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Open Problems from 2020 Vision for Dynamics Conference in Będlewo

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1.1 Introduction

This chapter contains a collection of open problems presented during the "2020 Vision for Dynamics" conference held in Będlewo in the fall of 2019. We have collected open problems presented in the two scheduled problem sessions: those discussed during the conference talks, and those contributed outside these events. We are grateful to all contributors for taking part in the preparation of this chapter. We thank Yakov Pesin for writing Section 1.2.15. We also thank Corinna Ulcigrai for sharing with us her notes from one of the problem sessions.

1.2 Open Problems

1.2.1 Pierre Berger

Here are two kinds of interesting measures in the phase space for (non-uniformly) hyperbolic dynamics:

- Measures of maximal entropy;
- Sinai–Ruelle–Bowen or, more generally, Gibbs measures.

The following interesting properties suffice to motivate their studies. In many contexts, the measure of maximal entropy is equidistributed on the hyperbolic periodic points, while the Gibbs measure has its unstable dimension, which coincides with one of its supports.

Are there similar measures *in the parameter space*?

For holomorphic families of dynamics of polynomial maps of $\mathbb{C}$, the bifurcation measure μ^{biff} is the canonical counterpart of the maximal entropy measure: it is equidistributed on the set of parameters for which all the critical points are strictly

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preperiodic, among other equidistribution properties; see [19, 44, 20] for a survey. Such a measure can be consistently defined for any properly embedded family of polynomials. Interestingly, at the μ^{biff}-a.e. parameter the dynamics satisfy the Collet–Eckmann condition [25, 47].

Problem 1. *When is it possible to define a counterpart of μ^{biff} for the real family of dynamics of the interval?*

Let us also recall the following:

Problem 2. *Find a counterpart of the bifurcation measure for surface mappings (with similar equidistribution properties).*

By a theorem of Makarov [37], the bifurcation measure for the quadratic family has Hausdorff dimension 1 and its support is the Mandelbrot set. Nevertheless, by Shishikura's theorem [46], the Hausdorff dimension of the Mandelbrot set is 2.

Problem 3. *Define a natural measure on the parameter space of the quadratic family supported by the Mandelbrot set and with full Hausdorff dimension* $(= 2)$.

A solution of this problem would lead to a parametric counterpart of the Gibbs measure. This might be helpful for studying the Lebesgue measure of the Mandelbrot set (which is conjecturally 0 following Shishikura [46]).

1.2.2 Jairo Bochi

Problem 4. *Is there a C^∞ conservative (i.e., volume-preserving) Anosov diffeomorphism of $\mathbb{T}^3$ with a 2-dimensional unstable manifold whose top Lyapunov exponent (with respect to volume) is strictly bigger than the top Lyapunov exponent of the linear Anosov map in the same homotopy class?*

By a result of Brin, Burago, and Ivanov [7], the unstable bundle cannot admit a dominated splitting.

Added in proof: The question was recently answered affirmatively by Carrasco and Saghin [10].

1.2.3 Danijela Damjanović

The questions here aim at more precise understanding of what happens with symmetry groups under perturbations. Typically they should not be preserved, but can one describe precisely what happens with symmetry groups after perturbation?

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For a smooth diffeomorphism $f \in \mathrm{Diff}^\infty(\mathrm{M})$ of a compact smooth manifold M, let $Z^\infty(f) = \{g \in \mathrm{Diff}^\infty(\mathrm{M})\colon g \circ f = f \circ g\}$ denote the smooth centralizer of f in $\mathrm{Diff}^\infty(\mathrm{M})$. Generically in C^1 topology, it was conjectured by Smale and proved by Bonatti, Crovoisier, and Wilkinson [6] that $Z^\infty(f)$ contains only powers of f. However, for algebraic transformations, the (algebraic) centralizer can be large. One class of such examples is ergodic automorphisms of nilmanifolds. Such automorphisms are necessarily partially hyperbolic since ergodicity of a nilmanifold automorphism is given by ergodicity of the induced map of the abelianization, and ergodic toral automorphisms are always partially hyperbolic. Let f_0 be such an ergodic (partially hyperbolic) automorphism of a nilmanifold. Centralizers of such maps within the space of automorphisms can often be very well understood by using algebraic methods, especially under extra conditions of irreducibility, and predominantly these centralizers have rank greater than 2. In particular, these automorphisms are very atypical in the sense of Smale.

