Proceedings of the Edinburgh Mathematical Society (2002) **45**, 229–239 ©

A CHARACTERIZATION OF THE HARMONIC BLOCH SPACE AND THE HARMONIC BESOV SPACES BY AN OSCILLATION

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(Received 28 December 1999)

Abstract We characterize the Bloch space and the Besov spaces of harmonic functions on the open unit disc D by using the following oscillation:

$$\sup_{\beta(z,w)<r}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\right|,$$

where $\alpha+\beta=n$, $\alpha,\beta\in\mathbb{R}$ and $\hat{D}^{(n)}=(\partial^n/\partial^n z+\partial^n/\partial^n \bar{z})$.

Keywords: harmonic Bloch space; harmonic Besov space; oscillation

AMS 2000 *Mathematics subject classification:* Primary 46E15

1. Introduction

Denote by $H(D)$ the space of functions f that are analytic on the open unit disk D in the complex plane C. $\mathrm{d}A(z)$ denotes the normalized area measure on D. For $1\leqslant p<+\infty$, the Lebesgue space $L^p(D,\mathrm{d}\lambda)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disc D with

$$\|f\|_{L^p(d\lambda)}:=\left(\int_D|f(z)|^p\,\mathrm{d}\lambda(z)\right)^{1/p}<+\infty,$$

where $\mathrm{d}\lambda(z)=\mathrm{d}A(z)/(1-|z|^2)^2$. The Lebesgue space $L^{\infty}(D,\mathrm{d}\lambda)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disc D with $\sup_{z\in D}|f(z)|<+\infty$.

Let $\alpha>0$ and let $(1/\alpha)<p\leqslant\infty$. Denote by B_p^{α} the vector space of functions $f\in H(D)$ such that $z\to(1-|z|^2)^{\alpha}|f'(z)|\in L^p(D,\mathrm{d}\lambda)$, and by B_0^{α} the space of functions $f\in H(D)$ such that

$$(1-|z|^2)^{\alpha}|f'(z)|\to 0\quad(|z|\to 1).$$

For $p=1$, $\alpha=1$, the Besov space B_1 consists of analytic functions f on D such that $f(z)=\sum_{n=1}^{+\infty}a_n\varphi_{\lambda_n}(z)$ with $\sum_{n=1}^{+\infty}|a_n|<+\infty$, where $\{\lambda_n\}_{n=1}^{+\infty}\subset D$.

Define

$$B_p^{h,\alpha} = B_p^\alpha + \overline{B_p^\alpha} = \{f + \bar{g} : f, g \in B_p^\alpha\}$$

and

$$B_0^{h,\alpha} = B_0^\alpha + \overline{B_0^\alpha} = \{f + \bar{g} : f, g \in B_0^\alpha\}.$$

You might then like to observe that B_p^1, with $1 < p < \infty$, is the usual scale of Besov spaces, that is $B_p^1 = B_p$. Observe that B_∞^1 and B_0^1 are the usual scale of Bloch space B and the little Bloch space B_0, respectively.

For $z, w \in D$, let

$$\beta(z,w) := \tfrac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|},$$

where $\varphi_z(w) = (z - w)/(1 - \bar{z}w)$. For $0 < r < +\infty$, let $D(z) = D(z,r) = \{w \in D; \beta(z,w) < r\}$ denote the Bergman disc. $|D(z,r)|$ denotes the normalized area of $D(z,r)$ and $|D(z,r)|$ is comparable with $(1 - |z|^2)^2$.

For a differentiable complex function h on D and for $n = 0, 1, 2, 3, \ldots$, we define a differential operator $\hat{D}^{(n)}$ by $\hat{D}^{(n)}h = (\partial^n/\partial^n z + \partial^n/\partial^n \bar{z})h$. If $h = f + \bar{g}$ is a harmonic function on D such that f and g are analytic functions on D, then $\hat{D}^{(n)}h = f^{(n)} + \overline{g^{(n)}}$, for $n = 0, 1, 2, 3, \ldots$.

