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NONEXISTENCE OF MAXIMA FOR PERTURBATIONS OF SOME INEQUALITIES WITH CRITICAL GROWTH

ALEXANDER R. PRUSS

ABSTRACT. We study the question of nonexistence of extremal functions for perturbations of some sharp inequalities such as those of Moser-Trudinger (1971) and Chang-Marshall (1985). We shall show that for each critically sharp (in a sense that will be precisely defined) inequality of the form

$$\sup_{f\in\mathcal{F}} \int_I \Phi(|f(x)|)\,d\mu(x) < \infty, \tag{1}$$

where $\mathcal{F}$ is a collection of measurable functions on a finite measure space (I, μ) and Φ a nonnegative continuous function on $[0, \infty)$, we have a continuous Ψ on $[0, \infty)$ with $0 \leq \Psi \leq \Phi$, but with

$$\sup_{f\in\mathcal{F}} \int_I \Psi(|f(x)|)\,d\mu(x) \tag{2}$$

not being attained even if the supremum in (1) is attained. We then apply our results to the Moser-Trudinger and Chang-Marshall inequalities. Our result is to be contrasted with the fact shown by Matheson and Pruss (1994) that if $\Psi(t) = o\big(\Phi(t)\big)$ as $t \longrightarrow \infty$ then the supremum in (2) is attained. In the present paper, we also give a converse to that fact.

1. **The general results.** Fix a finite measure space (I, μ). For a nonnegative Φ on $[0, \infty)$ and f a measurable function on I, define

$$\Lambda_\Phi(f) = \int_I \Phi(|f(x)|)\,d\mu(x).$$

Let $\mathcal{F}$ be a collection of measurable functions on (I, μ). We shall throughout assume that $0 \in \mathcal{F}$ and that $\mathcal{F}$ is sequentially compact with respect to convergence in measure. Throughout when we refer to concepts such as compactness, semicontinuity or continuity with respect to convergence in measure we shall mean sequential compactness, sequential semicontinuity or sequential continuity, respectively, all with respect to convergence in measure.

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We say that an upper semicontinuous Φ is *critical* for $\mathcal{F}$, provided:

(i) Λ_Φ is upper semicontinuous on $\mathcal{F}\setminus\{0\}$ with respect to convergence in measure, and

(ii) Λ_Φ is not upper semicontinuous with respect to convergence in measure at $0 \in \mathcal{F}$.

Condition (ii) says that there is a sequence of $f_k \in \mathcal{F}$ converging to zero in measure, but with $\Lambda_\Phi(f_k)$ converging to some number (possibly $+\infty$) which strictly greater than $\Phi(0) = \Lambda_\Phi(0)$.

The following result then is of the same type as the work of Flores [6]. We write $\Gamma \circ \Phi$ for the composition of the functions Γ and Φ.

THEOREM 1. *Let Φ be continuous and nonnegative on $[0, \infty)$. Assume that Φ is critical for $\mathcal{F}$. Then there exists a nonnegative, convex, non-decreasing and infinitely differentiable function Γ on $[0, \infty)$ with $\Gamma(y) \leq y$ for every $y \in [0, \infty)$, $\lim_{y\to\infty} \frac{\Gamma(y)}{y} = 1$ and support bounded away from zero, such that $\Lambda_{\Gamma\circ\Phi}$ does not attain its supremum on $\mathcal{F}$.*

Moreover, if $\Phi(0) = 0$ then we may require that there be a sequence of $f_k \in \mathcal{F}$ converging to zero in measure such that $\limsup_k \Lambda_\Phi(f_k) = \sup_{f\in\mathcal{F}} \Lambda_{\Gamma\circ\Phi}(f_k)$.

A proof will be given in Section 3. It is not known whether the assumption of continuity of Φ can be weakened to upper semicontinuity. It is not hard to see that the continuity of a nonnegative Φ immediately implies the lower semicontinuity of Λ_Φ on all of $\mathcal{F}$ by Fatou's Lemma.

We say that Λ_Φ is bounded on $\mathcal{F}$ if $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$. If Φ is in addition critical for $\mathcal{F}$ then we say that the inequality $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$ is *critically sharp*. Theorem 1 then says that if Φ is continuous then even if a critically sharp inequality $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$ attains its maximum, still we may perturb Φ by a bounded factor and lose the attainment of a maximum.