Problem 5 (Local centralizer classification)**.** *Let f_0 be an ergodic (partially hyperbolic) automorphism of a nilmanifold. Let f be a C^1-small smooth perturbation of f_0. Can we give a finite list of possibilities for $Z^\infty(f)$ for every f in a small C^1 neighborhood of f_0 and show that every possibility on this list actually happens?*

If the centralizer $Z^\infty(f)$ for a perturbation f is (up to a finite index subgroup) isomorphic to $Z_{\mathrm{aff}}(f_0)$, and $Z_{\mathrm{aff}}(f_0)$ is a group containing at least $\mathbb{Z}^2$, then we can hope in some cases that $Z^\infty(f)$ being large can give some more information on f. To make the question more precise, we may even assume that the action $Z^\infty(f)$ is homotopic to the linear part of $Z_{\mathrm{aff}}(f_0)$.

Problem 6 (Local centralizer rigidity)**.** *Assume that $Z^\infty(f)$ is homotopic to the linear part of $Z_{\mathrm{aff}}(f_0)$. For which f_0 does this imply that f is C^∞ conjugate to f_0?*

For some initial steps in these directions we refer the reader to [17] and references therein, and [28]. One of the results proved in [17] is the following dichotomy:

Theorem 1 ([17])**.** *Let f_0 be the time-1 map of the geodesic flow on a closed, negatively curved, locally symmetric manifold. The centralizer $Z_\infty(f_0)$ is $\mathbb{R}$. Let f be a C^1, volume-preserving perturbation of f_0. Then $Z_\infty(f)$ is either virtually $\mathbb{Z}$ or it contains a smooth flow and is virtually $\mathbb{R}$.*

Related to this, we ask the following:

Problem 7. *Is it true that a sufficiently small perturbation of any volume-preserving Anosov flow either has a virtually trivial centralizer, or embeds in a smooth flow?*

1.2.4 Manfred Denker

Problem 8. *In cases where a normal conditional local limit theorem holds, prove convergence of the local time of an integer-valued function to the local time of Brownian motion (at least if the dynamics is Gibbs–Markov).*

Background information can be extracted from [18], which may be seen as a first attempt to describe local times in dynamical systems.

Problem 9. *Can one formulate a suitable theory for the local time for real-valued functions and in the case of flows?*

A particular case in this problem is that of geodesic flows.

1.2.5 Bassam Fayad

The question of existence of a smooth, area-preserving diffeomorphism of the disc that is mixing with zero (metric or topological) entropy was raised by Fayad and Katok in [23].

Problem 10. *Prove that a conservative (area-preserving) transitive diffeomorphism of the disk with zero topological entropy is not mixing.*

Problem 11. *Does there exist a conservative diffeomorphism of the disk which has zero topological entropy but positive metric entropy?*

Problem 12. *On interval maps, there exists a dichotomy: an interval map on* $[0,1]$ *that fixes the boundary has growth that is either exponential or below* n^2*. Is this true for zero-entropy maps on surfaces?*

The next question is about affine parabolic abelian actions.

Problem 13. *Take two parabolic commuting affine maps of the torus:* $f = A + \alpha$ *and* $g = B + \beta$*. Their linear parts* A *and* B *are given by commuting unipotent matrices in* $SL(n,\mathbb{Z})$*, and* α *and* β *are in* $\mathbb{R}^n$*. Consider the* $\mathbb{Z}^2$ *action generated by* f *and* g *on the torus* $\mathbb{T}^n$*. When is such an action locally rigid? More precisely, do the local rigidity results on commuting perturbations of rigid translations on the torus (which follow Moser's result [38] on the circle, such as [15], [49]) extend to some of these parabolic actions?*

Progress has been made toward this problem under extra conditions on the linear part of the action, namely when one of the action generators is a step 2 parabolic map. Then a form of local rigidity is proved by Damjanović, Fayad, and Saprykina in [16].

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1.2.6 Giovanni Forni

Problem 14 (Katok)**.** *Prove that there exists a G_δ dense set of polygons with weakly mixing billiard flow. Prove that there exists a mixing polygon.*

Kerckhoff, Masur, and Smillie [32] proved ergodicity for a G_δ dense set of polygons. To prove mixing, one suggestion is to look at an arc of a circle around a point, consider its orbit, and see whether the push-forward of circles equidistributes on the surface. This has a fairly simple proof for the torus, but for other rational polygons the argument does not work: when the surface has singularities, the arcs at time T are expected to have length $1/T$.

