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REGULAR MULTILINEAR OPERATORS ON $C(K)$ SPACES

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The purpose of this paper is to characterise the class of *regular* continuous multilinear operators on a product of $C(K)$ spaces, with values in an arbitrary Banach space. This class has been considered recently by several authors in connection with problems of factorisation of polynomials and holomorphic mappings. We also obtain several characterisations of a compact dispersed space K in terms of polynomials and multilinear forms defined on $C(K)$.

1. INTRODUCTION AND NOTATION

Let K be a compact Hausdorff space. $C(K)$ is the space of scalar valued continuous functions on K, Σ will denote the σ-algebra of the Borel sets of K and $B(\Sigma)$ will stand for the space of Σ-measurable functions on K which are the uniform limit of elements of Σ-simple functions.

As it is well known, the Riesz representation theorem gives a representation of the operators on $C(K)$ as integrals with respect to Radon measures, and this has been very fruitfully used in the study of the properties of $C(K)$ spaces. In a series of papers (see specially [**6, 7**]), Dobrakov developed a theory of *polymeasures*, functions defined on a product of σ-algebras which are separately measures, that can be used to obtain a Riesz-type representation theorem for multilinear operators defined on a product of $C(K)$ spaces.

Before going any further, we make clear our notation: If X is a Banach space, X^* will denote its topological dual and B_X its closed unit ball. $\mathcal{L}^k(E_1 \dots, E_k; Y)$ will be the Banach space of all the continuous k-linear mappings from $E_1 \times \cdots \times E_k$ into Y, and $\mathcal{P}\left({}^kX; Y\right)$ the space of continuous k-homogeneous polynomials from X to Y, that is, the class of mappings $P : X \to Y$ of the form $P(x) = T(x, \dots, x)$, for some $T \in \mathcal{L}^k(X, \dots X; Y)$. When $Y = \mathbb{K}$, we shall omit it. We shall use the convention $.^{[i]}.$ to mean that the i-th coordinate is not involved.

We shall denote the semivariation of a measure μ by $\|\mu\|$ and also the semivariation of a polymeasure γ by $\|\gamma\|$. (For the general theory of polymeasures see [**6**], or

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[**14**].) We recall that a polymeasure is called *regular* if it is separately regular and it is called *countably additive* if it is separately countably additive. We shall denote the set of bounded semivariation polymeasures defined in $\Sigma_1 \times \cdots \times \Sigma_k$ with values in X by $\operatorname{bpm}(\Sigma_1, \ldots, \Sigma_k; X)$. $\operatorname{rcapm}(\Sigma_1, \ldots, \Sigma_k; X)$ stands for the subset of the regular countably additive polymeasures and $\operatorname{bsv} -\omega^* - \operatorname{rcapm}(\Sigma_1, \ldots, \Sigma_k; X^*)$ for the subset of $\operatorname{bpm}(\Sigma_1, \ldots, \Sigma_k; X^*)$ composed of those polymeasures that verify that for each $x \in X$, $x \circ \gamma \in \operatorname{rcapm}(\Sigma_1, \ldots, \Sigma_k; \mathbb{K})$. As customary we shall denote by $\operatorname{rca}(\Sigma; X)$ the set of regular countably additive measures from Σ into X.

With these notations at hand we can state for further reference the following theorem from [4], which extends and completes previous results of Pelczynski [**11**] and Dobrakov [**7**]:

Theorem 1.1. [4] *Let $K_1, \ldots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k\Big(C(K_1), \ldots, C(K_k); X\Big)$. Then there is a unique $\overline{T} \in \mathcal{L}^k\Big(B(\Sigma_1), \ldots, B(\Sigma_k), X^{**}\Big)$ which extends T and is $\omega^* - \omega^*$ separately continuous (the ω^*-topology that we consider in $B(\Sigma_i)$ is the one induced by the ω^*-topology of $C(K_i)^{**}$). As well, we have*

