Proceedings of the Edinburgh Mathematical Society (2015) **58**, 169–181
DOI:10.1017/S0013091513000746

ON COMMUTING VARIETIES OF NILRADICALS OF BOREL SUBALGEBRAS OF REDUCTIVE LIE ALGEBRAS

SIMON M. GOODWIN[1] AND GERHARD RÖHRLE[2]

[1]*School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK* (s.m.goodwin@bham.ac.uk)
[2]*Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany* (gerhard.roehrle@rub.de)

(Received 10 September 2012)

Abstract Let G be a connected reductive algebraic group defined over an algebraically closed field $\Bbbk$ of characteristic 0. We consider the commuting variety $\mathcal{C}(\mathfrak{u})$ of the nilradical $\mathfrak{u}$ of the Lie algebra $\mathfrak{b}$ of a Borel subgroup B of G. In case B acts on $\mathfrak{u}$ with only a finite number of orbits, we verify that $\mathcal{C}(\mathfrak{u})$ is equidimensional and that the irreducible components are in correspondence with the *distinguished* B-orbits in $\mathfrak{u}$. We observe that in general $\mathcal{C}(\mathfrak{u})$ is not equidimensional, and determine the irreducible components of $\mathcal{C}(\mathfrak{u})$ in the minimal cases where there are infinitely many B-orbits in $\mathfrak{u}$.

Keywords: commuting varieties; Borel subalgebras; Lie algebras; algebraic groups

2010 *Mathematics subject classification:* Primary 20G15
Secondary 17B45

1. Introduction

Let G be a connected reductive algebraic group defined over an algebraically closed field $\Bbbk$ of characteristic 0, and let $\mathfrak{g} = \operatorname{Lie} G$ be its Lie algebra. Richardson proved that the commuting variety

$$\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}$$

of $\mathfrak{g}$ is irreducible (see [**10**]). This fact was generalized to positive good characteristic by Levy in [**8**]. In [**9**], Premet showed that the commuting variety $\mathcal{C}(\mathcal{N}) = \mathcal{C}(\mathfrak{g}) \cap (\mathcal{N} \times \mathcal{N})$ of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ is equidimensional, where the irreducible components are in correspondence with the distinguished nilpotent G-orbits in $\mathcal{N}$; this theorem was also proved in good positive characteristic.

In this paper we consider the commuting variety of the Lie algebra of the unipotent radical of a Borel subgroup of G. To explain this further we introduce some notation. Let B be a Borel subgroup of G with unipotent radical U, and write $\mathfrak{b}$ and $\mathfrak{u}$ for the Lie algebras of B and U, respectively. The *commuting variety of* $\mathfrak{u}$ is

$$\mathcal{C}(\mathfrak{u}) = \{(x, y) \in \mathfrak{u} \times \mathfrak{u} \mid [x, y] = 0\}.$$

For $e \in \mathfrak{u}$, we write $\mathfrak{c}_{\mathfrak{b}}(e)$ and $\mathfrak{c}_{\mathfrak{u}}(e)$ for the centralizer of e in $\mathfrak{b}$ and $\mathfrak{u}$, respectively. We define

$$\mathcal{C}(e) = \overline{B \cdot (e, \mathfrak{c}_{\mathfrak{u}}(e))} \subseteq \mathcal{C}(\mathfrak{u})$$

to be the Zariski closure of the B-saturation of $(e, \mathfrak{c}_{\mathfrak{u}}(e))$ in $\mathcal{C}(\mathfrak{u})$; it is easy to see that $\mathcal{C}(e)$ is irreducible and $\dim \mathcal{C}(e) = \dim B - \dim \mathfrak{c}_{\mathfrak{b}}(e) + \dim \mathfrak{c}_{\mathfrak{u}}(e)$. We say that $e \in \mathfrak{u}$ is *distinguished* provided that $\mathfrak{c}_{\mathfrak{b}}(e) = \mathfrak{c}_{\mathfrak{u}}(e)$, and note that for e distinguished we have $\dim \mathcal{C}(e) = \dim B$.

