BULL. AUSTRAL. MATH. SOC. 55B20
VOL. 9 (1973), 55-60.

Remark on a theorem of E.H. Brown

Michael D. Alder

The proliferation of classifying spaces in recent years owes much to the theorem of Edgar H. Brown, Jr on the representability of homotopy functors. Since the theorem only gives a representation for functors defined on the category of spaces having the homotopy type of a CW complex, there is some interest in finding conditions under which the domain category may be enlarged. It appears that a version of the theorem holds for any small full subcategory of Htp, the category of topological spaces and homotopy classes of continuous maps, but that the resulting classifying space is generally intractable.

1. Introduction

In [2], Brown proved that a contravariant functor $\Gamma : HC \to S$ satisfying a form of cocontinuity condition is representable, and moreover is represented by a complex. Here HC is the category of CW complexes and homotopy classes of continuous maps, S is the category of sets. Dold [3] has shown that the fibre-bundle functors are representable on the category HP of paracompact spaces and homotopy classes of continuous maps. The purpose of this note is to show that for any contravariant functor satisfying the conditions of Brown's Theorem and defined on Htp, for every small full subcategory M of Htp, $\Gamma|M$ is representable. In particular therefore all the appropriate functors are representable on the category of separable metric spaces, for instance. This result is of limited utility, since the classifying space is generally 'huge' in the sense of [1].

We give some preliminaries on notation and definitions.

Received 1 March 1973.

Notation 1.1. *Htp* is the category of topological spaces and homotopy classes of continuous maps. S is the category of sets and set maps; *Top* the category of topological spaces and continuous maps. Lower case roman, f, g will denote morphisms in *Top* or S, the homotopy classes corresponding to f, g in *Top* are written $[f], [g]$, morphisms in *Htp*.

DEFINITION 1.2. Given $f, g : X \to Y$ in any category, a *wequaliser* is a pair (A, h) with $h : A \to X$ satisfying $f \circ h = g \circ h$, and whenever $j : B \to X$ satisfies $f \circ j = g \circ j$, there is a (not necessarily unique) map $\alpha : B \to A$ with $h \circ \alpha = j$. Dually a *cowequaliser* of $f, g : X \to Y$ is a pair $(Z, h) : h : Y \to Z$, which is a coequaliser except for the removal of the uniqueness of factorisation condition; (*cf.* [6], weak kernel).

REMARK 1.3. It is well known that *Htp* does not in general admit equalisers or coequalisers [5], but it does have a standard cowequaliser, namely, the mapping torus [4].

DEFINITION 1.4. $\Gamma : Htp \to S$ is a *homotopy functor* iff Γ is contravariant and

(1) there is no space X with $\Gamma X = \emptyset$;

(2) $\Gamma\left(\coprod_J X_j\right) \approx \prod_J \Gamma X_j$ with the obvious action of Γ on the maps; this may be described by the phrase 'Γ takes coproducts to products';

(3) Γ takes cowequalisers to wequalisers, with the obvious and conventional interpretation of the phrase.

Notation 1.5. We use lower case greek letters to denote transfinite ordinals; $|X|$ denotes the cardinality of the set X; we suppose the set theory of the appendix to [7].

2. Construction of the classifying space

This section consists of a proof of:

PROPOSITION 2.1. *Let* M *be any small, full subcategory of* *Htp*, *and* $\Gamma : Htp \to S$ *a homotopy functor. Then* $\Gamma|M$ *is representable.*

https://doi.org/10.1017/S0004972700042866 Published online by Cambridge University Press

Proof. Htp_Γ is the category with objects (X, x) with $X \in Htp$ and $x \in \Gamma X \in S$, and morphisms $f : (X, x) \to (Y, y)$ those morphisms of Htp $[f]$ such that $\Gamma[f](y) = X$.