1. $\|T\| = \|\overline{T}\|$.
2. *For every $\Big(g_1, \overset{[i]}{\ldots}, g_k\Big) \in B(\Sigma_1) \times \overset{[i]}{\ldots} \times B(\Sigma_k)$ there is a unique X^{**}-valued bounded ω^*-Radon measure $\gamma_{g_1, \overset{[i]}{\ldots}, g_k}$ on K_i (that is, a X^{**}-valued finitely additive bounded vector measure on the Borel subsets of K_i, such that for every $x^* \in X^*$, $x^* \circ \gamma_{g_1, \overset{[i]}{\ldots}, g_k}$ is a Radon measure on K_i), satisfying*
$$\int g_i \, d\gamma_{g_1, \overset{[i]}{\ldots}, g_k} = \overline{T}(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_k), \ \forall g_i \in B(\Sigma_i).$$
3. *$\overline{T}$ is $\omega^* - \omega^*$ sequentially continuous (that is, if $(g_i^n)_{n \in \mathbb{N}} \subset B(\Sigma_i)$, for $i = 1, \ldots k$, and $g_i^n \xrightarrow{\omega^*} g_i$, then $\lim\limits_{n \to \infty} \overline{T}(g_1^n, \ldots, g_k^n) = \overline{T}(g_1, \ldots, g_k)$ in the $\sigma(X^{**}, X^*)$ topology).*

Also, if we define $\gamma : B(\Sigma_1) \times \cdots \times B(\Sigma_k) \mapsto X^{**}$ by
$$\gamma(A_1, \ldots, A_k) := \overline{T}(\chi_{A_1}, \ldots \chi_{A_k}),$$
then γ is a polymeasure of bounded semivariation that satisfies

(a) $\|T\| = \|\gamma\|$.
(b) $T(f_1, \ldots, f_k) = \int (f_1, \ldots, f_k) d\gamma \ \Big(f_i \in C(K_i)\Big)$
(c) *For every $x^* \in X^*$, $x^* \circ \gamma$ is a regular (scalar) polymeasure and the map $x^* \mapsto x^* \circ \gamma$ is continuous for corresponding weak-$*$ topologies in X^* and $\Big(C(K_1)\overline{\otimes}_\pi \cdots \overline{\otimes}_\pi C(K_k)\Big)^*$.*

Conversely, if $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \mapsto X^{**}$ is a polymeasure which satisfies (c), then it has finite semivariation and formula (b) defines a k-linear continuous operator from $C(K_1) \times \cdots \times C(K_k)$ into X for which (a) holds.

Therefore the correspondence $T \leftrightarrow \gamma$ *is an isometric isomorphism between* $\mathcal{L}^k(C(K_1),\dots,C(K_k);X)$ *and the polymeasures in* $\mathrm{bsv}-\omega^*-\mathrm{rcapm}\,(\Sigma_1\times\cdots\times\Sigma_k;X^{**})$ *that satisfy condition (c).*

Our aim now is to exploit both representation theories, measures and polymeasures, to study multilinear operators on $C(K)$ spaces. In this paper we present some results in this direction.

2. The Main Results

The following definition can be found in [**6**] or in [**14**].

Definition 2.1: A polymeasure $\gamma:\Sigma_1\times\cdots\times\Sigma_k\mapsto X$ is said to be uniform in the i^{th} variable if it is countably additive and the measures

$$\left\{\gamma(A_1,\dots,A_{i-1},\cdot,A_{i+1},\dots A_k)\in\mathrm{ca}\,(\Sigma_i;X):\left(A_1,\overset{[i]}{\dots},A_k\right)\in\Sigma_1\times\overset{[i]}{\cdots}\times\Sigma_k\right\}$$

are uniformly countably additive.

A polymeasure is said to be uniform if it is uniform in every variable.

It is easy to check that given a natural number r, $1<r<k$ and r indices $1\leqslant j(1)<j(2)<\ldots<j(r)\leqslant k$, and given fixed $h_{j(p)}\in B\left(\Sigma_{j(p)}\right)$, $p=1\ldots r$, we can construct the multilinear operator

$$T_{h_{j(1)},\dots h_{j(r)}}:\prod_{\substack{1\leqslant q\leqslant k\\ q\notin(j(1)\dots j(r))}}C(K_q)\mapsto X$$

defined by $T_{h_{j(1)},\dots h_{j(r)}}\left(h_{q(1)},\dots,h_{q(k-r)}\right):=\overline{T}(h_1,\dots,h_k)$ whose associated polymeasure we shall call $\gamma_{h_{j(1)},\dots h_{j(r)}}$.

Given a bounded polymeasure $\gamma:\Sigma_1\times\cdots\times\Sigma_k\mapsto X$ and a fixed number i, $1\leqslant i\leqslant k$, we can construct in a natural way the measure $\phi_i:\Sigma_i\mapsto\mathrm{bpm}\left(\Sigma_1,\overset{[i]}{\cdots},\Sigma_k;X\right)$ defined by $\phi_i(A_i):=\gamma_{A_i}$. The fact that ϕ_i is bounded, indeed $\|\phi_i\|=\|\gamma\|$, and the following lemma are easy to check.

