooth k-varieties X and local embeddings $W \hookrightarrow \mathbb{A}^n_k$ to give a canonical construction of residual complexes $\mathcal{K}^\bullet_X$ for the family of all k-schemes W of finite type. In case of a Cohen–Macaulay variety X, Sastry's residual complex is canonically isomorphic to the above. Whenever we talk of the residual complex of a non-reduced k-scheme of finite type, we think of Sastry's realization.

Assume now that we have a finite morphism $f: X \to W$ of reduced and equidimensional Cohen-Macaulay k-varieties of dimension d resp. n. In this case the above description of residual complexes allows an explicit description of the trace

$$\mathrm{tr}_f\colon f_*\mathcal{C}^\bullet_X[d] \to \mathcal{C}^\bullet_W[n]$$

of [RD], VI, Section 4 in the following cases (see also [EZ$_2$]):

(i) Assume that f maps the generic point of each of the irreducible components of X to the generic point of an irreducible component of W (so that in particular $d = n$). Then the trace $\sigma_{X/W}$ on the level of meromorphic differential forms (cf. [KD], Sect. 16) exists, and it induces a map

$$\mathrm{Tr}_f\colon f_*\mathcal{C}^\bullet_X[d] \cong \mathcal{C}^\bullet(W, \mathcal{A}^\bullet(\omega^n_{W/Y}), \omega^d_{X/Y})[d] \to \mathcal{C}^\bullet_W[n]$$

which in terms of local sections can be described as follows

$$\mathrm{Tr}_f \begin{bmatrix} \omega/g \\ f_1, \ldots, f_l \end{bmatrix} = \begin{bmatrix} \sigma_{X/W}(\omega/g) \\ f_1, \ldots, f_l \end{bmatrix}.$$

Obviously it is a morphism of complexes.

(ii) Let $f: X \hookrightarrow W$ be a regular immersion, assume that no irreducible component of X is completely contained in the singular locus of W and set $h := n - d$. By (2.9) we get for each $p \in \mathbb{N}$ a unique map

$$\begin{aligned}\delta^p_{X/W}\colon & f_*\mathcal{C}^p(X, \mathcal{A}^\bullet(\mathcal{O}_X), \det \mathcal{N}_{X/W} \otimes_{\mathcal{O}_X} f^*\omega^n_{W/Y}) \\ & \to \mathcal{C}^{p+h}(X, \mathcal{A}^\bullet(\omega^d_{X/Y}), \omega^d_{X/Y})\end{aligned}$$

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and we define

$$\mathrm{Tr}_f\colon f_*\mathcal{C}^\bullet_X[d]\to\mathcal{C}^\bullet_W[n]$$

to be the composition $\mathrm{Tr}_f=\delta^\bullet_{X/W}\circ f_*\mathcal{C}^\bullet(X,\mathcal{A}^\bullet(\omega^d_{X/Y}),(\mathrm{Res}_{W/X})^{-1})$. Note that Tr_f is a morphism of complexes, as $\delta^\bullet_{X/W}$ is a morphism of complexes.

In the general situation assume that f can be factored as $f\colon X\xrightarrow{g}Z\xrightarrow{h}W$ (at least locally) with g as in (i) and h as in (ii). Then we define $\mathrm{Tr}_f:=\mathrm{Tr}_h\circ h_*\mathrm{Tr}_g$. This map is independent of the choice of the factorization by the adjunction formalism of Section 3, and we obtain

(4.6) THEOREM. *The morphism*

$$\mathrm{Tr}_f\colon f_*\mathcal{C}^\bullet_X[d]\to\mathcal{C}^\bullet_W[n]$$

is the trace of Grothendieck duality theory (cf. [RD], VI, Sect. 4).