Matheson and Pruss [8, Thm. 5] have shown that if Φ is any nonnegative measurable function with Λ_Φ bounded on $\mathcal{F}$, then, for every upper semicontinuous Ψ with $\Psi(t) = o\big(\Phi(t)\big)$ as $t \longrightarrow \infty$, we have Λ_Ψ upper semicontinuous with respect to convergence in measure on $\mathcal{F}$, and in particular attaining its maximum there. Theorem 1 shows that $o\big(\Phi(t)\big)$ cannot be replaced by $O\big(\Phi(t)\big)$ in that result, even under the assumption that Φ attains its maximum on $\mathcal{F}$. Matheson and Pruss's result can also be interpreted as saying that a critically sharp inequality $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$ cannot be improved by replacing Φ by some Ψ with $\Phi(t) = o\big(\Psi(t)\big)$ as $t \longrightarrow \infty$ since $\sup_{f\in\mathcal{F}} \Lambda_\Psi(f)$ will then fail to be finite.

We have the following partial converse to Matheson and Pruss's result. As before, $\mathcal{F}$ is a collection of measurable functions on a finite measure space (I, μ), with $0 \in \mathcal{F}$ and $\mathcal{F}$ being compact with respect to convergence in measure.

THEOREM 2. *Let Φ be continuous and nonnegative, and suppose that Λ_Φ is continuous on $\mathcal{F}$ with respect to convergence in measure. Then there exists a nonnegative, convex and non-decreasing $\Gamma \in C^\infty[0, \infty)$ with $\frac{\Gamma(y)}{y} \longrightarrow \infty$ as $y \longrightarrow \infty$ and $\Lambda_{\Gamma\circ\Phi}$ bounded on $\mathcal{F}$. Moreover, we may require that $\frac{d\Gamma(y)}{dy} \longrightarrow \infty$ as $y \longrightarrow \infty$.*

A proof will be given in Section 3. As in the case of Theorem 1, it is not known whether the assumption of continuity can be weakened to upper semicontinuity.

COROLLARY. *Let Φ be continuous and nonnegative with $\Phi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$ and suppose that Λ_Φ is continuous on $\mathcal{F}$ with respect to convergence in measure. Then there is a continuous and nonnegative Ψ with $\Phi(t) = o\big(\Psi(t)\big)$ as $t \longrightarrow \infty$ and with Λ_Ψ continuous on $\mathcal{F}$ with respect to convergence in measure, and, in particular, bounded there. Moreover, if Φ is convex* (respectively, non-decreasing, or convex non-decreasing), *then Ψ can be taken to also be convex* (respectively, non-decreasing, or convex non-decreasing).

PROOF OF COROLLARY. First assume that we do not need Ψ to be non-decreasing or convex. Let $\Psi(t) = \sqrt{\Gamma\big(\Phi(t)\big) \cdot \Phi(t)}$, where Γ is as in Theorem 2. Then it follows that $\Psi(t) = o\Big(\Gamma\big(\Phi(t)\big)\Big)$ as $t \longrightarrow \infty$, so that by [8, Thm. 5], it follows from the boundedness of $\Lambda_{\Gamma\circ\Phi}$ that Λ_Ψ is continuous on $\mathcal{F}$ with respect to convergence in measure. On the other hand, we also have $\Phi(t) = o\big(\Psi(t)\big)$ as $t \longrightarrow \infty$.

Now, if we do want Ψ to be non-decreasing and/or convex, then choose Γ as in the "moreover" of Theorem 2. Let

$$\tilde{\Gamma}(y) = \int_0^y \sqrt{1 + \frac{d\Gamma(t)}{dt}}\, dt.$$

Since $\frac{d\Gamma(t)}{dt} \longrightarrow \infty$ as $t \longrightarrow \infty$, it follows from L'Hôpital's Rule that $\tilde{\Gamma}(y) = o\big(\Gamma(y)\big)$ and that $y = o\big(\Gamma(y)\big)$, both as $y \longrightarrow \infty$. Furthermore, if Γ is infinitely differentiable, then so is $\tilde{\Gamma}$, and if Γ is convex then so is $\tilde{\Gamma}$. Then the desired result follows upon setting $\Psi = \tilde{\Gamma} \circ \Phi$, and applying [8, Thm. 5] as before in order to obtain the continuity of Λ_Ψ on $\mathcal{F}$ with respect to convergence in measure. (Of course we also need to use the general fact that if F is non-decreasing and convex while G is convex, then $F \circ G$ is convex.) ∎

We also have the following result which is complementary to Theorem 1 but which will turn out to be much easier to prove. The proof is again given in Section 3.