The following theorem explains why B_1 as defined above is compatible with the other Besov spaces B_p $(1 < p \leqslant \infty)$.

Theorem A (see p. 90 in [6]). *If $f \in H(D)$, $1 \leqslant p \leqslant +\infty$ and $n \geqslant 2$ is an integer, then $f \in B_p$ if and only if $(1 - |z|^2)^n f^{(n)}(z) \in L^p(D, \mathrm{d}\lambda)$.*

In [**5**] the author proved the following Theorems B and C for the Bloch space B and the Besov space B_p by using Theorem A, respectively.

Theorem B. *Let n be a positive integer, and α, β real numbers with $\alpha + \beta = n$. Then for $f \in H(D)$ and for $r \in (0, +\infty)$, $f \in B$ if and only if*

$$\sup_{z \in D} \sup_{w \in D(z,r), z \neq w} (1 - |z|^2)^\alpha (1 - |w|^2)^\beta \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z - w} \right| < +\infty.$$

Theorem C. *Let n be a positive integer, and α, β real numbers with $\alpha + \beta = n$. Then for $f \in H(D)$ and for $r \in (0, +\infty)$, for $p > 1$ in the case of $n = 1$, and for $p \geqslant 1$ in the case of $n \geqslant 2$, $f \in B_p$ if and only if*

$$\int_D \left(\sup_{w \in D(z,r)} (1 - |z|^2)^\alpha (1 - |w|^2)^\beta \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z - w} \right| \right)^p \mathrm{d}\lambda(z) < +\infty.$$

In this paper we will give the analogous result which characterize the spaces $B_\infty^{h,1}$, $B_0^{h,1}$ and $B_p^{h,1}$.

In [**3**], Holland and Walsh proved the following theorem in the case of $n = 1$. And we proved it in the case of $n = 2$ in [**5**].

Theorem D. *Let $f \in H(D)$ and $n = 1, 2$. Then $f \in B$ if and only if*

$$\sup_{z \neq w} (1-|z|^2)^{n/2}(1-|w|^2)^{n/2} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| < +\infty.$$

In this paper we will also generalize Theorem D.

In § 4 we will also characterize the α-harmonic Bloch space $B_\infty^{h,\alpha}$ and the little α-harmonic Bloch space $B_0^{h,\alpha}$ and the α-harmonic Besov space $B_p^{h,\alpha}$ as well as the harmonic Bloch space $B_\infty^{h,1}$ and the little harmonic Bloch space $B_0^{h,1}$ and the harmonic Besov space $B_p^{h,1}$, respectively.

Throughout this paper, C_i, K_i for $i = 0, 1, 2$, C, K will denote positive constants whose values are not necessarily the same at every occurrence.

2. The harmonic Bloch space

To prove the theorems above, we use the following lemmas.

Lemma 2.1. *Let $h \in H(D) + \overline{H(D)}$. Let $n \geqslant 2$ be an integer and $1 \leqslant p \leqslant +\infty$. Then $h = f + \bar{g} \in B_p^{h,1}$ if and only if*

$$(1-|z|^2)^n(|f^{(n)}(z)| + |g^{(n)}(z)|) \in L^p(D, \mathrm{d}\lambda).$$

Proof. This follows from Theorem A. □

Lemma 2.2. *Let $h \in H(D) + \overline{H(D)}$. Let $n \geqslant 2$ be an integer and $1 \leqslant p \leqslant +\infty$. Then $h = f + \bar{g} \in B_0^{h,1}$ if and only if*

$$\lim_{|z| \to 1^-} (1-|z|^2)^n(|f^{(n)}(z)| + |g^{(n)}(z)|) = 0.$$

Proof. This follows from Theorem 5.2.6 of [**6**]. □

Some of the techniques used to prove the following theorem were inspired by Colonna [**1**].