The result by Chaika and Hubert [12] shows that for almost all translation surfaces there is a sequence of times for which the circle equidistributes, but the sequence depends on the point, so the result does not imply a (relative) weak mixing result for the geodesic flow on the translation surface.

Recently, the first part of the problem (on weak mixing) has been solved in the preprint [11].

Problem 15. *Prove (polynomial) bounds on the decay of correlations for time-changes of higher step nilflows.*

Mixing with polynomial decay of correlations for smooth observables is known only in a few cases. For results about rate of mixing, see Fayad [21] for Kochergin flows, Forni–Ulcigrai [24] for time changes of horocycle flows, Forni–Kanigowski [23] for Heisenberg nilflows, and Ravotti for surface locally Hamiltonian flows with non-degenerate saddles (this case has logarithmic decay).

To get precise decay of correlations one needs an estimate for the shear, especially one needs lower bounds for ergodic integrals of the "roof" function, namely the time-change function. Such estimates are hard to establish for higher-step nilflows.

1.2.7 Anton Gorodetski

The following problem was popularized by logician Matt Foreman. The general question is

Can one prove that the set of dynamical systems is impossible to classify?

More precisely:

Problem 16. *Consider $\mathit{Diff}^\infty(M)$ with $\dim M \geq 2$. Then*

$$\mathit{Diff}^\infty(M) \times \mathit{Diff}^\infty(M)$$

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is a complete metric space. Let $E \subseteq \mathit{Diff}^\infty(M) \times \mathit{Diff}^\infty(M)$ be the set of pairs of diffeomorphisms that are topologically conjugate. Can you prove that E is not Borel?

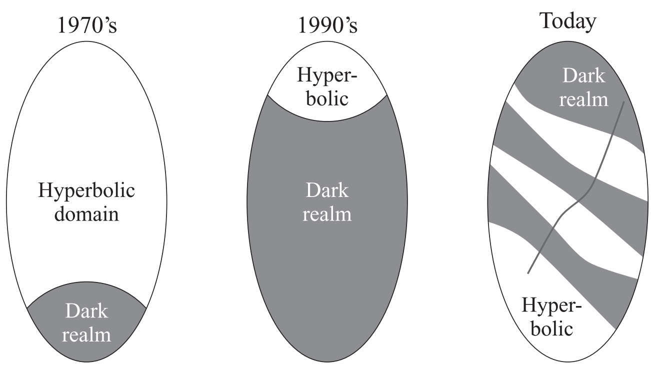
If the answer is yes, then this would imply that $\mathit{Diff}^\infty(M)$ is not classifiable, in the sense that there is no "countable" way to determine whether two diffeomorphisms are conjugate; see [22] for details.

The next problem concerns the "geometry" of hyperbolicity in a space of dynamical systems. The perception of the "size" of hyperbolicity has changed over the decades; see the figure.

Let $f: M \to M$ be a smooth dynamical system (e.g., an irrational rotation of the circle). Let $A_t: M \to SL(2,\mathbb{R})$ be a typical 1-parameter family of cocycles (the blue curve in the following figure). Let

$$\Sigma = \{t\colon\ (f, A_t) \text{ is not uniformly hyperbolic}\}$$

be the intersection of this curve with the set of (parameters of) non-hyperbolic cocycles.



For any specific dynamical system one can ask the following question:

Problem 17. *What is the structure of the set Σ for a typical 1-parameter family of cocycles over a given dynamical system? Does it have to be a finite number of intervals? A Cantor set? A Cantorval? In particular, in the case of $SL(2,\mathbb{R})$ cocycles over an irrational rotation of the circle, is it true that typically the set Σ is either empty or a Cantor set of positive measure?*

The problem is motivated by spectral theory of ergodic discrete Schrödinger operators, where the family of Schrödinger cocycles is parameterized by energy, and due to Johnson's theorem [31] the set Σ corresponds to the spectrum of an operator. In the case of irrational rotation of the circle, the problem can be considered as a dynamical analog of the Ten Martini Problem in spectral theory [3]. Also in that context, the case of C^0 cocycles over a strictly ergodic dynamical system (e.g., odometer) was studied in [1].