THEOREM 3. *Let Φ be upper semicontinuous and suppose that Λ_Φ is upper semicontinuous on $\mathcal{F}\backslash\{0\}$ with respect to convergence in measure. Furthermore, assume that Λ_Φ is bounded on $\mathcal{F}$. Then there exists a compactly supported nonnegative $\Gamma \in C^\infty(0, \infty)$ such that $\Lambda_{\Gamma+\Phi}$ attains its maximum over $\mathcal{F}$.*

2. **Applications.**

2.1 *Application to the Moser-Trudinger inequality.* Moser [10] showed that if $\mathcal{F}$ is the collection of real-valued absolutely continuous functions f on $[0, \infty)$ with $f(0) = 0$ and

$$\int_0^\infty \big(f'(t)\big)^2\, dt \le 1,$$

then

$$\sup_{f\in\mathcal{F}} \int_0^\infty e^{f^2(t)-t}\, dt < \infty. \tag{3}$$

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This sharpened an inequality of Trudinger. Carleson and Chang [2] then showed that the supremum is actually achieved at some $f \in \mathcal{F}$. Letting (I, μ) be $[0, \infty)$ with measure $d\mu(t) = e^{-t}dt$ and setting $\Phi(y) = e^{y^2}$, we see that (3) is of the form

$$\sup_{f \in \mathcal{F}} \Lambda_\Phi(f) < \infty.$$

As implicitly noted by Carleson and Chang [2, p. 117], $\mathcal{F}$ is compact with respect to uniform convergence on compact subsets of $[0, \infty)$, and in particular with respect to convergence in measure. (This can be seen from the fact that the collection of f' for $f \in \mathcal{F}$ is the unit ball of the Hilbert space $L^2[0, \infty)$ and hence is weakly compact.)

Furthermore, Φ is critical for $\mathcal{F}$. For, if $f_n \longrightarrow f$ in measure, then, choosing a further subsequence if necessary, we may assume that f_n converges to f uniformly on compact subsets of $[0, \infty)$. Then, the work of Carleson and Chang [2, pp. 117–118] shows that $\limsup_n \Lambda_\Phi(f_n) \leq \Lambda_\Phi(f)$ if $f \neq 0$. Hence Λ_Φ is upper semicontinuous on $\mathcal{F} \backslash \{0\}$ with respect to convergence in measure.

On the other hand, Λ_Φ fails to be upper semicontinuous at $0 \in \mathcal{F}$. To prove this we look at Moser's broken line functions, proceeding much like in [10]. Let $\beta(t) = \min(t, 1)$ and put $f_n(t) = \sqrt{n}\beta(t/n)$. Then clearly $f_n \in \mathcal{F}$ and

$$\int_0^\infty e^{f_n^2(t)-t}\, dt = \int_0^n e^{t^2/n-t}\, dt + \int_n^\infty e^{n-t}\, dt \geq \int_0^n e^{-t}\, dt + 1.$$

Now, the right hand side converges to 2 as $n \longrightarrow \infty$. On the other hand, it is easy to verify that $f_n \longrightarrow 0$ in measure and $\Lambda_\Phi(0) = 1$ so that Λ_Φ indeed fails to be upper semicontinuous at $0 \in \mathcal{F}$.

Hence $\Phi(t) = e^{t^2}$ is critical for $\mathcal{F}$. The following result which was conjectured by McLeod and Peletier [9] then follows immediately from Theorem 1.

THEOREM 4. *There exists a convex, non-decreasing and smooth function* Γ *with* $0 \leq \Gamma(y) \leq y$ *for every* $y \in [0, \infty)$ *and with* $\lim_{y\to\infty} \frac{\Gamma(y)}{y} = 1$, *such that the supremum*

$$\sup_{f \in \mathcal{F}} \int_0^\infty \Gamma(e^{f^2(t)})e^{-t}dt \tag{4}$$

is not achieved over $\mathcal{F}$.

Of course it should be noted that (4) is finite. Theorem 4 shows that the existence of the extremal for Moser's inequality is in some way accidental, relying on non-asymptotic properties of the function e^{t^2}.

2.2 *Application to the Chang-Marshall inequality.* Let $\mathfrak{B}$ be the collection of holomorphic functions f on the unit disc D with $f(0) = 0$ and Dirichlet integral

$$\frac{1}{\pi}\int\int_D |f'(x+iy)|^2\, dx\, dy \leq 1.$$

Let $\Phi(t) = e^{t^2}$ and let (I, μ) be the unit circle with normalized Lebesgue measure. Then, Chang and Marshall [3] proved that Λ_Φ is bounded on $\mathfrak{B}$. See [7] for an alternate potential-theoretic proof, and [5] for an interesting generalization and a stronger inequality. It is not known whether Λ_Φ achieves its supremum over $\mathfrak{B}$, but it was conjectured by Andreev and Matheson [1] that it does, and in fact that it achieves it at the identity function. This last conjecture has been numerically verified by the author of the present paper for over 40 million quasi-random polynomials of degree 6. See also [4] and [8] for more information on the question.