Theorem 2.3. *Let $r \in (0, +\infty)$. Fix an integer $n \geqslant 1$ and a pair of real numbers α, β such that $\alpha + \beta = n$. Let $h \in H(D) + \overline{H(D)}$. Then $h \in B_\infty^{h,1}$ if and only if*

$$\sup_{z \in D} \sup_{\beta(z,w) < r, z \neq w} (1-|z|^2)^\alpha (1-|w|^2)^\beta \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right| < +\infty.$$

Proof. Let $h \in H(D) + \overline{H(D)}$. To prove the necessity, put

$$E_{\hat{D}^{(n-1)}h} := \sup_{z \in D} \sup_{\beta(z,w) < r, z \neq w} (1-|z|^2)^\alpha (1-|w|^2)^\beta \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right|.$$

https://doi.org/10.1017/S001309159900142X Published online by Cambridge University Press

Let $w = z + r_1 e^{i\theta} \in D$. Then

$$\begin{aligned}\left|\frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w}\right| &= \left|\frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(z + r_1 e^{i\theta})}{z - (z + r_1 e^{i\theta})}\right| \\ &= \frac{|\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(z + r_1 e^{i\theta})|}{r_1} \\ &\to |(\hat{D}^{(n-1)}h)_x(z)\cos\theta + (\hat{D}^{(n-1)}h)_y(z)\sin\theta| \quad (r_1 \to 0).\end{aligned}$$

Putting

$$E_{\hat{D}^{(n-1)}h}(z) := \limsup_{w\to z}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w}\right|,$$

then we have

$$E_{\hat{D}^{(n-1)}h}(z) = \max_{\theta}(1-|z|^2)^n|(\hat{D}^{(n-1)}h)_x(z)\cos\theta + (\hat{D}^{(n-1)}h)_y(z)\sin\theta|.$$

By Lemma 1 of [**1**], we have

$$\begin{aligned}E_{\hat{D}^{(n-1)}h}(z) &= \tfrac{1}{2}(1-|z|^2)^n(|(\hat{D}^{(n-1)}h)_x(z) + i(\hat{D}^{(n-1)}h)_y(z)| \\ &\qquad + |(\hat{D}^{(n-1)}h)_x(z) - i(\hat{D}^{(n-1)}h)_y(z)|) \\ &= (1-|z|^2)^n(|f^{(n)}(z)| + |g^{(n)}(z)|).\end{aligned}$$

Since $E_{\hat{D}^{(n-1)}h}(z) \leqslant E_{\hat{D}^{(n-1)}h}$ $(z \in D)$, hence we have

$$\sup_{z\in D}(1-|z|^2)^n(|f^{(n)}(z)| + |g^{(n)}(z)|) \leqslant B_{\hat{D}^{(n-1)}h} < +\infty.$$

By Lemma 2.1, we have $h \in B^{h,1}_{\infty}$.

To prove the sufficiency, suppose that $h = f + \bar{g} \in B^{h,1}_{\infty}$. Then

$$\begin{aligned}&\sup_{\beta(z,w)<r}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w}\right| \\ &\quad\leqslant \sup_{\beta(z,w)<r}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w}\right| \\ &\qquad + \sup_{\beta(z,w)<r}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{g^{(n-1)}(z) - g^{(n-1)}(w)}{z-w}\right|.\end{aligned}$$

Since $h = f + \bar{g} \in B^{h,1}_{\infty}$, $f \in B$ and $g \in B$. By Theorem 2.7 in [**5**],

$$\sup_{z\in D}\sup_{w\in D(z,r), z\neq w}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w}\right| < +\infty,$$

$$\sup_{z\in D}\sup_{w\in D(z,r), z\neq w}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{g^{(n-1)}(z) - g^{(n-1)}(w)}{z-w}\right| < +\infty.$$

https://doi.org/10.1017/S001309159900142X Published online by Cambridge University Press

Hence we have

$$\sup_{z\in D}\sup_{\beta(z,w)<r,z\neq w}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\right|<+\infty.$$

This completes the proof of Theorem 2.3. □

The following corollary does not hold for $n\geqslant 3$ because $h(z)=\log(1-z)\in B_{\infty}^{h,1}$ is a counterexample and it will generalize Theorem D.