1.2.8 Colin Guillarmou

Let g_0 be a fixed metric of negative curvature on M. Denote by $\mathcal{C}$ the set of free homotopy classes. For $c \in \mathcal{C}$, define *marked length spectrum* $L_g : \mathcal{C} \to \mathbb{R}^+$ that maps c to the length of the shortest geodesic γ_c^g in the class c.

Define $\mathcal{L}_g : \mathcal{C} \to \mathbb{R}^+$ by

$$\mathcal{L}_g(c) := L_g(c)/L_{g_0}(c),$$

where $c \in \mathcal{C}$.

Consider the map

$$\Phi : g \in C^\infty(M; S_+^2 T^* M) \mapsto \mathcal{L}_g \in L^\infty(\mathcal{C}).$$

Problem 18. *What can be said about the range of Φ? Is the range a submanifold in some appropriate sense?*

1.2.9 Federico Rodriguez Hertz

Let X be a compact manifold (or a finite CW-complex, or any space that has a nice finite-dimensional homology theory). Let $T : X \to X$ be a homeomorphism. We say a matrix $A \in GL(n, \mathbb{R})$ is *quasi-unipotent* if all eigenvalues have modulus one.

Problem 19. *If T is minimal, is $T^* : H^k(X) \to H^k(X)$ quasi-unipotent for all k?*

Using results of Mañé , it is known that if T^* is hyperbolic, then T is not minimal. But if T^* is partially hyperbolic only, this is unknown.

Note that the minimal positive entropy example of Herman does not help here, because it is homotopic to the identity.

1.2.10 Steven Hurder

A Cantor action $(\mathfrak{X}, G, \Phi)$ is called *stable* if the chain $\left\{ \widehat{K}_\ell : \ \ell \geq 1 \right\}$ is bounded, that is, if there exists ℓ_0 so that $\widehat{K}_\ell = \widehat{K}_{\ell+1}$ for $\ell \geq \ell_0$. The action is called *wild* if the chain $\left\{ \widehat{K}_\ell : \ \ell \geq 1 \right\}$ is unbounded.

A monodromy action $(\mathfrak{X}, G, \Phi)$ is *locally quasi-analytic (LQA)* if there exists $\epsilon > 0$ so that, if U adapted and $\mathrm{diam}_{\mathfrak{X}}(U) < \epsilon$, then for all clopen $V \subset U$ we have

$$\Phi(g)\big|_V = Id \quad \Longrightarrow \quad \Phi(g)\big|_U = Id \qquad \text{for all } g \in G_U,$$

that is, the action of $\mathcal{H}_U^\Phi$ on U is *topologically free*.

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Problem 20. *How to classify Cantor actions of a finitely generated nilpotent group G? Can one use invariants of the associated cross-product C^*-algebra? Is a classification possible in terms of the representations of G?*

Problem 21. *If an action is wild, when is the action non-LQA?*

Problem 22. *For which number fields and polynomials f is the action of the absolute Galois group* $\mathrm{Gal}_{\mathrm{arith}}(f)$*, on the boundary of the tree of iterated solutions, non-LQA?*

Problem 23. *If G is a higher-rank lattice and the action is effective, must it be stable?*

Problem 24. *If G is a higher-rank lattice and the action is wild, must it be flat wild?*

For more details on Cantor actions, we refer the reader to [29], [30].

1.2.11 Yulij Ilyashenko

Consider diffeomorphisms of a closed manifold. An attractor is "thick" if both the attractor and its complement are of positive measure.

Problem 25 (Genericity of existence of thick attractors – simplified version)**.** *Is there an open set in Diff(M) such that every f in this set has a thick attractor?*

There is a theorem from the 1940s saying that in the space of diffeomorphisms of a manifold with boundary there is an open set of diffeomorphisms that preserve the boundary and these diffeomorphisms have thick attractors. So, genericity of existence of thick attractors is established for manifolds with boundary.

There are at least three kinds of "attractors":

1. The *maximal attractor* (or traditional attractor) for a map $f : X \to X$ is
$$\bigcap_{n=0}^{\infty} f^n(X).$$
This idea of an attractor is not reasonable for some maps. For example, a map of circle with one semi-stable (parabolic) fixed point could be considered to have that point as an attractor, but not with this definition.
2. If X is a metric space, the *Milnor attractor* of $f : X \to X$ is the minimal closed set with the property that
$$d(f^n x, A) \to 0 \text{ for a.e. } x.$$
The point of the definition is that A is the biggest global attractor.
3. There are also *statistical attractors*.