Cima and Matheson [4] have shown (see also [8, Cor. 3] for a generalization of this result) that Λ_Φ is weakly continuous on $\mathfrak{B}\backslash\{0\}$. Also, $\mathfrak{B}$ is the unit ball of the Dirichlet space which is a Hilbert space. Then, as [8] notes, it follows trivially from [1, Lemma 3], which says that L^p norms on the unit circle are weakly continuous on $\mathfrak{B}$, that Λ_Φ is continuous on $\mathfrak{B}\backslash\{0\}$ also with respect to the convergence in measure topology on the unit circle, and it also follows from [1, Lemma 3] that $\mathfrak{B}$ is compact with respect to convergence in measure since it is weakly compact by Banach-Alaoglu.

On the other hand, Cima and Matheson [4] have shown that Λ_Φ fails to be weakly upper semicontinuous at $0 \in \mathfrak{B}$ (see also another proof given as a part of [8, Proof of Thm. 1]), and it follows that it is not weakly upper semicontinuous with respect to convergence in measure there. Hence $\Phi(t) = e^{t^2}$ is critical for $\mathfrak{B}$. Then, even though we do not know whether Λ_Φ achieves its maximum over $\mathfrak{B}$, we do have the following result which follows from Theorems 1 and 3.

THEOREM 5. *There exist two $C^\infty[0, \infty)$ functions Ψ_1 and Ψ_2 such that for every $t \in [0, \infty)$ we have $0 \le \Psi_1(t) \le e^{t^2} \le \Psi_2(t)$ and Λ_{Ψ_i} is bounded on $\mathfrak{B}$ for $i = 1, 2$, but Λ_{Ψ_1} does not achieve its supremum over $\mathfrak{B}$ while Λ_{Ψ_2} does achieve* its *maximum over $\mathfrak{B}$. One may take Ψ_1 to be convex and non-decreasing with $\lim_{t\to\infty} e^{-t^2}\Psi_1(t) = 1$.*

Alec Matheson has kindly communicated to the author that he and Joseph Cima had strongly suspected the truth of this result.

3. **Proofs.** If f and ϕ are measurable on $[0, \infty)$, then write

$$\|g\|_{L^1(\phi)} \stackrel{\text{def}}{=} \int_0^\infty |f\phi| = \int_0^\infty |f(x)\phi(x)|\,dx.$$

Then, the main step in the construction of Ψ for Theorem 1 is encapsulated in the following result.

LEMMA 1. *Let $\mathcal{G}$ be a subset of $L^1[0, \infty)$ containing the zero function, such that for each finite number T we have $\sup_{g\in\mathcal{G}} \|g \cdot 1_{[0,T]}\|_{L^\infty} < \infty$. Assume that for every sequence g_n of elements of $\mathcal{G}$, there exists a subsequence g_{n_k} which converges in measure to some $g \in L^1[0, \infty)$ such that either g is almost everywhere null or else has $\|g\|_{L^1} \ge \limsup_k \|g_{n_k}\|_{L^1}$. Suppose further that $\|\cdot\|_{L^1}$ fails to be upper semicontinuous at $0 \in \mathcal{G}$ with respect to convergence in measure.*

Then, there exists a nonnegative non-decreasing function $\phi \le 1$ on $[0, \infty)$ such that $\lim_{x\to\infty} \phi(x) = 1$, with $\|\cdot\|_{L^1(\phi)}$ not attaining its maximum on $\mathcal{G}$. Furthermore, ϕ may be taken to be in $C^\infty[0, \infty)$, with support bounded away from 0. *Moreover if* $\sup_{g\in\mathcal{G}} \|g\|_{L^1} <$

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∞ *then we may also require that there be a sequence* $g_k \in \mathcal{G}$ *such that* $g_k \longrightarrow 0$ *almost everywhere and* $\limsup_k \|g_k\|_{L^1} = \sup_{g\in\mathcal{G}} \|g\|_{L^1(\phi)}$.

Assuming the lemma for now, we may proceed to prove Theorem 1.

PROOF OF THEOREM 1. Without loss of generality assume that $\Phi(0) = 0$. For $f \in \mathcal{F}$ and $t \in [0, \infty)$, let $m_f(t) = \mu\{x : \Phi(|f(x)|) > t\}$. Let $\mathcal{G} = \{m_f : f \in \mathcal{F}\}$. We shall apply the lemma to $\mathcal{G}$. Let us verify its conditions. Clearly, every element of $\mathcal{G}$ is pointwise bounded by $\mu(I) < \infty$. Furthermore, for $m_f \in \mathcal{G}$ we have