Proof. By the transitivity of traces and duality theory for finite and generically flat morphisms we only need to consider the case that f is a regular immersion. First we need to show that $\mathcal{C}^\bullet_X[d]=f^\triangle\mathcal{C}^\bullet_W[n]$ in the notation of [RD], VI, Section 4, i.e., that we have a canonical isomorphism

$$\mathcal{C}^\bullet_X[d]\cong\mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_X,\mathcal{C}^\bullet_W[n])=\mathrm{Ann}_{\mathcal{I}_X}(\mathcal{C}^\bullet_W[n])$$

(viewed as sheaves on X), where $\mathcal{I}_X$ denotes the ideal of X in W, and then we have to prove that via this isomorphism Tr_f can be viewed as the map ‘evaluation at 1’

$$\mathrm{ev}_1\colon\mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_X,\mathcal{C}^\bullet_W[n])\to\mathcal{C}^\bullet_W[n].$$

From (2.3) and (2.9) we conclude that our Tr_f identifies $\mathcal{C}^\bullet_X[d]$ canonically with the submodule $\mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_X,\mathcal{C}^\bullet_W[n])$ of $\mathcal{C}^\bullet_W[n]$, and in local terms it is now an easy calculation to verify that via this identification Tr_f and ev_1 coincide. □

(4.7) *Remark.* Assume now that $f:X\to W$ maps the generic points of the irreducible components of X to smooth points of W and that X is generically flat over its scheme-theoretic image. In this situation we can achieve a factorization as desired at least generically:

After replacing X by its scheme-theoretic image in W we may assume that f is a regular immersion. Then by the prime basis theorem (cf. [KW], (2.6)) there exists an open subset $U\subseteq W$ of W, containing the generic points of all irreducible components of X such that $U\cap X\hookrightarrow U$ is a regular immersion.

If in addition $W=\mathbb{A}^n_k$ and $\mathrm{char}(k)=0$, then the prime basis theorem globalizes to show that there exists a reduced and equidimensional subscheme $Z\subseteq W$ of dimension $\dim(X)$ such that $Z\cup X$ (in its reduced induced subscheme structure) is a global complete intersection in W (cf. [Web], (7.4)). In particular we can – in

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this situation – factor $f: X \to W$ globally as desired. If $\mathrm{char}(k) = p > 0$ this is in general only possible after a suitable field extension k'/k (cf. [Web], (4.5)).

Traces are of great importance in connection with fundamental classes in the sense of El Zein. For this let us recall the definition of the de Rham-residue complex according to El Zein [EZ$_2$], Section 3 and Yekutieli [Ye$_2$] first:

Assume from now on that $\mathrm{char}(k) = 0$. Given a scheme X/Y of finite type with residual complex $(\mathcal{K}_X^{\bullet}, \delta_X^{\bullet})$ we set

$$\mathcal{K}_X^{p,q} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}^{-p}, \mathcal{K}_X^q).$$

Denoting by $d^{\vee} := \mathrm{Dual}(d_{X/Y})$ the map on $\mathcal{K}_X^{\bullet,\bullet}$ induced by $d_{X/Y}$ (i.e., the differential operator dual to $d_{X/Y}$) and by $\overline{\delta}$ the morphism defined by $\delta_X^{\bullet}$, the partial derivatives of $\mathcal{K}_X^{\bullet,\bullet}$ are given by

$$d := (-1)^{p+1} d^{\vee} \colon \mathcal{K}_X^{p,q} \to \mathcal{K}_X^{p+1,q}, \quad \delta := (-1)^{p+q+1}\overline{\delta} \colon \mathcal{K}_X^{p,q} \to \mathcal{K}_X^{p,q+1}.$$

These definitions make $\mathcal{K}_X^{\bullet,\bullet}$ a double complex, and by $(\mathrm{Tot}(\mathcal{K}_X^{\bullet,\bullet}), D_X^{\bullet})$ we denote the associated simple complex. For a finite morphism $f: W \to X$ we let $\mathrm{tr}_f : f_*\mathcal{K}_W^{\bullet,\bullet} \to \mathcal{K}_X^{\bullet,\bullet}$ be the trace map induced by the canonical morphism $\Omega_{X/Y}^{\bullet} \to f_*\Omega_{W/Y}^{\bullet}$ and the trace $\mathrm{Tr}_f \colon f_*\mathcal{K}_W^{\bullet} \to \mathcal{K}_X^{\bullet}$ of duality theory (cf. (4.6)). Then tr_f is a morphism of double complexes. We note that we follow Yekutieli's conventions [Ye$_2$], which differ from El Zein's [EZ$_2$] by a shift in indices and a sign.