Below, we have an analogue of Premet's theorem from [**9**] for the case when B acts on $\mathfrak{u}$ with a finite number of orbits.

Theorem 1.1. *Suppose that B acts on $\mathfrak{u}$ with a finite number of orbits. Let $e_1, \dots, e_r$ be representatives of the distinguished B-orbits in $\mathfrak{u}$. Then,*

$$\mathcal{C}(\mathfrak{u}) = \mathcal{C}(e_1) \cup \cdots \cup \mathcal{C}(e_r)$$

is the decomposition of the commuting variety $\mathcal{C}(\mathfrak{u})$ into its irreducible components. In particular, $\mathcal{C}(\mathfrak{u})$ is equidimensional of dimension $\dim B$.

The cases when B acts on $\mathfrak{u}$ with a finite number of orbits are known, due to work by Bürgstein and Hesselink [**2**] and Kashin [**5**]. This is the case precisely when the length $\ell(\mathfrak{u})$ of the descending central series of $\mathfrak{u}$ is at most 4. Thus, if $\mathfrak{g}$ is simple, this is the case precisely when $\mathfrak{g}$ is of type A_1, A_2, A_3, A_4 or B_2.

We also consider the cases where $\ell(\mathfrak{u}) = 5$, so, for $\mathfrak{g}$ simple, $\mathfrak{g}$ is of type A_5, B_3, C_3, D_4 or G_2. In these minimal cases where there are infinitely many B-orbits in $\mathfrak{u}$, we describe the irreducible components of $\mathcal{C}(\mathfrak{u})$ in § 4. We note that, in these cases, $\mathcal{C}(\mathfrak{u})$ is no longer equidimensional. In fact, we observe that $\mathcal{C}(\mathfrak{u})$ is *never* equidimensional when there are infinitely many B-orbits in $\mathfrak{u}$ (see Lemma 4.1). This demonstrates that the situation is considerably more subtle in the infinite orbit case and there does not appear to be an obvious parametrization of the irreducible components.

Our methods are also applicable to the case where $\mathfrak{u}$ is the Lie algebra of the unipotent radical of a parabolic subgroup P of G. There are examples of such situations where P acts with finitely many orbits on $\mathfrak{u}$ yet $\mathcal{C}(\mathfrak{u})$ is not equidimensional (see Remark 3.1).

For simplicity, we assume that $\operatorname{char} \Bbbk = 0$ (or at least that $\operatorname{char} \Bbbk$ is sufficiently large), though, with additional work, it is strongly expected that the results remain true in good characteristic.

Finally, we note that Keeton investigated irreducibility and normality of the commuting variety $\mathcal{C}(\mathfrak{b})$ of $\mathfrak{b}$ in [**6**]. For instance, he gave, for $\mathfrak{g}$ of given classical type A, B, C or D, upper and lower bounds on the rank of $\mathfrak{g}$ for the irreducibility (and normality) of $\mathcal{C}(\mathfrak{b})$.

2. Generalities about commuting varieties

For this section, we work in the following setting. Let P be a connected algebraic group over $\Bbbk$, and let U be a normal subgroup of P; we write $\mathfrak{p}$ and $\mathfrak{u}$ for the Lie algebras of P and U, respectively. The group P acts on $\mathfrak{p}$ and $\mathfrak{u}$ via the adjoint action. For $x \in \mathfrak{p}$ and any subgroup H of P, we denote the H-orbit of x in $\mathfrak{p}$ by $H \cdot x$, the centralizer of x in H by $C_H(x)$, and the centralizer of x in $\mathfrak{h} = \operatorname{Lie} H$ by $\mathfrak{c}_{\mathfrak{h}}(x)$.