$$\|m_f\|_{L^1} = \int_0^\infty m_f(t)\,dt = \Lambda_\Phi(f).$$

Then, using the lack of upper semicontinuity of Λ_Φ at zero, we may choose a sequence $f_k \in \mathcal{F}$ such that $f_k \longrightarrow 0$ in measure and $\limsup_k \Lambda_\Phi(f_k) > \Lambda_\Phi(0) = 0$. Passing to a subsequence if necessary, we may assume that $f_k \longrightarrow 0$ almost everywhere. Then, $\limsup_k \Phi(|f_k|) = \Phi(0) = 0$ almost everywhere, by the continuity of Φ. Hence, $\Phi(|f_k|) \longrightarrow 0$ almost everywhere, too, and hence also in measure. Thus, for every $t > 0$ we have $m_{f_k}(t) \longrightarrow 0$, and in particular $m_{f_k} \longrightarrow 0$ in measure while $\limsup_k \|m_{f_k}(t)\|_{L^1} = \limsup_k \Lambda_\Phi(f_k) > 0$, so that $\|\cdot\|_{L^1}$ fails to be upper semicontinuous at zero in $\mathcal{G}$.

Now, given any sequence m_{f_n} of elements of $\mathcal{G}$, we may choose a subsequence $m_{f_{n_k}}$ such that f_{n_k} converges in measure, using the compactness with respect to convergence in measure of $\mathcal{F}$. Choosing a further subsequence if necessary, we may assume f_{n_k} converges almost everywhere. If the limit is almost everywhere zero then we are done. On the other hand, if $f_{n_k} \longrightarrow f$ where f does not vanish almost everywhere then we first of all have $\lim_k \Phi(|f_{n_k}|) = \Phi(|f|)$ by continuity of Φ, and secondly, by the upper semicontinuity of Λ_Φ with respect to convergence in measure away from zero, we have $\limsup_k \Lambda_\Phi(|f_{n_k}|) \leq \Lambda_\Phi(|f|)$. Then, since $\Phi(|f_{n_k}|) \longrightarrow \Phi(|f|)$ in measure, it follows that $m_{f_{n_k}} \longrightarrow m_f$ almost everywhere (in fact at all points of $[0, \infty)$ other than the at most countably many discontinuities of m_f), as can be easily verified. Also, $\|m_f\|_{L^1} \geq \limsup_k \|m_{f_{n_k}}\|_{L^1}$. Hence, the conditions for the lemma are satisfied.

Choose ϕ as in Lemma 1. Let

$$\Gamma(y) = \int_0^y \phi(x)\,dx.$$

It is easy to verify that $\|m_f\|_{L^1(\phi)} = \Lambda_{\Gamma\circ\Phi}(f)$. Then the Theorem follows from the conclusions of Lemma 1. For example, the convexity of Γ follows from the fact that ϕ is monotone non-decreasing. ∎

LEMMA 2. *Let* $\mathcal{G}$ *be a subset of* $L^1[0, \infty)$ *containing the zero function, such that for each finite number* T *we have* $\sup_{g\in\mathcal{G}} \|g \cdot 1_{[0,T]}\|_{L^\infty} < \infty$. *Assume that for every sequence* g_n *of elements of* $\mathcal{G}$, *there exists a subsequence* g_{n_k} *which converges in measure to some* $g \in L^1[0, \infty)$ *such that either* g *is almost everywhere null or else has* $\|g\|_{L^1} \geq \limsup_k \|g_{n_k}\|_{L^1}$. *Suppose further that* $\|\cdot\|_{L^1}$ *is uniformly bounded on all of* $\mathcal{G}$ *and upper semicontinuous with respect to convergence in measure at* $0 \in \mathcal{G}$.

Then, there exists a nonnegative and non-decreasing function $\phi \geq 1$ *on* $[0, \infty)$ *with* $\lim_{x\to\infty} \phi(x) = \infty$ *and* $\|\cdot\|_{L^1(\phi)}$ *bounded on* $\mathcal{G}$.

Theorem 2 then follows from Lemma 2 in the same way as Theorem 1 had followed from Lemma 1. We now proceed to prove our two lemmata.

PROOF OF LEMMA 1. First suppose $\mathcal{G}$ is not uniformly bounded in L^1 norm. Let ϕ be a nonnegative non-decreasing $C^\infty[0,\infty)$ function whose support is bounded away from zero and which has $\phi(x) = 1$ for all $x \geq 1$. Then let g_k be a sequence of elements of $\mathcal{G}$ with $\|g_k\|_{L^1} \longrightarrow \infty$. Passing to a subsequence we can assume that for some $g \in L^1[0,\infty)$ we have $g_k \longrightarrow g$ in measure. If g is almost everywhere null, then it is easy to see that the proof is complete since by the bounded convergence theorem (which is applicable because the $\{g_k\}$ are almost everywhere uniformly bounded on $[0, 1]$ by the hypotheses of the Lemma) we have $\int_0^1 |g_k| \longrightarrow 0$ so that $\|g_k\|_{L^1(\phi)} \geq \int_1^\infty |g_k| = \|g_k\|_{L^1} - \int_0^1 |g_k|$ and the right hand side tends to ∞, so that $\|g_k\|_{L^1(\phi)} \longrightarrow \infty$ as desired. Choosing a subsequence if necessary, then, we may assume that $g_k \longrightarrow 0$ almost everywhere and the Lemma follows. On the other hand, if g is not almost everywhere null then $\|g\|_{L^1} \geq \limsup_k \|g_k\|_{L^1}$ by our hypotheses. But, the right hand side is infinite, and this contradicts the fact that $g \in L^1$.