(4.8) *Remark.* Assume that X/Y is smooth and equidimensional of dimension n. Then $\mathcal{K}_X^{\bullet} = \mathcal{C}_X^{\bullet}[n]$, and the determinantal pairing on $\Omega_{X/Y}^{\bullet}$ defines a canonical isomorphism of complexes

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^i, \mathcal{K}_X^{\bullet}) \cong \mathcal{C}^{\bullet}(X, \mathcal{A}_X^{\bullet}, \Omega_{X/Y}^{n-i})[n+i],$$

where $\mathcal{A}_X^{\bullet}$ denotes the system of denominators of locally $\mathcal{O}_X$-active sequences.

According to [EZ$_2$], III, for each k-scheme Z of finite type, there exists a unique fundamental class $c_Z \in \mathcal{K}_Z^{\bullet,\bullet}$ (see also [Li$_1$], Sect. 3). In case of a reduced and equidimensional variety Z of dimension d it is given by the canonical map

$$c_{Z/Y}^d \colon \Omega_{Z/Y}^d \to \omega_{Z/Y}^d \subseteq \mathcal{M}_Z(\Omega_{Z/Y}^d) = \mathcal{K}_Z^{-d}$$

of [KW], Section 5. In fact in case Z/Y is smooth, $c_{Z/Y}^d$ is the canonical map from holomorphic to meromorphic differential forms, and for general reduced and equidimensional Z/Y it satisfies the trace property of [EZ$_2$], (3.1)(ii). Thus by the unicity of fundamental classes we conclude that $c_{Z/Y}^d = c_Z$.

Suppose now that X/Y is smooth and equidimensional of dimension n. To understand and study subschemes $\iota : W \hookrightarrow X$ it is important to know their

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fundamental classes c_W and in particular their images $\mathrm{tr}_\iota(c_W)$ in $\mathcal{K}_X^{\bullet,\bullet}$. For this we may assume that W is reduced and equidimensional, say of dimension d (cf. [EZ$_2$], (3.1)), and we set $h := n - d$. In this case $\mathrm{tr}_\iota(c_W)$ only depends on a suitable neighborhood of the generic points of the irreducible components of W in X. Thus and by the prime basis theorem we may assume that $X = \operatorname{Spec} R$ is affine, $W = \operatorname{Spec} R/I$, and that I is generated by a regular sequence $f_1, \ldots, f_h$. Then

$$\mathrm{tr}_\iota(c_W)\colon \Omega^d_{X/Y} \to \mathcal{C}^h(X, \mathcal{A}^\bullet_X, \Omega^n_{X/Y})$$

is given by $\mathrm{tr}_\iota(c_W)(\omega) = \left[\begin{smallmatrix}\omega \mathrm{d}f_1 \cdots \mathrm{d}f_h \\ f_1, \ldots, f_h\end{smallmatrix}\right]$. Hence as an element of $\mathcal{H}om_{\mathcal{O}_X}(\Omega^d_{X/Y}, \mathcal{C}^\bullet_X)$ $= \mathcal{C}^\bullet(X, \Omega^{n-d}_{X/Y})$ we have