Corollary 2.4. *Let $h\in H(D)+\overline{H(D)}$. Then for $n=1,2$, $h\in B_{\infty}^{h,1}$ if and only if*

$$\sup_{z\neq w}(1-|z|^2)^{n/2}(1-|w|^2)^{n/2}\left|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\right|<+\infty.$$

Proof. The necessity follows easily from Theorem 2.3. To prove the sufficiency, suppose that $h\in B_{\infty}^{h,1}$. When $n=1$, we have

$$\begin{aligned}\sup_{z\neq w}&(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\left|\frac{h(z)-h(w)}{z-w}\right|\\&\leqslant\sup_{z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\left|\frac{f(z)-f(w)}{z-w}\right|\\&\qquad+\sup_{z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\left|\frac{g(z)-g(w)}{z-w}\right|.\end{aligned}$$

Since $h\in B_{\infty}^{h,1}$, $f\in B$ and $g\in B$. By Theorem D we have

$$\sup_{z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\left|\frac{f(z)-f(w)}{z-w}\right|<\infty,$$
$$\sup_{z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\left|\frac{g(z)-g(w)}{z-w}\right|<\infty.$$

When $n=2$, we have

$$\begin{aligned}\sup_{z\neq w}&(1-|z|^2)(1-|w|^2)\left|\frac{\hat{D}h(z)-\hat{D}h(w)}{z-w}\right|\\&\leqslant\sup_{z\neq w}(1-|z|^2)(1-|w|^2)\left|\frac{f'(z)-f'(w)}{z-w}\right|+\sup_{z\neq w}(1-|z|^2)(1-|w|^2)\left|\frac{g'(z)-g'(w)}{z-w}\right|.\end{aligned}$$

Since $h\in B_{\infty}^{h,1}$, $f\in B$ and $g\in B$. By Theorem D we have

$$\sup_{z\neq w}(1-|z|^2)(1-|w|^2)\left|\frac{f'(z)-f'(w)}{z-w}\right|<\infty,$$
$$\sup_{z\neq w}(1-|z|^2)(1-|w|^2)\left|\frac{g'(z)-g'(w)}{z-w}\right|<\infty.$$

This completes the proof of Corollary 2.4. □

Corollary 2.5. *Let $r \in (0, +\infty)$. Fix an integer $n \geqslant 1$ and a pair of real numbers α, β such that $\alpha + \beta = n$. Let $h \in H(D) + \overline{H(D)}$. Then $h \in B_0^{h,1}$ if and only if*

$$\lim_{|z|\to 1^-} \sup_{\beta(z,w)<r, z\neq w} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right| = 0.$$

Proof. This is an immediate consequence of Lemma 2.2 and Theorem 2.3. □

The following corollary does not hold for $n \geqslant 3$ because $h(z) = (1-z)^{1/4} \in B_0^{h,1}$ is a counterexample. And it will generalize Theorem 2 in [**4**] and Corollary 4.3 in [**5**].

Corollary 2.6. *Let $h \in H(D) + \overline{H(D)}$. Then for $n = 1, 2$, $h \in B_0^{h,1}$ if and only if*

$$\lim_{|z|\to 1^-} \sup_{w\in D, z\neq w} (1-|z|^2)^{n/2}(1-|w|^2)^{n/2} \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right| = 0.$$

Proof. This follows easily from Lemma 2.2, Corollary 2.5, Theorem 2 in [**4**], and Corollary 4.3 in [**5**]. □

3. The harmonic Besov spaces

The following lemmas are used to prove Theorem 3.3.