Now, assume that

$$M \stackrel{\text{def}}{=} \sup_{g\in\mathcal{G}} \|g\|_{L^1} < \infty.$$

Let

$$\lambda = \sup_{\substack{\{g_k\}\subseteq \mathcal{G} \\ g_k \to 0}} \limsup_k \|f_k\|_{L^1}, \tag{5}$$

where the supremum is to be understood as taken over all sequences $\{g_k\}$ in $\mathcal{G}$ tending to zero in measure. Since $\|\cdot\|_{L^1}$ fails to be upper semicontinuous at $0 \in \mathcal{G}$ with respect to convergence in measure, we have $\lambda > 0$. Obviously, $\lambda \leq M$.

Replacing $\mathcal{G}$ by $\{|g| : g \in \mathcal{G}\}$ if necessary, we may assume all functions in $\mathcal{G}$ are nonnegative. Choose $0 < \alpha < 1$ such that $\alpha M < \lambda$.

For $g \in \mathcal{G}$, let

$$\tau_g = \inf\Big\{\tau \geq 0 : \int_\tau^\infty g \leq \alpha \int_0^\infty g\Big\},$$

so that

$$\int_{\tau_g}^\infty g = \alpha \int_0^\infty g \tag{6}$$

and

$$\int_0^{\tau_g} g = \int_0^\infty g - \int_{\tau_g}^\infty g = (1-\alpha)\int_0^\infty g. \tag{7}$$

Now define

$$\mathcal{G}_x = \{g : \tau_g \geq x\}.$$

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Let

$$M_x = \sup_{g \in \mathcal{G}_x} \int_0^\infty g.$$

I claim that

$$\limsup_{x \longrightarrow \infty} M_x \leq \lambda. \tag{8}$$

To show this, it suffices to prove that for any sequences $x_k \longrightarrow \infty$ and $g_k \in \mathcal{G}_{x_k}$ such that $\int_0^\infty g_k$ converges, we have $\lim_k \int_0^\infty g_k \leq \lambda$. Fix such sequences x_k and g_k. Passing to subsequences, if necessary, by our hypotheses we may assume that g_k either converges to 0 in measure, or else it converges in measure to some nonzero $g \in L^1[0, \infty)$ with $\|g\|_{L^1} \geq \limsup_k \|g_k\|_{L^1}$. If it converges to 0 in measure then $\limsup_k \int_0^\infty g_k \leq \lambda$ by definition of λ. Otherwise note that since $g_k \in \mathcal{G}_{x_k}$, we have $\int_{x_k}^\infty g_k \geq \alpha \|g_k\|_{L^1}$. Let $h_k(x) = g_k(x) \cdot 1_{\{x \geq x_k\}}$. We have $h_k \longrightarrow 0$ pointwise since $x_k \longrightarrow \infty$. By Fatou's Lemma then,

$$\liminf_k \int_0^\infty (g_k - h_k) \geq \int_0^\infty g.$$

But

$$\int_0^\infty (g_k - h_k) = \int_0^{x_k} g_k \leq \int_0^{T_{g_k}} g_k = (1-\alpha)\|g_k\|_{L^1},$$

where we have used the fact that $g_k \in \mathcal{G}_{x_k}$ together with (7). Thus,

$$\liminf_k (1-\alpha)\|g_k\|_{L^1} \geq \|g\|_{L^1}.$$

But since $\alpha < 1$, this contradicts the facts that $\|g\|_{L^1} \geq \limsup_k \|g_k\|_{L^1}$ and that g does not almost everywhere vanish. Hence, the case where g_k does not converge to zero in measure is impossible, and the claim is proved.