Let P act diagonally on $\mathfrak{u} \times \mathfrak{u}$. The *commuting variety of* $\mathfrak{u}$ is the closed P-stable subvariety of $\mathfrak{u} \times \mathfrak{u}$, given by

$$\mathcal{C}(\mathfrak{u}) = \{(x, y) \in \mathfrak{u} \times \mathfrak{u} \mid [x, y] = 0\}.$$

We recall that the *modality* of U on $\mathfrak{u}$ is defined to be

$$\operatorname{mod}(U; \mathfrak{u}) = \max_{i \in \mathbb{Z}_{\geqslant 0}} (\dim \mathfrak{u}_i - i),$$

where $\mathfrak{u}_i = \{x \in \mathfrak{u} \mid \dim U \cdot x = i\}$.

Our first lemma gives an expression for the dimension of $\mathcal{C}(\mathfrak{u})$.

Lemma 2.1. *We have that* $\dim \mathcal{C}(\mathfrak{u}) = \dim U + \operatorname{mod}(U; \mathfrak{u})$.

Proof. Consider $\mathcal{C}(\mathfrak{u})_i = \mathcal{C}(\mathfrak{u}) \cap (\mathfrak{u}_i \times \mathfrak{u})$. Clearly, we have that $\dim \mathcal{C}(\mathfrak{u})_i = \dim \mathfrak{u}_i + (\dim U - i) = \dim U + (\dim \mathfrak{u}_i - i)$ and $\mathcal{C}(\mathfrak{u}) = \bigcup_{i \in \mathbb{Z}_{\geqslant 0}} \mathcal{C}(\mathfrak{u})_i$. Therefore,

$$\dim \mathcal{C}(\mathfrak{u}) = \max_{i \in \mathbb{Z}_{\geqslant 0}} \{\dim U + (\dim \mathfrak{u}_i - i)\} = \dim U + \operatorname{mod}(U; \mathfrak{u}).$$

□

For $e \in \mathfrak{u}$, we define

$$\mathcal{C}(e) = \overline{P \cdot (e, \mathfrak{c}_{\mathfrak{u}}(e))} \subseteq \mathcal{C}(\mathfrak{u})$$

to be the Zariski closure of the P-saturation of $(e, \mathfrak{c}_{\mathfrak{u}}(e))$ in $\mathcal{C}(\mathfrak{u})$. It is easy to see that $\mathcal{C}(e)$ is a closed irreducible P-stable subvariety of $\mathcal{C}(\mathfrak{u})$ of dimension

$$\dim \mathcal{C}(e) = \dim P \cdot e + \dim \mathfrak{c}_{\mathfrak{u}}(e) = \dim P - (\dim \mathfrak{c}_{\mathfrak{p}}(e) - \dim \mathfrak{c}_{\mathfrak{u}}(e)). \tag{2.1}$$

We define an action of $\mathrm{GL}_2(\Bbbk)$ on $\mathfrak{u} \times \mathfrak{u}$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot (x, y) = (\alpha x + \beta y, \gamma x + \delta y)$$

(see part (1) of the proof of [**9**, Proposition 2.1]). Since any pair of linear combinations of two commuting elements from $\mathfrak{u}$ gives again a pair of commuting elements from $\mathfrak{u}$, it follows that $\mathrm{GL}_2(\Bbbk)$ acts on $\mathcal{C}(\mathfrak{u})$ and, furthermore, since $\mathrm{GL}_2(\Bbbk)$ is connected, it must stabilize each irreducible component of $\mathcal{C}(\mathfrak{u})$. This proves the following lemma.