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Problem 26. *Is the non-coincidence of different definitions of attractors generic? To what extent is the situation of having one kind of attractor be strictly smaller than another generic?*

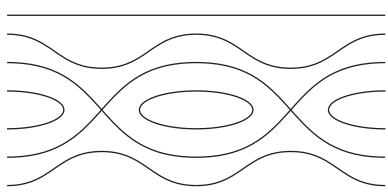
Recently, Ivan Shilin proved in [45] that non-coincidence of maximal and Milnor attractor is topologically generic in some domain.

Problem 27. *Is the coexistence of an infinite number of attracting periodic orbits "truly prevalent"?*

Newhouse showed coexistence of an infinite number of attracting periodic orbits in some open set of diffeomorphisms. Pierre Berger found open sets of dynamics where coexistence of an infinite number of attracting periodic orbits is typical in the sense of Arnold. But this notion of typicality has some drawbacks: if you apply it to $\mathbb{R}^n$, then there are metrically typical subsets of $\mathbb{R}^n$ with zero measure. The strongest notion of typicality is called prevalence and was introduced by Hunt, Sauer, and Yorke [43].

1.2.12 Raphaël Krikorian

Problem 28. *Take a twist real-analytic* (C^{ω}) *symplectic diffeomorphism of an annulus with* $h_{\mathrm{top}}(f) = 0$*. Does it follow that (up to a finite quotient)* f *is the time-*1 *map of a real-analytic flow (as in the figure)?*



The following question is related to the problems stated in Section 17. Consider quasi-periodic cocyles on $\mathbb{T}^d \times SL(2,\mathbb{R})$ given by $(x,y) \mapsto (x+\alpha, A(x)y)$. A cocycle is *non-uniformly hyperbolic (NUH)* if it has nonzero Lyapunov exponents. A cocycle is *almost reducible* if it cannot be conjugated to a constant but you can come arbitrarily close:

$$(\cdot, B_n) \circ (\alpha, A) \circ (\cdot, B_n)^{-1} \to (\alpha, \mathrm{const}).$$

A cocycle is *stable* if you cannot approximate it by NUH cocycles.

Problem 29. *Does stable imply almost reducible?*

Next problem is asking about possible generalization of the Herman–Yoccoz theorem to tori of any dimension.

Recall the Herman–Yoccoz theorem on $\mathbb{T}$. Let $f\colon \mathbb{T} \to \mathbb{T}$ be orientation-preserving and smooth diffeomorphism of the circle. We can define rotation number $\rho(f)$. By Denjoy's theorem, since f is at least C^2, we know that, if $\rho(f)$ is irrational, then f is topologically conjugate to rotation by $\rho(f)$. The Herman–Yoccoz theorem says that if f is smooth and $\rho(f)$ is Diophantine, then the conjugacy h is smooth.

The following question concerns generalization of the Herman–Yoccoz theorem to $f\colon \mathbb{T}^d \to \mathbb{T}^d$ with $d \geq 2$.

Problem 30. *Suppose $f \in \mathit{Diff}_0^\infty(\mathbb{T}^d)$ is homotopic to the identity and is conjugate to a translation T_α by vector α via a homeomorphism h. If α is Diophantine, is h necessarily smooth?*

A local result is that this is true if $|f - T_\alpha| < \epsilon(\alpha)$. The semi-local version would be to have ϵ not depending on α, but this is not known.

1.2.13 François Ledrappier

Let M be a Banach manifold, $f\colon M \to M$ be a diffeomorphism, and let μ be an invariant probability measure with compact support. Assume that $\int \log^+ \|D_x f\|\, d\mu(x)$ is finite. Then, by the subadditive ergodic theorem,

$$\frac{1}{n} \log \|D_x f^n\| \to \text{some } \lambda \qquad \mu\text{-almost everywhere.}$$

Problem 31 (Katok, 1987)**.** *Does $\lambda = 0$ imply $h_\mu(f) = 0$?*

When $f\colon M \to M$ is a C^1-diffeomorphism of a compact Riemannian manifold and μ is an f-invariant probability measure on M, the Ruelle inequality gives an upper bound on entropy $h_\mu(f)$ in terms of Lyapunov exponents, which in particular implies that the answer to the preceding question is yes in this case.