Lemma 3.1. *Let $f \in H(D)$ and let $0 < r < +\infty$. Then for some constant $K > 0$*

$$\sup_{\beta(z,w)<r} \left| \frac{f(z) - f(w)}{z-w} \right| \leqslant \frac{K}{|D(z,2r)|} \int_{D(z,2r)} \left| \frac{f(u) - f(z)}{u-z} \right| \mathrm{d}A(u).$$

Proof. In fact, we have for all analytic functions g on D

$$|g(w)| \leqslant \frac{C}{|D(w,r)|} \int_{D(w,r)} |g(u)| \,\mathrm{d}A(u).$$

Applying $g(u) = (f(u) - f(z))/(u-z)$, then

$$\left| \frac{f(w) - f(z)}{w-z} \right| \leqslant \frac{C}{|D(w,r)|} \int_{D(w,r)} \left| \frac{f(u) - f(z)}{u-z} \right| \mathrm{d}A(u)$$

for all analytic functions f on D. Since $D(w,r) \subset D(z,2r)$ for $w \in D(z,r)$ and there is a constant $K > 0$ such that $(1/|D(w,r)|) \leqslant (K/|D(z,2r)|)$, we have

$$\sup_{w\in D(z,r)} \left| \frac{f(w) - f(z)}{w-z} \right| \leqslant \frac{CK}{|D(z,2r)|} \int_{D(z,2r)} \left| \frac{f(u) - f(z)}{u-z} \right| \mathrm{d}A(u).$$

□

Lemma 3.2. *Let $f \in H(D)$ and let $0 < r < +\infty$. Then for some constant $K > 0$*

$$\frac{1}{|D(z,r)|} \int_{D(z,r)} (1-|z|^2) \left| \frac{f(u) - f(z)}{u-z} \right| \mathrm{d}A(u) \leqslant K \int_{D(z,r)} (1-|w|^2)|f'(w)| \,\mathrm{d}\lambda(w).$$

Proof. The following inequality follows from Theorem 5.6 of [**2**]:

$$\int_{sD}\left|\frac{g(u)-g(0)}{u}\right|\mathrm{d}A(u)\leqslant C\int_{sD}|g'(u)|(1-|u|^2)\,\mathrm{d}A(u)$$

for all analytic functions g on D, where $sD=\{w\in D;\ |w|<s\}=D(0,r)$, $s=\tanh r$. Applying g to $f\circ\varphi_z$, we have

$$\int_{D(0,r)}\left|\frac{(f\circ\varphi_z)(w)-(f\circ\varphi_z)(0)}{w}\right|\mathrm{d}A(w)$$
$$\leqslant C\int_{D(0,r)}|(f\circ\varphi_z)'(w)|(1-|w|^2)\,\mathrm{d}A(w). \qquad (**)$$

Then by using $(**)$, changing the variable and noting that $(1-|z|^2)$ is comparable with $|1-\bar{w}z|$, $(1-|w|^2)$ and $|D(z,r)|^{1/2}$ when $w\in D(z,r)$, there exist constants C_1, C_2, K, K_1, $K_2>0$ (independent of f) such that

