ON FUNDAMENTAL OPERATIONS IN GROUPS

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By an *operation* in equationally definable class of group we mean here the element of a free group in this class generated by set $\{x, y\}$.

An operation $\omega(x, y)$ is called *fundamental* in a class of groups K if for every group $G \in K$ the operation xy^{-1} can be expressed in terms of ω.

Higman and Neumann raised in [1] the problem: Is there any binary operation other than xy^{-1}, $x^{-1}y$, yx^{-1}, $y^{-1}x$ fundamental in the class of all groups?*

To show that the above operations are unique fundamental in class G it suffices to show that for every $k = 1, 2, \cdots$ they are unique fundamental in the class N_k of all nilpotent groups of class k. In fact free groups are residually nilpotent.

I do not know if the converse is true, that is, if a word ω, fundamental in the class N of all nilpotent groups must be fundamental in every group H; although the free groups are subdirect products of the nilpotent groups, it can happen that term T_k which expresses xy^{-1} in N_k in terms of ω depends on k.

The authors of [1] noted that their problem has negative solution for the class N_1 of all abelian groups. The first proof was published by Padmanabhan [3]. Our aim is to to prove the same for N_2. Since we need the theorem about N_1 we shall state it now and give a proof shorter than that of [3].

THEOREM 1. *The only operations fundamental in the class of all abelian groups are $x - y$ and $y - x$.*

PROOF. Every binary operation in an abelian group is of the form $mx + ny$. Suppose that $\omega(x, y) = mx + ny$ is fundamental in N_1. Then we must have

* The interpretation of the problem of Higman and Neumann became a source of controversy. In [2] Hulanicki and Świerczkowski gave an example of a group in which there are fundamental operation other then those mentioned above. Padmanabhan has observed [3] that there is an easier example of such group. However, it should be remarked that the example of Hulanicki and Świerczkowski was only a by-product of investigations concerning some problems of Marczewski regarding weak automorphisms of a group.

$|m| = |n| = 1$. In fact, if for example $|m| > 1$, then in the group Z_m (cyclic group of order m) the operation $mx + ny$ depends on at most, one variable and thus their superpositions depends on at most one variable. Similarly in other cases, the operation $x + y$ is not fundamental because it leads by superpositions only to operations with non negative coefficient. To see that $-x - y$ is not fundamental, let us observe that the following property of linear combinations is preserved under the operation of superposition:

$$\textit{sum of coefficients is equal } 1 \pmod k,$$

But for $m = n = -1$ the sum of coefficients of ω is equal 1 (mod 3). Thus Theorem 1 is proved.

THEOREM 2. *The only operations fundamental in the class of all nilpotent groups of class 2 are: right division, left division, and their transposes.*

PROOF. Let (x, y) denote the commutator of x and y i.e. the word $x^{-1}y^{-1}xy$. In the class N_2 the following well known identities hold:

$$(x,(y,z)) = ((x,y),z) = e$$

$$(x^n, y) = (x,y)^n$$

$$(xy, z) = (xz, yz)$$

As was observed in [2], using these identities every binary operation in N_2 can be represented in the form $x^\alpha y^\beta (x,y)^\gamma$.

Let $\omega(x,y) = x^\alpha y^\beta (x,y)^\gamma$ be a fundamental operation in N_2. Since abelian groups are nilpotent of class 2, by Theorem 1 we have $\alpha = 1$ $\beta = -1$ (we are omitting symmetric cases).

Thus $\omega(x,y) = xy^{-1}(x,y)^\gamma$.

If $\gamma = 1$ we have $\omega(x,y) = xy^{-1}(x,y) = y^{-1}x$.

Suppose $0 \neq \gamma \neq 1$.

We shall show that in the class N^γ of groups satisfying identities $(x,y)^{2\gamma-1} = ((x,y),z) = e$ the following holds.

$$(*) \qquad \omega^{-1}(x,y) = \omega(x^{-1}, y^{-1}).$$

As a matter of fact

$$\omega^{-1}(x,y) \cdot \omega^{-1}(x^{-1},y^{-1}) = [xy^{-1}(x,y)^\gamma]^{-1}[x^{-1}y(x^{-1},y^{-1})^\gamma]^{-1}$$
$$= (x,y)^{-\gamma}yx^{-1}(x^{-1},y^{-1})^{-\gamma}y^{-1}x = yx^{-1}y^{-1}x(x,y)^{-2\gamma} = (y^{-1},x)(x,y)^{-2\gamma}$$
$$(x,y)^{-2\gamma+1} = e.$$

Let us observe that (*) means that the operation $h(x) = x^{-1}$ is an automorphism of the algebra (G, ω). Thus if G belongs to N^γ, the operation $h(x) = x^{-1}$

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is its automorphism. But if ω is fundamental, $h(x)$ is the automorphism of the group G. The only groups for which x^{-1} is an automorphism are abelian. But if $0 \neq \gamma \neq 1$ it is easy to show that $N_1 \subsetneq N^\gamma \subseteq N_2$. Thus if ω is fundamental we must have $\gamma = 0$ or $\gamma = 1$.

References

[1] G. Higman and B. H. Neumann, 'Groups as groupoids with one law', *Publicationes Mathematicae* 2 (1952), 215–221.

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