Lemma 2.2. *With the above notation, a countably additive polymeasure* γ *is uniform in the* i^{th} *variable if and only if* ϕ_i *is countably additive. The same is true if in this statement "countably additive" is replaced by "regular".*

Let $E_1,\dots,E_k,X$ be Banach spaces. Each $T\in\mathcal{L}^k(E_1,\dots,E_k;X)$ generates in a natural way k linear operators

$$T_i:E_i\mapsto\mathcal{L}^{k-1}\left(E_1,\overset{[i]}{\dots},E_k;X\right),\ \ i=1,\dots,k$$

defined by $T_i(x_i)\left(x_1,\overset{[i]}{\dots},x_k\right):=T(x_1,\dots,x_k)$ for each $x_j\in E_j$, $j=1,\dots,k$.

We shall state now a definition:

Definition 2.3: A k-linear mapping $T \in \mathcal{L}^k(E_1, \dots, E_k; X)$ is said to be *regular* if every mapping T_i defined above is weakly compact.

When X is the scalar field, the above definition was given in [**3**]. In general, given an operator ideal $\mathcal{U}$, we can define the *$\mathcal{U}$-regular* k-linear mappings as those such that the corresponding T_i belong to $\mathcal{U}$ for every $1 \leqslant i \leqslant k$. When $\mathcal{U}$ is the ideal of compact operators, such mappings have been considered in [**8**], and for a general closed injective operator ideal $\mathcal{U}$ in [**9**]. In every case a non-linear version of the factorisation theorem of Davies, Figiel, Johnson and Pelczynsky (see [**5**, pages 250, 259]) through operators in $\mathcal{U}$ is obtained for such multilinear mappings. These results are then applied to get some factorisation theorems for holomorphic mappings.

We are ready now to prove the following characterisation of the uniform polymeasures.

Theorem 2.4. *Let $K_1, \dots, K_k$ be compact Hausdorff spaces, let X be a Banach space and let $T \in \mathcal{L}^k\big(C(K_1), \dots, C(K_k); X\big)$. Let $\gamma : \Sigma_1 \times \dots \times \Sigma_k \to X^{**}$ be the polymeasure associated to it according to Theorem 1.1. Then γ is uniform if and only if T is regular. In that case the measures ϕ_i defined before Lemma 2.2 are the measures canonically associated to the operators T_i.*

Proof: Let us first assume that γ is uniform (in particular this means that γ is regular countably additive and therefore X-valued, see [**7**]). According to Lemma 2.2 this means that for each $i = 1, \dots, k$, $\phi_i \in \operatorname{rca}\big(\Sigma_i; \operatorname{rcapm}\big(\Sigma_1, \overset{[i]}{\dots} \Sigma_k; X\big)\big)$. Since $\operatorname{rcapm}\big(\Sigma_1, \overset{[i]}{\dots} \Sigma_k; X\big) \subset \mathcal{L}^{k-1}\big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\big)$ (see Theorem 1.1) we get that $\phi_i \in \operatorname{rca}\big(\Sigma_i; \mathcal{L}^{k-1}\big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\big)\big)$. Then we can consider the operator $H_{\phi_i} \in \mathcal{L}\big(C(K_i); \mathcal{L}^{k-1}\big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\big)\big)$ associated to ϕ_i by the Riesz representation theorem (vector valued case; see [**5**, Theorem VI.2.1]). Since ϕ_i is countably additive we know that H_{ϕ_i} is weakly compact ([**5**, Theorem VI.2.5]). We consider now $H_{\phi_i}^{**}$, the bitranspose of H_{ϕ_i}. Since H_{ϕ_i} is weakly compact we get that $H_{\phi_i}^{**}$ is $\mathcal{L}^{k-1}\big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\big)$-valued. It is easy to see that for every $A_i \in \Sigma_i$, and for every $\big(f_1, \overset{[i]}{\dots}, f_k\big) \in C(K_1) \times \overset{[i]}{\dots} \times C(K_k)$,

$$\begin{aligned} H_{\phi_i}^{**}(A_i)\big(f_1, \overset{[i]}{\dots}, f_k\big) &= \big\langle \phi_i(\chi_{A_i}), \big(f_1, \overset{[i]}{\dots}, f_k\big)\big\rangle = \int \big(f_1, \overset{[i]}{\dots}, f_k\big)\, d\gamma_{A_i} \\ &= \overline{T}(f_1, \dots f_{i-1}, \chi_{A_i}, f_{i+1}, \dots f_k). \end{aligned}$$

