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Cohomology of locally trivial groupoids and Lie algebroids

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The contribution of this thesis is in three main parts: the construction and calculation of a natural cohomology theory for locally trivial topological groupoids, the development of the cohomology theory of exact Lie algebroids and the construction of a spectral sequence related to the Leray-Serre spectral sequence of an underlying principal bundle, and the introduction of the concept of "transition form" for exact Lie algebroids, which is the key to many technical results and to the integrability of Lie algebroids. This work forms part of a programme to prove the converse of Lie's third theorem for Lie groupoids and Lie algebroids (Pradines [8]) by generalizing the classical proof of van Est [2], [3].

Chapter I is intended as a detailed introduction to topological groupoids, Lie groupoids, and Lie algebroids with the emphasis on the role of local triviality. Exact Lie algebroids play for locally trivial Lie groupoids the role that the Lie algebra plays for a Lie group; the Atiyah sequence [1] of a principal bundle is the Lie algebroid of the corresponding groupoid. We give proofs for some basic results that have not been proved in the literature, and in §3.4 introduce the concept of transition form for exact Lie algebroids; the transition forms of an exact Lie algebroid are analogous to the transition functions of a locally trivial Lie groupoid and play a similarly central role. We develop this concept sufficiently to prove a necessary technical result (3.4.6), but its full analysis must await another occasion.

Chapter II constructs a cohomology theory for locally trivial, locally

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compact topological groupoids, with coefficients in vector bundles, generalizing the construction of 'continuous' cohomology for topological groups of Hochschild and Mostow [5] and the discrete groupoid cohomology of Higgins [4]. It is calculated to be naturally isomorphic to the cohomology of any vertex group, and is thus independent of the twistedness of the groupoid. The second cohomology space is accordingly realized as those 'rigid' extensions which arise from extensions of a vertex group; the cohomological machinery now yields the unexpected result that all extensions, satisfying some natural weak conditions, are in fact rigid. This chapter has been published as [7].

Postponing the corresponding theory for Lie groupoids, we proceed in Chapter III to develop the cohomology theory of exact Lie algebroids $L \rightarrowtail A \twoheadrightarrow TB$ with coefficients in vector bundles E. Using cohomology set up by Rinehart [9] we give in §§2, 3 the standard interpretations of $H^2(A, E)$ and $H^3(A, E)$ in terms of extension classes and obstruction classes, paying particular attention to the restriction map $H^*(A, E) \to H^*(L, E)$, which is no longer an isomorphism as in the rigid theory of Chapter II. The exact sequence $L \rightarrowtail A \twoheadrightarrow TB$ leads in §4 to a natural spectral sequence $H^s\left(TB, H^t(L, E)\right) \Rightarrow H^n(A, E)$ of Hochschild and Serre [6] type. For a Lie groupoid Ω with compact vertex groups, and trivial coefficients, this reduces to the (de Rham) Leray-Serre spectral sequence for the principal bundle $\Omega_w\left(B, \Omega_w^w\right)$. It follows that the Lie algebroid cohomology $H^*(A\Omega, B\times\mathbf{R})$ is isomorphic to the de Rham cohomology $H^*\left(\Omega_w, \mathbf{R}\right)$, for Ω with compact vertex groups. In §5 the results of §§2, 3 are applied to prove that an exact Lie algebroid on a contractible base admits a flat connection, in analogy with the result that a principal bundle on such a base is trivializable. This result establishes the existence of transition forms for abstract Lie algebroids.

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