Then if M_Γ is the obvious subcategory of Htp_Γ, we seek an object $(T, t) \in Htp_\Gamma$ such that $\forall (X, x) \in M_\Gamma$, $\exists ! \ \alpha \in Htp_\Gamma$, $\alpha : (X, x) \to (T, t)$. Such an object is said to *terminate* M_Γ, and it is plain that $[t] : [-, T] \to \Gamma$ defined by

$$[t](X) : [X, T] \rightsquigarrow \Gamma X , \qquad f \rightsquigarrow \Gamma f(t)$$

is a natural equivalence for all X in M precisely when (T, t) terminates M.

We seek to construct such an object. Let $F : Htp_\Gamma \to Htp$ denote the forgetful functor $(X, x) \rightsquigarrow X$, $f \rightsquigarrow [f]$. Put $(L, l)_1 = \coprod_{M_\Gamma} (X, x)$, the coproduct of every object in M_Γ. It is easy to see this is well defined (since Γ takes coproducts to products, Htp_Γ has arbitrary coproducts, and M_Γ is a set).

Let ω be the least (infinite) limit ordinal such that

$$|\omega| > \left| \bigcup_M X \right| .$$

We proceed inductively to obtain a long sequence $\{(L, l)_\alpha : \alpha < \omega\}$ of objects in Htp_Γ. We shall construct particular spaces L_α such that for $\alpha \le \beta$, $L_\alpha \subset L_\beta$ with a canonical inclusion $i_{\alpha\beta} : L_\alpha \to L_\beta$ so that $[i_{\alpha\beta}]$ is a morphism of the long sequence.

Suppose we have a sequence $\{(L, l)_\alpha : \alpha < \zeta\}$ for some ordinal ζ, with maps $[i_{\alpha\beta}]$ such that for some $i_{\alpha\beta} \in [i_{\alpha\beta}]$, $i_{\alpha\beta} : L_\alpha \to L_\beta$ is an inclusion, for all $\alpha \le \beta < \zeta$; ($i_{\beta\beta}$ the identity $\forall \beta$).

Case 1. If ζ is not a limit ordinal, let $J_{\zeta-1}$ be the set of morphisms $J_{\zeta-1} = \{[j] : (L, l)_1 \to (L, l)_{\zeta-1}\}$ and define

$$[a], [b] : \coprod_{J_{\zeta-1}} [(L, l_1)]_j \to (L, l)_{\zeta-1}$$

by

$$[a] \mid [(L, l)_1]_j = [i_{1,\zeta-1}] \ , \quad \forall\ j \in J_{\zeta-1} \ ,$$

$$[b] \mid [(L, l)_1]_j = [j] \qquad , \quad \forall\ j \in J_{\zeta-1}$$

Taking the diagram on Htp which is the image by F of the above diagram, we take the standard cowequaliser to obtain a space L_ζ and a map $i_{\zeta-1,\zeta} : L_{\zeta-1} \to L_\zeta$ that is an inclusion.

Put

$$i_{\alpha,\zeta} = i_{\zeta-1,\zeta} \circ i_{\alpha,\zeta-1} \ , \ \forall\ \alpha \leq \zeta-1 \ .$$

Since Γ takes cowequalisers to wequalisers, there is an object l_ζ in $\Gamma(L_\zeta)$ such that

$$\Gamma[i_{\alpha,\zeta}](l_\zeta) = l_\alpha$$

(the oɔject is nɔt necessarily unique), and the sequence extends to the ordinal ζ .

Case 2. If ζ is a limit ordinal, let Δ be the diagram in Htp_Γ with objects the set $\{(L, l)_\alpha : \alpha < \zeta\}$ and morphisms the $\{[i_{\alpha\beta}], \alpha \leq \beta < \zeta\}$. $F\Delta$ in Htp admits a 'standard cowlimit', the standard cowequaliser of the maps

$$A, B : \coprod_{Z} L_{\alpha\beta} \to \coprod_{\alpha<\zeta} L_\alpha \ ,$$

where the domain of A and B is the coproduct of as many copies of L_α as there are ordinals β such that $\alpha < \beta < \zeta$ (for each $\alpha < \zeta$).