$$\begin{aligned}
\frac{1}{|D(z,r)|}\int_{D(z,r)}&(1-|z|^2)\left|\frac{f(z)-f(w)}{z-w}\right|\mathrm{d}A(w)\\
&\leqslant C_1\int_{D(z,r)}(1-|z|^2)\left|\frac{f(w)-f(z)}{\varphi_z(w)}\right||1-\bar{z}w|\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\,\mathrm{d}A(w)\\
&=C_1\int_{D(0,r)}(1-|z|^2)\left|\frac{f\circ\varphi_z(w)-f\circ\varphi_z(0)}{w}\right||1-\bar{z}\varphi_z(w)|\,\mathrm{d}A(w)\\
&\leqslant K\int_{D(0,r)}(1-|z|^2)\frac{(1-|z|^2)}{|1-\bar{z}w|}|(f\circ\varphi_z)'(w)|(1-|w|^2)\,\mathrm{d}A(w)\\
&=K\int_{D(0,r)}(1-|z|^2)\frac{(1-|z|^2)^2}{|1-\bar{z}w|^3}|f'(\varphi_z(w))|(1-|w|^2)\,\mathrm{d}A(w)\\
&\leqslant K_1\int_{D(0,r)}(1-|z|^2)|f'(\varphi_z(w))|\,\mathrm{d}A(w)\\
&\leqslant K_2\int_{D(z,r)}(1-|z|^2)|f'(w)|\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\,\mathrm{d}A(w)\\
&\leqslant C_2\int_{D(z,r)}(1-|w|^2)|f'(w)|\,\mathrm{d}\lambda(w).
\end{aligned}$$

□

Theorem 3.3. *Let $r\in(0,+\infty)$. Fix an integer $n\geqslant 1$ and a pair of real numbers α, β such that $\alpha+\beta=n$. Let $h\in H(D)+\overline{H(D)}$. Then for $p>1$ in the case of $n=1$, and for $p\geqslant 1$ in the case of $n\geqslant 2$, $h\in B_p^{h,1}$ if and only if*

$$\int_D\left(\sup_{\beta(z,w)<r}(1-|z|^2)^\alpha(1-|w|^2)^\beta\left|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\right|\right)^p\mathrm{d}\lambda(z)<+\infty.$$

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Proof. Let $h \in H(D) + \overline{H(D)}$. Suppose that

$$\int_D \left(\sup_{\beta(z,w)<r} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right| \right)^p \mathrm{d}\lambda(z) < +\infty.$$

Let

$$E_{\hat{D}^{(n-1)}h}(z) := \limsup_{w\to z} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right|.$$

Then, by the proof of Theorem 2.3, we have

$$E_{\hat{D}^{(n-1)}h}(z) = (1-|z|^2)^n (|f^{(n)}(z)| + |g^{(n)}(z)|).$$

Since

$$E_{\hat{D}^{(n-1)}h}(z) \leqslant \sup_{\beta(z,w)<r} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right|,$$

hence we see

$$\begin{aligned} &\int_D (1-|z|^2)^{np} (|f^{(n)}(z)| + |g^{(n)}(z)|)^p \,\mathrm{d}\lambda(z) \\ &\qquad \leqslant \int_D \left(\sup_{\beta(z,w)<r} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right| \right)^p \mathrm{d}\lambda(z). \end{aligned}$$

By Lemma 2.1, we have $h \in B_p^{h,1}$.

To prove the sufficiency, suppose $h \in B_p^{h,1}$. Since $(1-|z|^2)$ is comparable with $(1-|w|^2)$ for $\beta(z,w) < r$, for some constant $C > 0$

$$\begin{aligned} &\sup_{\beta(z,w)<r} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{\hat{D}^{(n-1)}h(z) - \hat{D}^{(n-1)}h(w)}{z-w} \right| \\ &\quad \leqslant \sup_{\beta(z,w)<r} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| \\ &\qquad\quad + \sup_{\beta(z,w)<r} (1-|z|^2)^{\alpha}(1-|w|^2)^{\beta} \left| \frac{g^{(n-1)}(z) - g^{(n-1)}(w)}{z-w} \right| \\ &\quad \leqslant C \sup_{\beta(z,w)<r} (1-|z|^2)^{n} \left| \frac{f^{(n-1)}(z) - f^{(n-1)}(w)}{z-w} \right| \\ &\qquad\quad + C \sup_{\beta(z,w)<r} (1-|z|^2)^{n} \left| \frac{g^{(n-1)}(z) - g^{(n-1)}(w)}{z-w} \right| \\ &\quad \leqslant \frac{K_1}{|D(z,2r)|} \int_{D(z,2r)} (1-|z|^2)^{n} \left| \frac{f^{(n-1)}(u) - f^{(n-1)}(z)}{u-z} \right| \mathrm{d}A(u) \\ &\qquad\quad + \frac{K_2}{|D(z,2r)|} \int_{D(z,2r)} (1-|z|^2)^{n} \left| \frac{g^{(n-1)}(u) - g^{(n-1)}(z)}{u-z} \right| \mathrm{d}A(u). \end{aligned}$$