Now define

$$\psi(x) = \Big(\frac{x}{1+x}\Big)\Big(1 \wedge \inf_{g \in \mathcal{G}_x \setminus \{0\}} \frac{\lambda - \int_{T_g}^\infty g}{\int_0^{T_g} g}\Big).$$

I claim that $\psi(x) \longrightarrow 1$ as $x \longrightarrow \infty$. To prove this, consider the function

$$h(t) = \frac{\lambda - \alpha t}{(1-\alpha)t},$$

which is easily seen to be decreasing for $t \in [0, M]$ since $\alpha M < \lambda$, and which satisfies $h(\lambda) = 1$. By (6) and (7) we then have

$$\psi(x) = \Big(\frac{x}{1+x}\Big)\Big(1 \wedge \inf_{g \in \mathcal{G}_x \setminus \{0\}} h\Big(\int_0^\infty g\Big)\Big).$$

But for $g \in \mathcal{G}_x$ we have $\int_0^\infty g \leq M_x$ so that by the monotonicity of h on $[0, M]$ we have

$$\psi(x) \geq \frac{x}{1+x}\big(1 \wedge h(M_x)\big).$$

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Now by (8) we have $\liminf_{x\to\infty} h(M_x) \geq h(\lambda) = 1$ and hence $\lim_{x\to\infty} \psi(x) = 1$ as desired.

Note that ψ is measurable as it is non-decreasing. It is easy to verify that $\|g\|_{L^1(\psi)} < \lambda$ for every $g \in \mathcal{G}$. For, given $g \in \mathcal{G}$ with $\|g\|_{L^1} \neq 0$, we have

$$\int_0^\infty g\psi < \int_0^{\tau_g} g\psi + \int_{\tau_g}^\infty g \leq \psi(\tau_g) \int_0^{\tau_g} g + \int_{\tau_g}^\infty g \leq \lambda - \int_{\tau_g}^\infty g + \int_{\tau_g}^\infty g = \lambda,$$

where we have used the monotonicity and choice of ψ. The first inequality came from the facts that $\psi < 1$ everywhere and that $\int_{\tau_g}^\infty g > 0$ if $\|g\|_{L^1} \neq 0$.

If we do not need ϕ to be $C^\infty[0,\infty)$ or to have support bounded away from zero, then just let $\phi = \psi$. Otherwise, since ψ is a non-decreasing function $[0,\infty)$ with limit 1, we may easily choose a non-decreasing $C^\infty[0,\infty)$ function ϕ with support bounded away from 0 and with the properties that $0 \leq \phi \leq \psi$ everywhere and that $\phi(x) \longrightarrow 1$ as $x \longrightarrow \infty$. We will then necessarily still have $\|g\|_{L^1(\phi)} < \lambda$ for each $g \in \mathcal{G}$.

We shall now show that

$$\sup_{g\in\mathcal{F}} \|g\|_{L^1(\phi)} = \lambda. \tag{9}$$

To do this, fix $\varepsilon > 0$. By definition of λ, let g_k be a sequence in $\mathcal{G}$ with $g_k \longrightarrow 0$ in measure and $\|g_k\|_{L^1} \longrightarrow \lambda$. Choose T sufficiently large that $\phi(x) \geq 1-\varepsilon$ for $x \geq T$. Since $g_k \longrightarrow 0$ in measure and, by our hypotheses, the $g_k \cdot 1_{[0,T]}$ are uniformly bounded in L^∞, the bounded convergence theorem tells us that $\int_0^T g_k \longrightarrow 0$ as $k \longrightarrow \infty$. Since $\|g_k\|_{L^1} \longrightarrow \lambda$, we may choose K sufficiently large that $\int_T^\infty g_k \geq \lambda - \varepsilon$ for $k \geq K$. Then, for $k \geq K$ we have

$$\|g_k\|_{L^1(\phi)} \geq \int_T^\infty g_k\phi \geq (1-\varepsilon)\int_T^\infty g_k \geq (1-\varepsilon)(\lambda-\varepsilon).$$

Hence $\liminf_k \|g_k\|_{L^1(\phi)} \geq (1-\varepsilon)(\lambda-\varepsilon)$ for every $\varepsilon > 0$, and so we see that indeed $\liminf_k \|g_k\|_{L^1(\phi)} \geq \lambda$ and (9) follows from this and the already proved inequality $\|g\|_{L^1(\phi)} < \lambda$ valid for every $g \in \mathcal{G}$.