Therefore,

$$H_{\phi_i}^{**}(g_i)\big(f_1, \overset{[i]}{\dots}, f_k\big) = \overline{T}(f_1, \dots f_{i-1}, g_i, f_{i+1}, \dots f_k),$$

for every Σ_i-simple function g_i and for every $\big(f_1, \overset{[i]}{\dots}, f_k\big) \in C(K_1) \times \overset{[i]}{\dots} \times C(K_k)$. From continuity, we get the same relation for every $g_i \in B(\Sigma_i)$. In particular, when we choose $f_i \in C(K_i)$ we get

$$H_{\phi_i}^{**}(f_i)\big(f_1, \overset{[i]}{\dots}, f_k\big) = \overline{T}(f_1, \dots f_{i-1}, f_i, f_{i+1}, \dots f_k)$$

$$= T(f_1, \dots f_{i-1}, f_i, f_{i+1}, \dots f_k) = T_i(f_i)\Big(f_1, \overset{[i]}{\dots}, f_k\Big).$$

Obviously this means that $T_i = H_{\phi_i}$ and, therefore, that T_i is weakly compact.

Let us now assume that T is regular. Then, for every $i = 1 \dots k$, $T_i \in \mathcal{L}\Big(C(K_i)\Big)$; $\mathcal{L}^{k-1}\Big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\Big)$ is weakly compact and so the measure μ_i associated to it by the Riesz representation theorem is countably additive and $\mathcal{L}^{k-1}\Big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\Big)$-valued [**5**, Theorem VI.2.5]. We shall check now that for every $i = 1 \dots k$, $\mu_i = \phi_i$. Then, the proof will be finished just by looking at Lemma 2.2.

Let T_i^{**} be the bitranspose of T_i. For each $A_i \in \Sigma_i$ let $(f_i^\alpha)_{\alpha \in I}$ be a net in $C(K_i)$ such that $f_i^\alpha \overset{\omega^*}{\to} \chi_{A_i}$. T_i^{**} is known to be ω^*-ω^* continuous; since T_i is weakly compact we get that T_i^{**} is $\mathcal{L}^{k-1}\Big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\Big)$-valued. Both of these facts together imply that $\Big(T_i^{**}(f_i^\alpha)\Big)_{\alpha \in I}$ converges weakly to $T_i^{**}(\chi_{A_i})$. For fixed $\Big(f_1, \overset{[i]}{\dots}, f_k\Big) \in C(K_1) \times \overset{[i]}{\dots} \times C(K_k)$ and $x^* \in X^*$, the linear form

$$\theta : \mathcal{L}^{k-1}\Big(C(K_1), \overset{[i]}{\dots}, C(K_k); X\Big) \mapsto \mathbb{K}$$

defined by $\theta(S) := \Big\langle S\Big(f_1, \overset{[i]}{\dots}, f_k\Big), x^*\Big\rangle$ is clearly continuous and therefore

$$\theta\Big(T_i^{**}(f_i^\alpha)\Big) \to \theta\Big(T_i^{**}(\chi_{A_i})\Big) = \Big\langle T_i^{**}(\chi_{A_i})\Big(f_1, \overset{[i]}{\dots}, f_k\Big), x^*\Big\rangle.$$

Also

$$\theta\Big(T_i^{**}(f_i^\alpha)\Big) = \Big\langle T_i^{**}(f_i^\alpha)\Big(f_1, \overset{[i]}{\dots}, f_k\Big), x^*\Big\rangle = \Big\langle T(f_1, \dots f_{i-1}, f_i^\alpha, f_{i+1}, \dots f_k), x^*\Big\rangle.$$