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The last inequality follows from Lemma 3.1. By using Lemma 3.2, we have the following:

$$\frac{1}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)\left|\frac{f^{(n-1)}(u)-f^{(n-1)}(z)}{u-z}\right|\mathrm{d}A(u) \leqslant C\int_{D(z,2r)}(1-|w|^2)|f^{(n)}(w)|\,\mathrm{d}\lambda(w).$$

Multiplying both sides by $(1-|z|^2)^{(n-1)}$ and then using the fact that $(1-|z|^2)$ is comparable with $(1-|w|^2)$, we have

$$\frac{1}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)^n\left|\frac{f^{(n-1)}(u)-f^{(n-1)}(z)}{u-z}\right|\mathrm{d}A(u) \leqslant K_3\int_{D(z,2r)}(1-|w|^2)^n|f^{(n)}(w)|\,\mathrm{d}\lambda(w).$$

By similar calculations, we also have

$$\frac{1}{|D(z,2r)|}\int_{D(z,2r)}(1-|z|^2)^n\left|\frac{g^{(n-1)}(u)-g^{(n-1)}(z)}{u-z}\right|\mathrm{d}A(u) \leqslant K_4\int_{D(z,2r)}(1-|w|^2)^n|g^{(n)}(w)|\,\mathrm{d}\lambda(w).$$

Put $K:=\max\{K_1K_3,K_2K_4\}$. Then we have

$$\sup_{\beta(z,w)<r}(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\right| \leqslant \frac{K}{|D(z,2r)|}\int_{D(z,2r)}(1-|w|^2)^n(|f^{(n)}(w)|+|g^{(n)}(w)|)\,\mathrm{d}\lambda(w).$$

Hence, since

$$\int_{D(z,2r)}\mathrm{d}\lambda(w)\leqslant C<+\infty$$

and $\chi_{D(w,2r)}(z)=\chi_{D(z,2r)}(w)$, by using Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}
\int_D\bigg(\sup_{\beta(z,w)<r}&(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}\left|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\right|\bigg)^p\,\mathrm{d}\lambda(z)\\
&\leqslant K^p\int_D\bigg(\int_{D(z,2r)}(1-|w|^2)^n(|f^{(n)}(w)|+|g^{(n)}(w)|\bigg)\,\mathrm{d}\lambda(w))^p\,\mathrm{d}\lambda(z)\\
&\leqslant C\int_D\int_{D(z,2r)}(1-|w|^2)^{np}(|f^{(n)}(w)|+|g^{(n)}(w)|)^p\,\mathrm{d}\lambda(w)\,\mathrm{d}\lambda(z)\\
&= C\int_D\bigg(\int_{D(w,2r)}\mathrm{d}\lambda(z)\bigg)(1-|w|^2)^{np}(|f^{(n)}(w)|+|g^{(n)}(w)|)^p\,\mathrm{d}\lambda(w)\\
&\leqslant C_1\int_D(1-|w|^2)^{np}(|f^{(n)}(w)|+|g^{(n)}(w)|)^p\,\mathrm{d}\lambda(w).
\end{aligned}$$

https://doi.org/10.1017/S001309159900142X Published online by Cambridge University Press

Since $f \in B_p^{h,1}$, we have, by Lemma 2.1,

$$\int_D (1-|w|^2)^{np}(|f^{(n)}(w)|+|g^{(n)}(w)|)^p \,\mathrm{d}\lambda(w) < +\infty.$$