The last sentence of the statement of the Lemma now follows upon taking a subsequence of the above g_k which converges almost everywhere to zero. Thus, the Lemma is proved. ∎

PROOF OF LEMMA 2. As in the proof of Lemma 1, we may assume without loss of generality that all functions of $\mathcal{G}$ are nonnegative. Let

$$M = \sup_{g\in\mathcal{G}} \int_0^\infty g.$$

By assumption this will be finite. Let

$$U_x = \sup_{g\in\mathcal{G}} \int_x^\infty g.$$

I claim that $U_x \longrightarrow 0$ as $x \longrightarrow \infty$. For, fix any $0 < \alpha < 1$, and define

$$\tau_g^\alpha = \inf\Big\{\tau \geq 0 : \int_\tau^\infty g \leq \alpha \int_0^\infty g\Big\}.$$

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Let $\mathcal{G}_x^\alpha = \{g \in \mathcal{G} : \tau_g^\alpha \geq x\}$. Exactly as in the proof of Lemma 1, we may show that if

$$M_x^\alpha = \sup_{g \in \mathcal{G}_x^\alpha} \int_0^\infty g,$$

then $M_x^\alpha \longrightarrow 0$ as $x \longrightarrow \infty$. (For, in the present case λ as defined by (5) will be zero, by the assumption of upper semicontinuity with respect to convergence in measure to zero.) Now, fix $\varepsilon > 0$ and choose $0 < \alpha < 1$ such that $\alpha M \leq \varepsilon$. Assume that x is sufficiently large that $M_x^\alpha \leq \varepsilon$. Then, for such x and $g \in \mathcal{G}$, we have $\int_x^\infty g \leq \int_0^\infty g \leq M_x^\alpha \leq \varepsilon$ providing $g \in G_x^\alpha$. On the other hand, if $g \notin G_x^\alpha$ then $\tau_g^\alpha < x$, so that $\int_x^\infty g \leq \int_{\tau_g^\alpha}^\infty g = \alpha \int_0^\infty g \leq \alpha M \leq \varepsilon$. Hence, in either case $\int_x^\infty g \leq \varepsilon$, and so $U_x \longrightarrow 0$ as desired.

Choose a sequence of finite nonnegative numbers $x_k \longrightarrow \infty$ with the property that $U_{x_k} \leq 2^{-k}$. Then set

$$\psi(x) = \mathrm{Card}\{k \in \mathbb{Z}^+ : x_k \leq x\}.$$

This is a non-decreasing nonnegative function on $[0, \infty)$, and we have $\psi(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Furthermore, if $g \in \mathcal{G}$ then it is easy to verify that

$$\|g\|_{L^1(\psi)} = \sum_{k=1}^{\infty} \int_{x_k}^{\infty} g \leq \sum_{k=1}^{\infty} U_{x_k} \leq 1,$$

by choice of x_k. In fact, we also have $\|\cdot\|_{L^1(1+\psi)}$ bounded on $\mathcal{G}$ since we had assumed that $\|\cdot\|_{L^1}$ is bounded on $\mathcal{G}$. Now, we may easily choose a non-decreasing function $\phi \in C^\infty[0, \infty)$ with the property that $1 \leq \phi(x) \leq 1 + \psi(x)$ and $\phi(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. The Lemma then follows. ∎

PROOF OF THEOREM 3. If all functions in $\mathcal{F}$ are zero almost everywhere, then we are done. Otherwise, fix $f_0 \in \mathcal{F}$ which is not almost everywhere null. Let $M = \sup_{f \in \mathcal{F}} \Lambda_\Phi(f)$. Choose a compactly supported $\Gamma \in C^\infty(0, \infty)$ such that $\Lambda_{\Gamma+\Phi}(f_0) > M$. Now, $\Lambda_{\Gamma+\Phi} = \Lambda_\Gamma + \Lambda_\Phi$. By the bounded convergence theorem, Λ_Γ is continuous on all of $\mathcal{F}$ with respect to convergence in measure, so that $\Lambda_{\Gamma+\Phi}$ is upper semicontinuous everywhere on $\mathcal{F}\backslash\{0\}$. Now, choose a sequence $f_n \in \mathcal{F}$ such that

$$\lim_{n \to \infty} \Lambda_{\Gamma+\Phi}(f_n) = \sup_{f \in \mathcal{F}} \Lambda_{\Gamma+\Phi}(f). \tag{10}$$

Then, the upper semicontinuity of $\Lambda_{\Gamma+\Phi}$ away from zero implies that there exists a maximum if there is a subsequence of the f_n which converges in measure to some nonzero function of $\mathcal{F}$. Hence, by the compactness property of $\mathcal{F}$, in order to obtain a contradiction we may assume that $f_n \longrightarrow 0$ in measure. But since Λ_Γ is continuous on $\mathcal{F}$, it follows that $\Lambda_\Gamma(f_n) \longrightarrow \Gamma(0) = 0$. Hence the left side of (10) cannot exceed M. Thus, it follows that $\Lambda_{\Gamma+\Phi}(f) \leq M$ for each $f \in \mathcal{F}$. But this cannot be true for $f = f_0$ because of our choice of Γ. Hence, we see that f_n cannot tend to zero in measure and we are done. ∎

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University of British Columbia
Vancouver, B.C. V6T 1Z2
e-mail: pruss@math.ubc.ca