Since $\overline{T}$ is separately ω^*-ω^* continuous we get that this last expression converges to $\Big\langle \overline{T}(f_1, \dots f_{i-1}, \chi_{A_i}, f_{i+1}, \dots, f_k), x^*\Big\rangle$. So we have obtained that for every $x^* \in X^*$, $\Big\langle \overline{T}(f_1, \dots f_{i-1}, \chi_{A_i}, f_{i+1}, \dots, f_k), x^*\Big\rangle = \Big\langle T_i^{**}(\chi_{A_i})\Big(f_1, \overset{[i]}{\dots}, f_k\Big), x^*\Big\rangle$. Therefore for every $A_i \in \Sigma_i$ and for every $\Big(f_1, \overset{[i]}{\dots}, f_k\Big) \in C(K_1) \times \overset{[i]}{\cdots} \times C(K_k)$,

$$\overline{T}(f_1, \dots f_{i-1}, \chi_{A_i}, f_{i+1}, \dots, f_k) = T_i^{**}(\chi_{A_i})\Big(f_1, \overset{[i]}{\dots}, f_k\Big) = \mu_i(A_i)\Big(f_1, \overset{[i]}{\dots}, f_k\Big).$$

But clearly

$$\overline{T}(f_1, \dots f_{i-1}, \chi_{A_i}, f_{i+1}, \dots, f_k) = \int \Big(f_1, \overset{[i]}{\dots}, f_k\Big)\, d\gamma_{A_i} = \phi_i(A_i)\Big(f_1, \overset{[i]}{\dots}, f_k\Big).$$

From here it follows that $\mu_i = \phi_i$ and the proof is over. ∎

Since every operator from $C(K_1)$ to $C(K_2)^*$ is weakly compact (see [**5**, Theorem VI.2.15]), we get immediately the following result (see [**6**]):

Corollary 2.5. *Every regular countably additive scalar bimeasure $\gamma : \Sigma_1 \times \Sigma_2 \to \mathbb{K}$ is uniform.*

From the above theorem we can derive the following propositions, useful to decide whether a polymeasure is or is not uniform. First we need a lemma.

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Lemma 2.6. *Let $T : C(K_1) \times \cdots \times C(K_k) \mapsto X$ be a regular k-linear operator. Let $(f_i^n)_{n\in\mathbb{N}} \subset C(K_i)$ be a weakly null sequence and let $\left((g_1^n)_{n\in\mathbb{N}}, \overset{[i]}{\ldots}, (g_k^n)_{n\in\mathbb{N}}\right) \subset B(\Sigma_1)\times \overset{[i]}{\ldots} \times B(\Sigma_k)$ be bounded sequences. Then, with the notation of Theorem 1.1, $\overline{T}\left(g_1^n, \ldots g_{i-1}^n, f_i^n, g_{i+1}^n, \ldots, g_k^n\right)$ converges in norm to zero.*

Proof: If T is regular, then the above defined operator T_i is weakly compact and therefore completely continuous, by the Dunford-Pettis property of $C(K_i)$. This means that $\left\|T_i(f_i^n)\right\| \to 0$. We observe now that, due to the uniqueness of the extension (1.1), for every $\left(g_1, \overset{[i]}{\ldots}, g_k\right) \subset B(\Sigma_1)\times \overset{[i]}{\ldots} \times B(\Sigma_k)$ and for every $f_i \in C(K_i)$, we have $\overline{T_i(f_i)}\left(g_1, \overset{[i]}{\ldots}, g_k\right) = \overline{T}(g_1, \ldots g_{i-1}, f_i, g_{i+1}, \ldots, g_k)$. By the equality of the norms of the operator and its extension, we can write $\left\|\overline{T_i(f_i^n)}\right\| \to 0$. This can also be written as

$$\sup_{g_j \in B_{B(\Sigma_j)}} \left\|\overline{T_i(f_i^n)}\left(g_1, \overset{[i]}{\ldots}, g_k\right)\right\| \to 0,$$

which means that

$$\sup_{g_j \in B_{B(\Sigma_j)}} \left\|\overline{T}(g_1, \ldots g_{i-1}, f_i^n, g_{i+1}, \ldots, g_k)\right\| \to 0$$

and finishes the proof. ∎

Proposition 2.7. *A regular countably additive polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \mapsto X$ is uniform in the i^{th} variable if and only if the measures*

$$\left\{\gamma_{g_1, \overset{[i]}{\ldots}, g_k} : \left(g_1, \overset{[i]}{\ldots}, g_k\right) \in B(\Sigma_1)\times \overset{[i]}{\cdots} \times B(\Sigma_k),\ \|g_j\| \leqslant 1\right\}$$

are uniformly countably additive.