When M is finite-dimensional but non-compact, F. Riquelme in [42] gave an example where f is a diffeomorphism of M and μ is a finite f-invariant measure such that $\lambda = 0$, but $h_\mu(f) > 0$; this example satisfies the integrability condition, but the support of μ is non-compact.

P. Thieullen in his PhD thesis [48] showed an estimate on entropy which implies that the answer to this question is yes in the case of Banach manifolds, under extra compactness assumptions (when the quasi-compactness exponent is negative).

1.2.14 Mariusz Lemańczyk

Problem 32 (Katok, 2004 or earlier)**.** *Is it true that the von Neumann flows have finite multiplicity?*

A von Neumann flow is a special flow over a rotation in which the roof function has finitely many discontinuities and the sum of jumps is not equal to zero. These von Neumann flows are now known to have a variation of Ratner's property.

Let $\{T_t\}_{t\in\mathbb{R}}$ be a flow with Ratner's property. The *essential centralizer* of such a flow is defined as $C(T_t)/\{T_t\}_{t\in\mathbb{R}}$, where $C(T_t)$ is the total centralizer in the space of all automorphisms of the given measure space.

Problem 33. *What is the essential centralizer of a flow with Ratner's property? Is it always finite? (Lemańczyk and Kanigowski [33] proved it is at most countable.)*

It is expected that the essential centralizer of flows with Ratner's property is not always finite.

Remark: Unipotent flows in $SL(2)$ have Ratner's property, but in $SL(3)$ they do not, essentially because their homogeneous centralizer is a group larger than the flow itself. They do have the generalized shearing property, namely that the fastest relative motion is along the same direction as the centralizer of the flow.

1.2.15 Yakov Pesin

I state and discuss here two long-standing open problems in smooth ergodic theory related to genericity of diffeomorphisms with nonzero Lyapunov exponents. They first appeared in [39] and were also mentioned in [4, 14, 40].

Let M be a compact, smooth Riemannian manifold without boundary and f a volume-preserving diffeomorphism of M. Recall that given a point $x \in M$ and a tangent vector $v \in T_xM$, the *Lyapunov exponent* $\chi_f(x, v)$ at x in the direction of v is given by

$$\chi_f(x, v) := \lim_{n\to\infty} \frac{1}{n} \log \|df_x^n v\|,\ x \in M,\ v \in T_xM.$$

Denote by μ the Riemannian volume on M and by $\mathrm{Diff}^r(M, \mu)$ the space of all C^r volume-preserving diffeomorphisms on M. We say that a diffeomorphism $g \in \mathrm{Diff}^r(M, \mu)$ has nonzero Lyapunov exponents on a set $A = A(g) \subset M$ of positive (respectively full) volume if

1. $\mu(A) > 0$ (respectively, $\mu(A) = 1$);
2. for μ-almost every $x \in A$ (with respect to volume) and every $v \in TM$ one has $\chi_g(x, v) \neq 0$;
3. there are $v_1, v_2 \in T_xM$ such that $\chi_g(x, v_1) < 0$ and $\chi_g(x, v_2) > 0$.

Problem 34. *Let f be a C^2 volume-preserving diffeomorphism of M. Assume that f has nonzero Lyapunov exponents on a set of full volume. Then there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^2(M,\mu)$ of f and a G_δ subset $\mathcal{B} \subset \mathcal{U}$ such that every $g \in \mathcal{B}$ has nonzero Lyapunov exponents on a set $A = A(g) \subset M$ of positive volume.*

Problem 35. *Let f be a C^2 volume-preserving diffeomorphism of M (possibly with some zero Lyapunov exponents on a set of positive or even full volume). Then arbitrarily close to f in $\mathrm{Diff}^2(M,\mu)$ there is a diffeomorphism g, which has nonzero Lyapunov exponents on a set $A = A(g) \subset M$ of positive volume.*

Remark 1. The requirement that the map f is of class C^2 can be relaxed by assuming that f is of class $C^{1+\alpha}$, namely it is of class C^1 and the differential df_x is Hölder continuous in x. It is, however, crucial that f is not simply of class C^1 due to the following two results:

1. *On any compact surface there is a dense G_δ subset $\mathcal{U} \subset \mathrm{Diff}^1(M,\mu)$ such that any $f \in \mathcal{U}$ is either (1) Anosov (and is ergodic) or (2) has zero Lyapunov exponents on a set of full volume*; see [5].
2. *On any compact manifold there is a dense G_δ subset $\mathcal{U} \subset \mathrm{Diff}^1(M,\mu)$ such that any $f \in \mathcal{U}$ is either (1) non-uniformly hyperbolic (and is ergodic; in particular, f has nonzero Lyapunov exponents on a set of full volume) or (2) has zero Lyapunov exponent*; see [27] for the case $\dim M = 3$ and [2] for the general case.