Lemma 2.2. *The action of* $\mathrm{GL}_2(\Bbbk)$ *on* $\mathcal{C}(\mathfrak{u})$ *preserves each irreducible component. In particular, each irreducible component is invariant under the involution*

$$\sigma\colon (x, y) \mapsto (y, x).$$

For the remainder of this section, apart from Remark 2.10, we assume that there are finitely many P-orbits in $\mathfrak{u}$, and we choose representatives $e_1, \dots, e_s$ of these orbits. We then have that

$$\mathcal{C}(\mathfrak{u}) = \mathcal{C}(e_1) \cup \dots \cup \mathcal{C}(e_s).$$

In particular, each irreducible component of $\mathcal{C}(\mathfrak{u})$ is of the form $\mathcal{C}(e_i)$ for some i.

We proceed with some elementary lemmas. We recall that under our assumption that P acts on $\mathfrak{u}$ with finitely many orbits, there exists a unique dense open P-orbit in $\mathfrak{u}$.

Lemma 2.3.

(i) *Let* $e, e' \in \mathfrak{u}$. *If* $\mathcal{C}(e) \subseteq \mathcal{C}(e')$, *then* $P \cdot e \subseteq \overline{P \cdot e'}$.

(ii) *If* $e \in \mathfrak{u}$ *is in the dense open* P*-orbit, then* $\mathcal{C}(e)$ *is an irreducible component of* $\mathcal{C}(\mathfrak{u})$.

Proof. Let $\pi_1 \colon \mathfrak{u} \times \mathfrak{u} \to \mathfrak{u}$ be the projection onto the first factor. Since $\mathcal{C}(e) \subseteq \mathcal{C}(e')$, we have that

$$\overline{P \cdot e} = \pi_1(\mathcal{C}(e)) \subseteq \pi_1(\mathcal{C}(e')) = \overline{P \cdot e'},$$

so (i) holds. Part (ii) follows from (i). □

The next lemma is used to show that certain $\mathcal{C}(e)$ are not irreducible components of $\mathcal{C}(\mathfrak{u})$.

Lemma 2.4. *Let* $e \in \mathfrak{u}$. *If* $\mathcal{C}(e)$ *is an irreducible component of* $\mathcal{C}(\mathfrak{u})$, *then* $\mathfrak{c}_{\mathfrak{u}}(e) \subseteq \overline{P \cdot e}$.

Proof. The argument of part (2) in the proof of [**9**, Proposition 2.1] also applies in our case; we repeat it here for the convenience of the reader. The projection $\pi_1 \colon \mathfrak{u} \times \mathfrak{u} \to \mathfrak{u}$ onto the first factor maps an irreducible component $\mathcal{C}(e)$ to $\overline{P \cdot e}$. Consequently, by Lemma 2.2, we have that

$$\mathfrak{c}_{\mathfrak{u}}(e) \subseteq (\pi_1 \circ \sigma)\mathcal{C}(e) = \overline{P \cdot e}.$$

□

We define

$$d = \min_{e \in \mathfrak{u}} \{\dim \mathfrak{c}_{\mathfrak{p}}(e) - \dim \mathfrak{c}_{\mathfrak{u}}(e)\},$$

so we have $\dim \mathcal{C}(\mathfrak{u}) = \dim P - d$, by (2.1). We say that $e \in \mathfrak{u}$ is *distinguished* for P if $\dim \mathfrak{c}_{\mathfrak{p}}(e) - \dim \mathfrak{c}_{\mathfrak{u}}(e) = d$. We assume that our representatives of the P-orbits in $\mathfrak{u}$ are chosen such that $e_1, \dots, e_r$ are the representatives of the distinguished orbits. The following lemma is immediate; we record it for ease of reference.