(4.9) PROPOSITION. *If $X = \operatorname{Spec} R$ is affine and smooth over k, and if $\iota\colon W \hookrightarrow X$ is a regular immersion, given by a regular sequence $f_1, \ldots, f_h$, then*

$$\mathrm{tr}_\iota(c_W) = \begin{bmatrix}\mathrm{d}f_1 \cdots \mathrm{d}f_h \\ f_1, \ldots, f_h\end{bmatrix} \in \mathcal{C}^\bullet(X, \mathcal{A}_X, \Omega^{n-d}_{X/Y}).$$

Note that for (a not necessarily smooth) X and for each $i \in \mathbb{N}$ we have

$$\mathcal{C}^i(X, \mathcal{A}^\bullet_X, \omega^n_{X/Y}) \cong \bigoplus_{\mathrm{ht}(x)=i} C^i_{\mathcal{A}_{X,x}}(\omega^n_{X/Y,x})$$

(viewed as a direct sum of skyscraper sheaves on X), inducing an isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega^d_{X/Y}, \mathcal{C}^h_X) \cong \bigoplus_{\mathrm{ht}(x)=h} \operatorname{Hom}_{\mathcal{O}_{X,x}}(\Omega^d_{X/Y,x}, C^h_{\mathcal{A}_{X,x}}(\omega^n_{X/Y,x}))$$

and for each $x \in X$ such that X/Y is smooth at x we have

$$\operatorname{Hom}_{\mathcal{O}_{X,x}}(\Omega^d_{X/Y,x}, C^h_{\mathcal{A}_{X,x}}(\omega^n_{X/Y,x})) \cong C^h_{\mathcal{A}_{X,x}}(\Omega^h_{X/Y,x}).$$

Thus any element $\alpha \in \mathcal{K}_X^{-d,-d}$ may be decomposed as $\alpha = (\alpha_x)_{\mathrm{ht}(x)=h}$ with components $\alpha_x \in \operatorname{Hom}_{\mathcal{O}_{X,x}}(\Omega^d_{X/Y,x}, C^h_{\mathcal{A}_{X,x}}(\omega^n_{X/Y,x}))$, resp., for smooth points x of X/Y, with $\alpha_x \in C^h_{\mathcal{A}_{X,x}}(\Omega^h_{X/Y,x})$. We get

(4.10) THEOREM. *Let $\iota\colon W \hookrightarrow X$ be an immersion from a reduced and equidimensional k-variety W of dimension d into a reduced and equidimensional k-variety X of dimension n, and assume that no irreducible component of W is completely contained in the singular locus of X. Set $h := n - d$. Then near the generic points of its irreducible components, W can be cut out by h sections $f_1, \ldots, f_h$, and we have*

$$\mathrm{tr}_\iota(c_W)_x = \begin{bmatrix}\mathrm{d}f_1 \cdots \mathrm{d}f_h \\ f_1, \ldots, f_h\end{bmatrix} \in C^h_{\mathcal{A}_{X,x}}(\Omega^h_{X/Y,x}),$$

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if x is the generic point of an irreducible component of W, and

$$\mathrm{tr}_\iota(c_W)_x = 0,$$

otherwise, i.e., if $x \notin X$ or if $\mathrm{ht}(x) \neq h$.

Proof. As $\mathrm{tr}_\iota(c_W)$ only depends on an open neighbourhood of the generic points of W in X, we may assume that X is smooth and that W is regularly embedded. Now the theorem follows from (4.9). □

(4.11) *Remark.* For a smooth k-variety X, El Zein [EZ$_2$] obtains similar formulas for the fundamental class of a subvariety, using the description of residual complexes via local cohomology modules.

(4.12) *Remark.* In case X/Y is smooth, the de Rham–residue double complex $\mathcal{K}_X^{\bullet,\bullet}$ induces an injective resolution of the de Rham complex $\Omega^\bullet_{X/Y}$, hence may be used to calculate the de Rham cohomology of X. In this case the above description of tr_ι gives a (for purposes of explicit calculation) very convenient representative of the (usual) class of W in $H^\bullet_{\mathrm{DR}}(X)$.

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