This completes the proof of Theorem 3.3. □

4. The α-harmonic Bloch space $B_\infty^{h,\alpha}$, the little α-harmonic Bloch space $B_0^{h,\alpha}$ and the α-harmonic Besov space $B_p^{h,\alpha}$

In §§ 2 and 3 we characterized the harmonic Bloch space $B_\infty^{h,1}$ and the little harmonic Bloch space $B_0^{h,1}$ and the harmonic Besov space $B_p^{h,1}$, but similar characterizations also hold for the α-harmonic Bloch space $B_\infty^{h,\alpha}$ and the little α-harmonic Bloch space $B_0^{h,\alpha}$ and the α-harmonic Besov space $B_p^{h,\alpha}$ by using the following Lemma E and Lemma F as well. Since most proofs are also similar to the case of $\alpha = 1$, we will only present results.

Lemma E. *Let $\alpha > 0$ and n an integer greater than or equal to 2. Let $h \in H(D) + \overline{H(D)}$. For $1 \leqslant p \leqslant +\infty$, $p(\alpha+n-1) > 1$, $h = f + \bar{g} \in B_p^{h,\alpha}$ if and only if*

$$(1-|z|^2)^{\alpha+n-1}(|f^{(n)}(z)|+|g^{(n)}(z)|) \in L^p(D, \mathrm{d}\lambda).$$

Lemma F. *Let $\alpha > 0$ and n an integer greater than or equal to 2. Let $h \in H(D) + \overline{H(D)}$. Then $h = f + \bar{g} \in B_0^{h,\alpha}$ if and only if*

$$(1-|z|^2)^{\alpha+n-1}(|f^{(n)}(z)|+|g^{(n)}(z)|) \to 0 \quad (|z| \to 1^-).$$

Theorem 4.1. *Let $\alpha > 0$ and $p > 0$. Fix an integer $n \geqslant 1$ and a pair of real numbers α_1, β_1 such that $\alpha_1 + \beta_1 - \alpha + 1 = n$ and let $r \in (0, +\infty)$. Then for $h \in H(D) + \overline{H(D)}$, $h \in B_\infty^{h,\alpha}$ if and only if*

$$\sup_{z\in D}\biggl(\sup_{w\in D(z,r), z\neq w}(1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}\biggl|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\biggr|\biggr) < +\infty.$$

Corollary 4.2. *Let $\alpha > 0$ and $p > 0$ and fix an integer $n \geqslant 1$ and a pair of real numbers α_1, β_1 such that $\alpha_1 + \beta_1 - \alpha + 1 = n$ and let $r \in (0, +\infty)$. Then for $h \in H(D) + \overline{H(D)}$, $h \in B_0^{h,\alpha}$ if and only if*

$$\lim_{|z|\to 1^-}\biggl(\sup_{w\in D(z,r), z\neq w}(1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}\biggl|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\biggr|\biggr) = 0.$$

Theorem 4.3. *Let $\alpha > 0$ and fix an integer $n \geqslant 1$ and a pair of real numbers α_1, β_1 such that $\alpha_1 + \beta_1 - \alpha + 1 = n$ and let $r \in (0, +\infty)$. Then for $h \in H(D) + \overline{H(D)}$, for $1 \leqslant p < +\infty$, $p(\alpha+n-1) > 1$, $h \in B_p^{h,\alpha}$ if and only if*

$$\int_D \biggl(\sup_{w\in D(z,r)}(1-|z|^2)^{\alpha_1}(1-|w|^2)^{\beta_1}\biggl|\frac{\hat{D}^{(n-1)}h(z)-\hat{D}^{(n-1)}h(w)}{z-w}\biggr|\biggr)^p \mathrm{d}\lambda(z) < +\infty.$$

Acknowledgements. The author expresses his sincere gratitude to Professor Takahiko Nakazi for his many helpful suggestions and his advice.

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