Lemma 2.5. $\mathcal{C}(e_1), \dots, \mathcal{C}(e_r)$ *are the irreducible components of* $\mathcal{C}(\mathfrak{u})$ *of maximal dimension.*

Assume from now on that there exists a complementary subalgebra $\mathfrak{l}$ of $\mathfrak{u}$ in $\mathfrak{p}$ and that U is unipotent. Let h be an element of the centre $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ such that $\mathfrak{p} = \bigoplus_{j \in \mathbb{Z}_{\geqslant 0}} \mathfrak{p}(j; h)$, where $\mathfrak{p}(j; h) = \{x \in \mathfrak{p} \mid [h, x] = jx\}$ and $\mathfrak{p}(1; h) \neq 0$; we call such an h *admissible*. Note that we have $\mathfrak{u} \subseteq \bigoplus_{j \in \mathbb{Z}_{\geqslant 1}} \mathfrak{p}(j; h)$. Since there are finitely many orbits of P in $\mathfrak{u}$, we see

https://doi.org/10.1017/S0013091513000746 Published online by Cambridge University Press

that there is a dense orbit of $C_P(h)$ in $\mathfrak{p}(1;h)$ and we let e be a representative of this orbit; we then say that e is *linked* to h. We define the irreducible P-stable subvariety

$$\mathcal{S}(h,e) = \overline{P\cdot(h,e)} \subseteq \mathfrak{p}\times\mathfrak{u}$$

of $\mathfrak{p}\times\mathfrak{u}$. We write $\mathfrak{c}_\mathfrak{p}(h,e) = \mathfrak{c}_\mathfrak{p}(h)\cap\mathfrak{c}_\mathfrak{p}(e)$ for the simultaneous centralizer of h and e in $\mathfrak{p}$.

Given a closed subvariety X of an affine space V, we write $\mathbb{K}(X)$ for the cone of X in V, as defined in [**7**, § II.4.2].

The following lemmas are analogues of results from [**9**, § 2]; the subsequent corollary is key in what follows.

Lemma 2.6. *Let h be admissible and let e be linked to h. Then,*

$$\mathbb{K}(\mathcal{S}(h,e)) \subseteq \mathcal{C}(\mathfrak{u})$$

and $\mathbb{K}(\mathcal{S}(h,e))$ is equidimensional of dimension $\dim P - \dim\mathfrak{c}_\mathfrak{p}(h,e)$. In particular, $\mathbb{K}(\mathcal{S}(h,e))$ lies in the union of some $\mathcal{C}(e_i)$ for which $\dim\mathcal{C}(e_i) \geqslant \dim P - \dim\mathfrak{c}_\mathfrak{p}(h,e)$.

Proof. We see that

$$\mathcal{S}(h,e) \subseteq \{(x,y)\in\mathfrak{p}\times\mathfrak{u} \mid [x,y]=y\}.$$

Therefore, by [**7**, § II.4.2, Theorem 2] and the definition of cones,

$$\mathbb{K}(\mathcal{S}(h,e)) \subseteq \mathbb{K}(\{(x,y)\in\mathfrak{p}\times\mathfrak{u} \mid [x,y]=y\}) = \{(x,y)\in\mathfrak{p}\times\mathfrak{u} \mid [x,y]=0\}.$$

We have $\mathcal{S}(h,e)\subseteq(h+\mathfrak{u})\times\mathfrak{u}$, and this implies that $\mathbb{K}(\mathcal{S}(h,e))\subseteq\mathfrak{u}\times\mathfrak{u}$. Hence,

$$\mathbb{K}(\mathcal{S}(h,e)) \subseteq (\mathfrak{u}\times\mathfrak{u})\cap\{(x,y)\in\mathfrak{p}\times\mathfrak{u} \mid [x,y]=0\} = \mathcal{C}(\mathfrak{u}).$$

By [**7**, § II.4.2, Theorem 2], we have that $\mathbb{K}(\mathcal{S}(h,e))$ is equidimensional. The final statement follows easily from the fact that the irreducible components of $\mathcal{C}(\mathfrak{u})$ are of the form $\mathcal{C}(e_i)$. □