Proof: One of the implications is clear. For the other, let us suppose that γ is uniform in the i^{th} variable. Were the measures $\left\{\gamma_{g_1, \overset{[i]}{\ldots}, g_k}; \left(g_1, \overset{[i]}{\ldots}, g_k\right) \in B(\Sigma_1)\times \overset{[i]}{\cdots} \times B(\Sigma_k)\right\}$ not uniformly countably additive, then there would exist $\varepsilon > 0$, a sequence $(A_i^n)_{n\in\mathbb{N}} \subset \Sigma_i$ of disjoint open sets and sequences $\left((g_1^n)_{n\in\mathbb{N}}, \overset{[i]}{\ldots}, (g_k^n)_{n\in\mathbb{N}}\right) \subset B(\Sigma_1)\times \overset{[i]}{\ldots} \times B(\Sigma_k)$ with $\|g_j^n\| \leqslant 1$ for each $n \in \mathbb{N}$ and for each $j = 1 \overset{[i]}{\ldots} k$, such that $\left\|\gamma_{g_1, \overset{[i]}{\ldots}, g_k}(A_i^n)\right\| > \varepsilon$. Then for each $n \in \mathbb{N}$ there would exist $f_i^n \in C(K_i)$ with $\operatorname{supp} f_i^n \subset A_i^n$ and $\|f_i^n\| \leqslant 1$ such that $\left\|\int f_i^n d\gamma_{g_1, \overset{[i]}{\ldots}, g_k}\right\| > \varepsilon$. This contradicts Lemma 2.6, since the sequence f_i^n converges weakly to 0. ∎

Proposition 2.8. *A regular countably additive polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \mapsto X$ is uniform in the i^{th} variable if and only if the measures*

$$\left\{\gamma_{f_1, \overset{[i]}{\ldots}, f_k}; \left(f_1, \overset{[i]}{\ldots}, f_k\right) \in C(K_1)\times \overset{[i]}{\cdots} \times C(K_k),\ \|f_j\| \leqslant 1\right\}$$

are uniformly countably additive.

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PROOF: In one direction the result follows from the previous proposition. For the other, we shall suppose without loss of generality that $i = k$. Let us suppose that the measures $\left\{\gamma_{f_1,\dots,f_{k-1}}; (f_1,\dots,f_{k-1}) \in C(K_1) \times \dots \times C(K_{k-1}),\ \|f_j\| \leqslant 1\right\}$ are uniformly countably additive. If γ is not uniform in the k^{th} variable then there exist a sequence $A_k^n \subset \Sigma_k$ of disjoint open sets and sequences $\left(A_j^n\right)_{n\in\mathbb{N}} \subset \Sigma_j$ for $j = 1\dots k-1$ such that $\left\|\gamma(A_1^n,\dots,A_k^n,)\right\| > \varepsilon$. Since γ is regular, $\gamma(\cdot, A_2^n,\dots A_k^n)$ is regular for each $n \in \mathbb{N}$ and therefore there exists a function $f_1^n \in C(K_1)$ with $\|f_1^n\| \leqslant 1$ such that $\left\|\int f_1^n d\gamma_{A_2^n,\dots,A_k^n}\right\| > \varepsilon$. Now $\gamma_{f_1^n,\cdot,\chi_{A_3^n},\dots,\chi_{A_k^n}}$ is also regular and therefore there exists a function $f_2^n \in C(K_2)$ with $\|f_2^n\| \leqslant 1$ such that $\left\|\int f_2^n d\gamma_{f_1^n,\chi_{A_3^n},\dots,\chi_{A_k^n}}\right\| > \varepsilon$. Continuing in the same way we obtain $k-1$ sequences of norm one functions $f_j^n \subset C(K_j)$, $j = 1\dots k-1$ such that $\left\|\gamma_{f_1^n,\dots,f_{k-1}^n}(A_k^n)\right\| > \varepsilon$ which contradicts the hypothesis. ∎

3. POLYMEASURES ON COMPACT DISPERSED SPACES

Recall that a compact Hausdorff space is said to be *dispersed* if it does not contain any non empty perfect set. In **[12]** a deep insight is given into the structure of dispersed spaces, proving among other results that K is dispersed if and only if $C(K)$ contains no copy of ℓ_1, if and only if $C(K)^*$ contains no copy of L_1. Also, in this case $C(K)^*$ can be identified with $\ell_1(\Gamma)$ for some Γ.

Some (if not all) of the following results are probably known, but we have not been able to find an explicit reference.

