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ON VON NEUMANN–JORDAN CONSTANTS

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Abstract

In this note, we provide an example of a Banach space X for which $\tilde{C}_{NJ}(X) = 1$ that is not isomorphic to any Hilbert space, where $\tilde{C}_{NJ}(X)$ denotes the infimum of all von Neumann–Jordan constants for equivalent norms of X.

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1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. The von Neumann–Jordan constant of X, denoted by $C_{NJ}(X)$, is the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

for all $x, y \in X$ such that $\|x\|^2 + \|y\|^2 \neq 0$. Classical results state that:

(i) $1 \leq C_{NJ}(X) \leq 2$ for any Banach space X, and X is a Hilbert space if and only if $C_{NJ}(X) = 1$ (Jordan and von Neumann [2]);

(ii) $C_{NJ}(L_p) = 2^{2/t-1}$, where $t = \min\{p, p'\}$ and $1/p + 1/p' = 1$ (see [1]).

The constant $\tilde{C}_{NJ}(X)$ is defined by

$$\tilde{C}_{NJ}(X) = \inf\{C_{NJ}(X, |\cdot|) : |\cdot| \text{ is a norm equivalent to } \|\cdot\|\}.$$

Let Y be a subspace of X. It is easily checked that $\tilde{C}_{NJ}(Y) \leq \tilde{C}_{NJ}(X)$.

Many results on the constants $C_{NJ}(X)$ and $\tilde{C}_{NJ}(X)$ for various X have been proved by Kato *et al.* [3–8]. In particular, Kato and Takahashi [5] showed that $\tilde{C}_{NJ}(X) < 2$

if and only if X is superreflexive. Moreover, in [8], they gave the following stronger result: $C_{NJ}(X) < 2$ if and only if X is uniformly nonsquare.

We are concerned with the question whether a Banach space X with $\tilde{C}_{NJ}(X) = 1$ is necessarily isomorphic to a Hilbert space. In this note, we provide a negative answer to this question, by giving an example of a Banach space X for which $\tilde{C}_{NJ}(X) = 1$ that is not isomorphic to any Hilbert space.

We denote ℓ_2-direct sums using the $\oplus$ symbol: we write, for example, both $\bigoplus_{n=1}^{\infty} X_n$ and $X \oplus Y$.

2. Main results

DEFINITION 2.1 [7]. The nth von Neumann–Jordan constant, where $n \geq 1$, is defined by

$$C_{NJ}^{(n)}(X) := \sup\left\{ \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^2 \Big/ \left(2^n \sum_{j=1}^{n} \|x_j\|^2 \right) : x_j \in X, \sum_{j=1}^{n} \|x_j\|^2 \neq 0 \right\}.$$

It is evident that $C_{NJ}^{(2)}(X) = C_{NJ}(X)$.

THEOREM 2.2. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces satisfying the following conditions:*

(i) *the dimension of each X_n is finite;*
(ii) $\sup_{m,n} C_{NJ}^{(m)}(X_n) = \infty$;
(iii) $\lim_{n\to\infty} C_{NJ}(X_n) = 1$.

Let X be the ℓ_2-direct sum $\bigoplus_{n=1}^{\infty} X_n$. Then $\tilde{C}_{NJ}(X) = 1$ and X is not isomorphic to any Hilbert space. In particular, $\tilde{C}_{NJ}(X) < C_{NJ}(X)$.

EXAMPLE. The following example satisfies conditions (i), (ii) and (iii) above. Suppose that $1 \leq p < 2$, and e_i are the unit coordinate vectors in ℓ_p^n, where $1 \leq i \leq n$ and $n \in \mathbb{N}$. Then

$$\frac{\sum_{\theta_j=\pm 1} \| \sum_{j=1}^{n} \theta_j e_j \|^2}{2^n \sum_{j=1}^{n} \|e_j\|^2} = \frac{\sum_{\theta_j=\pm 1} n^{2/p}}{2^n n} = \frac{2^n n^{2/p}}{n 2^n} = n^{2/p-1}.$$

Hence, $C_{NJ}^{(n)}(\ell_p^n) \geq n^{2/p-1}$. When $1 \leq p < 2$, we have $\lim_{n\to\infty} C_{NJ}^{(n)}(\ell_p^n) = \infty$, and so we can take a sequence $\{a_n\} \subseteq \mathbb{N}$ satisfying $C_{NJ}^{(a_n)}(\ell_{2-1/n}^{a_n}) > n$. We put $X_n = \ell_{2-1/n}^{a_n}$, then (i) and (ii) hold. As mentioned in the introduction, (iii) holds since $C_{NJ}(X_n) = 2^{1/(2n-1)}$.

LEMMA 2.3. *If X is isomorphic to a Hilbert space, then*

$$\sup_n C_{NJ}^{(n)}(X) < +\infty.$$

PROOF. We assume that the Banach space $(X, \|\cdot\|)$ is isomorphic to a Hilbert space $(X, |\cdot|)$. Then there exists $M \geq 1$ such that

$$\frac{1}{M}\|x\| \leq |x| \leq M\|x\| \quad \forall x \in X. \tag{2.1}$$

For all $x_1, x_2, \ldots, x_n \in X$ such that $\sum_{j=1}^n \|x_j\|^2 \neq 0$,

$$\sum_{\theta_j=\pm 1}\left|\sum_{j=1}^n \theta_j x_j\right|^2 = 2^n \sum_{j=1}^n |x_j|^2,$$

by the parallelogram law in Hilbert space. Using this equality and inequality (2.1) above,

$$\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^n \theta_j x_j\right\|^2 \leq M^2 \sum_{\theta_j=\pm 1}\left|\sum_{j=1}^n \theta_j x_j\right|^2 = M^2 2^n \sum_{j=1}^n |x_j|^2 \leq M^4 2^n \sum_{j=1}^n \|x_j\|^2.$$