Remark 2. In Problem 1, while the map f is assumed to have nonzero Lyapunov exponents on a set of full volume, one should expect its small perturbations to have nonzero Lyapunov exponents only on a set of positive (not necessarily full) volume. This is due to a phenomenon known as *essential coexistence*; see [14]. More precisely, a map f is said to exhibit an essential coexistence of regular and chaotic behavior if

1. M can be split into two invariant disjoint Borel subsets A and B of positive volume – the chaotic and regular regions for f;
2. for almost every $x \in A$ and every $v \in T_xM$ the Lyapunov exponent $\chi_f(x,v) \neq 0$;
3. for every $x \in B$ and every $v \in T_xM$ the Lyapunov exponent $\chi_f(x,v) = 0$;
4. $f|A$ is ergodic.

The entropy formula gives that $h_{\mathrm{top}}(f|A) > 0$ and the Margulis–Ruelle inequality implies that $h_{\mathrm{top}}(f|B) = 0$.

Essential coexistence is said to be of type I if the set A is dense in M and of type II otherwise. In the former case the regular and chaotic regions for f cannot be topologically separated. The essential coexistence phenomenon is an obstruction to having nonzero Lyapunov exponents on a set of full volume.

The concept of essential coexistence of type I was inspired by the work of Cheng–Sun [13], Herman [26], Xia [50], and Yoccoz [51], who have shown that on any manifold M and for any sufficiently large r one has what can be regarded as a discrete version of the classical KAM theory phenomenon in the volume-preserving category – there are open sets of volume-preserving C^r diffeomorphisms of M all of which possess positive volume sets of codimension-1 invariant tori; on each such torus the diffeomorphism is C^1 conjugate to a Diophantine translation; all Lyapunov exponents are zero at every point in any invariant torus. The set of invariant tori is nowhere dense and has positive volume.

Examples of essential coexistence of type II can be found in works of Przytycki [41] and of Liverani [34], where they constructed surface diffeomorphisms with nonzero Lyapunov exponents such that some of their arbitrary small perturbations have elliptic islands.

1.2.16 E. Arthur Robinson, Jr.

The flow corresponding to the substitution $\{a \to abbb, b \to a\}$, suspended by its left Perron–Frobenius eigenvector, has purely singular diffraction spectrum. We can think of this as a tiling (or set of tilings) of the real line by intervals whose lengths are 1 and the Perron–Frobenius eigenvalue.

Problem 36. *Does this flow actually have purely singular dynamical spectrum? (Is the measure of maximal spectral type singular to Lebesgue?) What about the spectrum of the discrete substitution $\{a \to abbb, b \to a\}$ itself?*

Note that the measure of maximal spectral type is continuous (except for the single atom corresponding to constant functions) because the flow is weakly mixing. However, it cannot be purely absolutely continuous (because no substitution with finite local complexity can be strongly mixing).

Showing that the dynamical spectrum equals the diffraction spectrum would amount to showing that a function supported on a set near the tile endpoints of the tiles has maximal spectral type.

Remark: This problem has now essentially been solved by Bufetov and Solomyak; see [8]. See also [9], which deals with the case of discrete substitutions.

Problem 37. *Is the Conway–Radin pinwheel tiling strongly mixing or not? It is easily shown to be weakly mixing. Is it topologically mixing?*

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Problem 38. *Is there a general approach to prove a primitive Hilbert cube substitution is uniquely ergodic?*

Problem 39. *Is there some kind of Markov partition for a geodesic flow on a compact surface of constant negative curvature that will make the horocycle flow into an infinite local complexity tiling substitution? Can this be done for Anosov flows in general?*

Note that in the case of finite local complexity tiling substitutions, there is always a Markov partition hiding in the background, often for an Anosov diffeomorphism, but in general for a "Smale space" in the terminology of Putnam. For example, Penrose tilings come from looking at a Markov partition for a hyperbolic automorphism of $\mathbb{T}^4$ restricted to its 2-dimensional stable manifold. Conversely, every Smale space gives rise to a tiling substitution (at least if we allow for disconnected tiles).