THEOREM 3.1. *For a compact Hausdorff space K, the following statements are equivalent:*

(a) *K is dispersed.*

(b_0) *For every $k \geqslant 1$, the space $\mathcal{L}^k\big(C(K)\big)$ is Schur.*

(b_1) *For some $k \geqslant 2$, the space $\mathcal{L}^k\big(C(K)\big)$ is Schur.*

(b_2) *For some $k \geqslant 2$, the space $\mathcal{P}\big({}^kC(K)\big)$ is Schur.*

(b_3) *For every $k \geqslant 2$, the space $\mathcal{P}\big({}^kC(K)\big)$ is Schur.*

(c_0) *For every $k \geqslant 1$, the space $\mathcal{L}^k\big(C(K)\big)$ is weakly sequentially complete.*

(c_1), (c_2), (c_3) *Same statements as* (b_1), (b_2), (b_3), *replacing Schur by weakly sequentialy complete.*

(d_0) *For every $k \geqslant 1$, $\mathcal{L}^k\big(C(K)\big)$ contains no copy of ℓ_∞.*

(d_1), (d_2), (d_3) *Same statements as* (b_1), (b_2), (b_3), *replacing Schur by the non containement of ℓ_∞.*

(e) *For every $k \geqslant 1$, $\mathcal{L}^k\big(C(K)\big)$ contains no copy of c_0.*

(e_1), (e_2), (e_3) *Same statements as* (d_1), (d_2), (d_3), *replacing ℓ_∞ by c_0.*

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PROOF: Since $\mathcal{L}^k\big(C(K)\big)$ is a dual space for every $k \geqslant 1$, every (d) statement is equivalent to the corresponding (e) statement. Also clearly (b_i) $\Rightarrow$ (c_i) $\Rightarrow$ (d_i), for every i, (b_0) $\Rightarrow$ (b_1) $\Rightarrow$ (b_2) and (b_0) $\Rightarrow$ (b_3) $\Rightarrow$ (b_2). Therefore, it remains to prove (a) $\Rightarrow$ (b_0) and (e_2) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b_0): We shall prove it by induction on k. For $k = 1$, it is clear since $C(K)^* \approx \ell_1(\Gamma)$. Suppose now that

$$\mathcal{L}^k\big(C(K)\big) = \Big(\widehat{\bigotimes}{}^k_\pi C(K)\Big)^* := X^*$$

(see [5, Corollary VIII.2.2]) is Schur. Then

$$\mathcal{L}^{k+1}\big(C(K)\big) = \mathcal{L}\big(C(K); X^*\big) = \big(C(K)\overline{\otimes}_\pi X\big)^*.$$

Since $C(K)$ contains no copy of ℓ_1 and has the Dunford-Pettis property, by the induction hypothesis it follows that all members of the last space are compact operators. Hence, since $C(K)^*$ has the approximation property,

$$\mathcal{L}^{k+1}\big(C(K)\big) = C(K)^*\overline{\otimes}_\varepsilon X^*$$

[5, Theorem VIII.3.6], which is a Schur space, since this property is stable by taking injective tensor products (see [13]).

(e_2) $\Rightarrow$ (a): If K is not dispersed, $C(K)^* \supset L_1 \supset \ell_2$. Consequently

$$\ell_2\overline{\otimes}_\varepsilon \ell_2 \subset C(K)^*\overline{\otimes}_\varepsilon C(K)^* \subset \big(C(K)\overline{\otimes}_\pi C(K)\big)^*$$

(topological inclusions), and it is well known that if (e_n) is the canonical basis of ℓ_2, then $(e_n \otimes e_n)$ is equivalent to the canonical basis of c_0 (see [10]). This means that $\mathcal{P}\big({}^2C(K)\big)$ contains a copy of c_0. Since $\mathcal{P}\big({}^2C(K)\big)$ is a (complemented) subspace of $\mathcal{P}\big({}^kC(K)\big)$, for every $k \geqslant 2$, it follows that the latter space contains a copy of c_0, too. $\square$

As we mentioned in Corollary 2.5, every scalar regular bimeasure on a compact Hausdorff space is uniform. This is not true for arbitrary polymeasures, as the following example from [2] shows: The 3-linear map $T : C\big([0,1]\big) \times C\big([0,1]\big) \times C\big([0,1]\big) \to \mathbb{C}$ defined by

$$T(f,g,h) := \sum_{i=1}^{\infty} f\Big(\frac{1}{2^i}\Big) \int_0^1 g r_i \, dx \int_0^1 h r_i \, dx,$$

where r_i is the standard ith Rademacher function, is not regular. See [2] for details.