Hence,

$$\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^n \theta_j x_j\right\|^2 \Bigg/ \left(2^n \sum_{j=1}^n \|x_j\|^2\right) \leq M^4,$$

and we conclude that $C_{NJ}^{(n)}(X) \leq M^4$. □

LEMMA 2.4. *Let $\{X_n\}$ be a sequence of Banach spaces; then*

$$C_{NJ}\left(\bigoplus_{n=1}^{\infty} X_n\right) = \sup\{C_{NJ}(X_n) \mid n \in \mathbb{N}\}.$$

PROOF. We first show that

$$C_{NJ}\left(\bigoplus_{n=1}^{\infty} X_n\right) \leq \sup\{C_{NJ}(X_n) \mid n \in \mathbb{N}\}. \tag{2.2}$$

To prove this, it is sufficient to show that when $C > 0$,

$$\sup\{C_{NJ}(X_n) \mid n \in \mathbb{N}\} \leq C \Longrightarrow C_{NJ}\left(\bigoplus_{n=1}^{\infty} X_n\right) \leq C.$$

Moreover, it suffices to show that this assertion holds for the case of two terms:

$$\max\{C_{NJ}(X_1), C_{NJ}(X_2)\} \leq C \Longrightarrow C_{NJ}(X_1 \oplus X_2) \leq C.$$

For all $x, y \in X_1 \oplus X_2$, we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$, and then

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \|x_1+y_1\|^2 + \|x_1-y_1\|^2 + \|x_2+y_2\|^2 + \|x_2-y_2\|^2 \\ &\leq 2C(\|x_1\|^2 + \|y_1\|^2) + 2C(\|x_2\|^2 + \|y_2\|^2) \\ &= 2C(\|x_1\|^2 + \|x_2\|^2) + 2C(\|y_1\|^2 + \|y_2\|^2) \\ &= 2C(\|x\|^2 + \|y\|^2),\end{aligned}$$

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and hence

$$\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \le C.$$

In the same way,

$$\frac{1}{C} \le \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)}.$$

Thus, $C_{NJ}(X_1 \oplus X_2) \le C$ and hence (2.2) holds.

The other inequality is obvious as $\bigoplus_{k=1}^{\infty} X_k \supseteq X_n$ for all $n \in \mathbb{N}$. □

COROLLARY 2.5. *For any two Banach spaces X and Y,*

$$\tilde{C}_{NJ}(X \oplus Y) = \max\{\tilde{C}_{NJ}(X), \tilde{C}_{NJ}(Y)\}.$$

PROOF. We first show that

$$\tilde{C}_{NJ}(X \oplus Y) \le \max\{\tilde{C}_{NJ}(X), \tilde{C}_{NJ}(Y)\}. \tag{2.3}$$

By the definition of $\tilde{C}_{NJ}$, for any $\varepsilon > 0$, there exist Banach spaces X' and Y', isomorphic to X and Y, such that

$$C_{NJ}(X') \le \tilde{C}_{NJ}(X) + \varepsilon \quad \text{and} \quad C_{NJ}(Y') \le \tilde{C}_{NJ}(Y) + \varepsilon.$$

Using Lemma 2.4,

$$\begin{aligned}\max\{\tilde{C}_{NJ}(X), \tilde{C}_{NJ}(Y)\} + \varepsilon &\ge \max\{C_{NJ}(X'), C_{NJ}(Y')\}\\ &= C_{NJ}(X' \oplus Y')\\ &\ge \tilde{C}_{NJ}(X \oplus Y).\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, (2.3) holds.

As mentioned in the introduction, the opposite inequality to (2.3) can easily be derived from the inclusion of both X and Y in $X \oplus Y$. □

PROOF OF THEOREM 2.2. By Corollary 2.5, for all $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{C}_{NJ}(X) &= \tilde{C}_{NJ}\left(\bigoplus_{k=1}^{n} X_k \oplus \bigoplus_{k=n+1}^{\infty} X_k\right)\\ &= \max\left\{\tilde{C}_{NJ}\left(\bigoplus_{k=1}^{n} X_k\right), \tilde{C}_{NJ}\left(\bigoplus_{k=n+1}^{\infty} X_k\right)\right\}\\ &\le \max\left\{\tilde{C}_{NJ}\left(\bigoplus_{k=1}^{n} X_k\right), C_{NJ}\left(\bigoplus_{k=n+1}^{\infty} X_k\right)\right\}.\end{aligned}$$

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Since $\bigoplus_{k=1}^{n} X_k$ is finite-dimensional, it is isomorphic to a Hilbert space and thus $\tilde{C}_{NJ}(\bigoplus_{k=1}^{n} X_k) = 1$. Further, by Lemma 2.4 and condition (iii),

$$\lim_{n\to\infty} C_{NJ}\left(\bigoplus_{k=n+1}^{\infty} X_k\right) = 1.$$

Hence $\tilde{C}_{NJ}(X) \leq 1$ and so $\tilde{C}_{NJ}(X) = 1$.

On the other hand, $X_n \subseteq X$ for each $n \in \mathbb{N}$ and condition (ii) holds, so

$$\sup\{C_{NJ}^{(m)}(X) \mid m \in \mathbb{N}\} \geq \sup\{C_{NJ}^{(m)}(X_n) \mid m, n \in \mathbb{N}\} = \infty.$$

Thus from Lemma 2.3, X is not isomorphic to any Hilbert space. □

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