We know that in case $d > 2$ Markov partitions for hyperbolic toral automorphisms "usually" have fractal boundaries. (The Penrose case mentioned earlier avoids this because it is essentially a product of two 2-dimensional automorphisms.)

Problem 40. *When does a hyperbolic toral automorphism have a Markov partition with connected partition elements? When does the resulting tiling have finite local complexity?*

1.2.17 Klaus Schmidt

The questions raised here are related to homoclinic points for abelian actions. We begin by looking at a few examples.

Example 1. The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ induces a map f on $\mathbb{T}^2$. The eigenlines of the matrix induce two f-invariant foliations with dense leaves on the torus. Each point x where these leaves intersect gives $\lim_{|n|\to\infty} f^n x = 0$ but will deviate significantly from 0 for many values of n.

Now suppose we have another hyperbolic toral automorphism g that commutes with f. Then f and g generate a $\mathbb{Z}^2$ action α on the torus. Given a $\underline{n} = (n_1, n_2) \in \mathbb{Z}^2$, consider $\alpha^{\underline{n}} = f^{n_1} g^{n_2} \in Aut(\mathbb{T}^2)$. We say that x is a *homoclinic* point if $\lim_{\|\underline{n}\| \to \infty} \alpha^{\underline{n}} x = 0$.

If a $\mathbb{Z}^2$ action by automorphisms of the torus is higher-rank, there are no nonzero homoclinic points. Also, higher-rank actions have zero action-entropy.

Example 2. Ledrappier's example: the shift-invariant subspace $X \subset (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{Z}^2)}$ given by

$$X = \left\{ (x_{k,\ell})\colon\ x_{k,\ell} + x_{k+1,\ell} + x_{k,\ell+1} \equiv 0 \bmod 2 \text{ for all } k, \ell \right\}.$$

There are no homoclinic points for the multi-dimensional shift σ on X (the $\underline{n}$-th coordinate of $\sigma^{\underline{m}}(x)$ is defined as $x_{\underline{m}+\underline{n}}$).

Example 3. If we replace the alphabet $\mathbb{Z}/2\mathbb{Z}$ with the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, but keep the same rule, we get a large space for X. (If we fix one row we can uniquely fill in one half of the plane.) This action has actually positive entropy; see [35] for the entropy calculation. Also, this action has interesting homoclinic points. Moreover, it also has so-called "summable homoclinic points" (SHP), meaning here that the conference to 0 happens at a reasonable speed (not exponential decay but in ℓ^1). See [36] for existence of SHP.

Example 4. We can also consider the following generalizations of Ledrappier's example. Let

$$X_f = \ker\big(f(\sigma)\big),$$

where, for example, $f(\sigma) = \mathrm{Id} + \sigma^{(1,0)} + \sigma^{(0,1)}$ for $f = 1 + z_1 + z_2$. For the following Laurent polynomials, we have

$$\begin{aligned} 1 + z_1 + z_2 &\text{ has SHP,} \\ 3 - z_1 - z_1^{-1} - z_2 - z_2^{-1} &\text{ does not have homoclinic points, no SHP,} \\ 4 - z_1 - z_1^{-1} - z_2 - z_2^{-1} &\text{ does have homoclinic points, has SHP,} \\ 5 - z_1 - z_1^{-1} - z_2 - z_2^{-1} &\text{ does have homoclinic points.} \end{aligned}$$

The existence of SHP is the measure of degree of non-expansiveness of the shift on X_f. The shift is expansive if and only if f is a Laurent polynomial seen as an element of $\ell^1(\mathbb{Z}^2)$ is invertible.

Expansiveness is equivalent to

$$U(f) = \left\{ \underline{c} \in \mathbb{T}^2 \colon f(\underline{c}) = 0 \right\}$$

being non-empty. The existence of SHP is equivalent to $\dim U(f) \le d - 2$.

Why should we be interested in the existence of SHP? Firstly, there are examples where the existence of SHP is the only way of proving certain dynamical properties, for example, a very strong version of specification. Also, the existence of SHP implies positive entropy.

Problem 41. *How can we prove properties such as the ones just mentioned,* without *using summable homoclinic points?*

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