Then $A|L_{\alpha\beta}$ is $[i_{\alpha\beta}]$, and $B|L_{\alpha\beta}$ is $[i_{\alpha,\alpha}]$ (the identity) followed by the canonical inclusions. Then, again, the hypotheses on Γ give an object $l_\zeta \in \Gamma(L_\zeta)$ such that

https://doi.org/10.1017/S0004972700042866 Published online by Cambridge University Press

$$(\Gamma[i_{\alpha\zeta}])(l_\zeta) = l_\alpha , \quad \forall \alpha < \zeta ,$$

where $(i_{\alpha\zeta})$ is the composite of the inclusion $L_\alpha \to \coprod_{\alpha<\zeta} L_\alpha$ with the canonical map into the standard cowlimit.

Thus we can construct the sequence for any ordinal, in particular it may be taken of length ω .

REMARK 2.1.1. The mapping torus construction referred to here as the 'standard cowequaliser' means that the space L_ω admits the decomposition

$$L_\omega : \bigcup_{\alpha<\omega} \left(\bigcup_{\alpha<\theta<\omega} L_\alpha \times I_\theta \right)$$

where I_θ is a copy of the unit interval indexed by the ordinals $\theta : \alpha < \theta < \omega$ and with $L_\alpha \times 0_\theta$ identified to L_α for all θ , $L_\alpha \times 1_\theta$ identified to L_θ by the canonical inclusions $i_{\alpha\theta}$.

Now we show that $(L, l)_\omega$ terminates M_Γ by a simple cardinality argument.

The inclusions $(X, x) \to (L, l)_1$ followed by $(i_{1,\omega})$ guarantee at least one morphism from each object of M to $(L, l)_\omega$. We show there is precisely one.

Let $[f], [g] : (X, x) \to (L, l)_\omega$ be morphisms in Htp_Γ . Choosing any representatives $f, g : X \to L_\omega$ we observe that each point of X is sent into some $L_\alpha \times I_\theta$, so

$$f(X) \subset \bigcup_{\alpha\in V} \left(\bigcup_{\theta\in W} (L_\alpha \times I_\theta) \right) ,$$

for some indexing sets V, W . Similarly, for gX ; (and we may choose the same V, W).

Now we have that $|V|, |W| \le |X|$; $|\alpha| < |\omega|$ so both fX and gX are in some 'left segment'

$$\bigcup_{\alpha<\zeta} \left(\bigcup_{\alpha<\theta<\zeta} (L_\alpha \times I_\theta) \right) \text{ of } L_\omega , \quad \zeta < \omega .$$

Without loss of generality we may take ζ to be a limit ordinal, when we have that $fX,\ gX \subset L_\zeta$.

But $[i_{\zeta,\zeta+1}] : (L, l)_\zeta \to (L, l)_{\zeta+1}$ coequalises maps from M_Γ by construction, and so $[f] = [g] : (X, x) \to (L, l)_\omega$.

Hence there is a unique map $(X, x) \to (L, l)_\omega$ and the result follows.

REMARK 2.2. Using the reduced mapping torus in Htp_* gives a similar result in this category: Remark 2.1.1 requires a small modification, but the category-theoretic form ensures that the rest goes through.

References

[1] Raoul Bott, "On topological obstructions to integrability", *Actes Congr. Internat. Math. 1970* **1** (1971), 27-36.

[2] Edgar H. Brown, Jr, "Abstract homotopy theory", *Trans. Amer. Math. Soc.* **119** (1965), 79-85.

[3] Albrecht Dold, "Partitions of unity in the theory of fibrations", *Ann. of Math.* (2) **78** (1963), 223-255.

[4] Albrecht Dold, *Halbexakte Homotopiefunktoren* (Lecture Notes in Mathematics, **12**. Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[5] B. Eckmann and P.J. Hilton, "Unions and intersections in homotopy theory", *Comment. Math. Helv.* **38** (1963/64), 293-307.

[6] Peter Freyd, "Stable homotopy", *Proc. Conf. Categorical Algebra, La Jolla,* 1965, 121-172 (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[7] John L. Kelley, *General topology* (Van Nostrand, Toronto, New York, London, 1955).

Department of Mathematics,
University of Western Australia,
Nedlands,
Western Australia.