In the next theorem we show that the uniformity of all the k-polymeasures for some (every) $k \geqslant 3$, characterises compact dispersed spaces. We shall denote by $\mathcal{K}(X;Y)$ and $\mathcal{W}(X;Y)$ the compact and weakly compact operators between X and Y, respectively.

THEOREM 3.2. *For a compact Hausdorff space K the following statements are equivalent:*

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(a) *K is dispersed.*

(f) *For every (some) $k \geqslant 2$, $\mathcal{L}\left(C(K); \mathcal{L}^k\big(C(K)\big)\right) = \mathcal{K}\left(C(K); \mathcal{L}^k\big(C(K)\big)\right)$.*

(g) *For every (some) $k \geqslant 2$, $\mathcal{L}\left(C(K); \mathcal{L}^k\big(C(K)\big)\right) = \mathcal{W}\left(C(K); \mathcal{L}^k\big(C(K)\big)\right)$.*

(h) *For every (some) $k \geqslant 3$, any scalar regular k-polymeasure on the product of the Borel σ-algebra of K, is uniform.*

Proof: (a) $\Rightarrow$ (f) was included in the proof of (a) $\Rightarrow$ (b_0) in Theorem 3.1, and clearly (f) $\Rightarrow$ (g). The equivalence of (g) and (h) follows from Theorem 2.4. Finally, let us prove that (g) implies (a): Let $k \geqslant 3$. If K is not dispersed, $C(K)$ is infinite dimensional and thus contains a copy of c_0 [**5**, Corollary VI.2.16]. On the other hand, by Theorem 3.1, $\mathcal{L}^{k-1}\big(C(K)\big)$ contains a copy of ℓ_∞. By the injectivity of this space, the inclusion map from c_0 into ℓ_∞ can be extended to the whole space $C(K)$, providing in this way a non weakly compact operator in $\mathcal{L}\left(C(K); \mathcal{L}^{k-1}\big(C(K)\big)\right)$. ∎

The equivalence of (a), (f) and (g) has been also obtained in [**1**], although with a different and, in our opinion, more involved proof.

References

[1] J. Alaminos, Y.S. Choi, S.G. Kim, and R. Payá, 'Norm attaining bilinear forms on spaces of continuous functions', (preprint).

[2] R. Aron, S.Y. Choi, and J.L.G. Llavona, 'Estimates by polynomials', *Bull. Austral. Math. Soc.* **52** (1995), 475–486.

[3] R. Aron and P. Galindo, 'Weakly compact multilinear mappings', *Proc. Edinburgh Math. Soc.* **40** (1997), 181–192.

[4] F. Bombal and I. Villanueva, 'Multilinear operators on spaces of continuous functions', *Funct. Approx. Comment. Math.* **25** (1998), 117–126.

[5] J. Diestel and J.J. Uhl, *Vector measures*, Mathematical Surveys **15** (American Math. Soc., Providence, R.I., 1977).

[6] I. Dobrakov On integration in Banach spaces, VIII (polymeasures), *Czech. Math. J.* **37** (1987), 487–506.

[7] I. Dobrakov, 'Representation of multilinear operators on $\times C_0(T_i)$', *Czech. Math. J.* **39** (1989), 288–302.

[8] M. González and J. Gutiérrez, 'Factorisation of weakly continuous holomorphic mappings', *Studia Math.* **118** (1996), 117–133.

[9] M. González and J. Gutiérrez, 'Injective factorisation of holomorphic mappings' (to appear).

[10] J.R. Holub, 'Tensor product bases and tensor diagonals', *Trans. Amer. Math. Soc.* **151** (1970), 563–579.

[11] A. Pelczynski, 'A theorem of Dunford-Pettis type for polynomial operators', *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys.* **11** (1963), 379–386.

[12] A. Pelczynsky and Z. Semadeni, 'Spaces of continuous functions (III). Spaces $C(\Omega)$ for Ω without perfect subsets', *Studia Math.* **18** (1959), 211–222.

[**13**] W. Ruess and D. Werner, 'Structural properties of operator spaces', *Acta Univ. Carolin. - Math. Phys.* **28** (1987), 127–136.

[**14**] I. Villanueva, 'Polimedidas y representación de operadores multilineales de $C(\Omega_1, X_1) \times \cdots \times C(\Omega_d, X_d)$', *Tesina de Licenciatura, Dpto. de Análisis Matemático, Fac. de Matemáticas, Universidad Complutense de Madrid* (1997).

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