Lemma 2.7. *Let $e\in\mathfrak{u}$ and suppose that there exists an admissible $\tilde{h}\in\mathfrak{z}(\mathfrak{l})$ with linked $\tilde{e}$, such that $[\tilde{h},e]=e$ and $[\mathfrak{c}_u(e),\tilde{h}]=\mathfrak{c}_\mathfrak{u}(e)$. Then, $(\mathfrak{c}_\mathfrak{u}(e),e)\subseteq\mathbb{K}(\mathcal{S}(\tilde{h},\tilde{e}))$.*

Proof. Let $H=C_P(\tilde{h})$. The H-orbit of $\tilde{e}$ is dense in $\mathfrak{p}(1;\tilde{h})$, and $(\tilde{h},H\cdot\tilde{e})\subseteq\mathcal{S}(\tilde{h},\tilde{e})$, so we obtain $(\tilde{h},\mathfrak{p}(1;\tilde{h}))\subseteq\mathcal{S}(\tilde{h},\tilde{e})$. Thus, $(\tilde{h},\Bbbk e)\subseteq\mathcal{S}(\tilde{h},\tilde{e})$, because $e\in\mathfrak{p}(1;\tilde{h})$. Consider the $C_U(e)$-orbit $C_U(e)\cdot\tilde{h}$ in $\tilde{h}+\mathfrak{c}_\mathfrak{u}(e)$. This is closed in $\tilde{h}+\mathfrak{c}_\mathfrak{u}(e)$, because $C_U(e)$ is unipotent. Since $[\mathfrak{c}_u(e),\tilde{h}]=\mathfrak{c}_\mathfrak{u}(e)$, we obtain that $C_U(e)\cdot\tilde{h}=\tilde{h}+\mathfrak{c}_u(e)$. Hence,

$$C_U(e)\cdot(\tilde{h},\Bbbk e) = (\tilde{h}+\mathfrak{c}_\mathfrak{u}(e),\Bbbk e)\subseteq\mathcal{S}(\tilde{h},\tilde{e}).$$

Taking cones, we get $\mathbb{K}(\tilde{h}+\mathfrak{c}_\mathfrak{u}(e),\Bbbk e)\subseteq\mathbb{K}(\mathcal{S}(\tilde{h},\tilde{e}))$, by [**7**, § II4.2. Theorem 2]. From the definition of cones we see that $\mathbb{K}(\tilde{h}+\mathfrak{c}_\mathfrak{u}(e),\Bbbk e)=(\mathfrak{c}_\mathfrak{u}(e),\Bbbk e)$, and the lemma follows. □

Corollary 2.8. *Let $e\in\mathfrak{u}$. Suppose that there exists an admissible $\tilde{h}\in\mathfrak{z}(\mathfrak{l})$ such that $[\tilde{h},e]=e$, but e is not linked to $\tilde{h}$. Then, $\mathcal{C}(e)$ is not an irreducible component of $\mathcal{C}(\mathfrak{u})$.*

https://doi.org/10.1017/S0013091513000746 Published online by Cambridge University Press

Proof. If $[\mathfrak{c}_u(e),\tilde{h}] = \mathfrak{c}_\mathfrak{u}(e)$, then we can apply Lemma 2.7 to deduce that $(\mathfrak{c}_\mathfrak{u}(e), e) \subseteq \mathbb{K}(\mathcal{S}(\tilde{h},\tilde{e}))$, where $\tilde{e}$ is linked to $\tilde{h}$. Then, by Lemma 2.6 we have that $\mathbb{K}(\mathcal{S}(\tilde{h},\tilde{e}))$ is contained in a union of $\mathcal{C}(e_i)$ of dimension at least $\dim P - \dim \mathfrak{c}_\mathfrak{p}(\tilde{h},\tilde{e})$. Since these $\mathcal{C}(e_i)$ are stable under P and σ (see Lemma 2.2), we see that $\mathcal{C}(e)$ is contained in their union. We note that the conditions $[\tilde{h}, e] = e$ and $[\mathfrak{c}_u(e),\tilde{h}] = \mathfrak{c}_\mathfrak{u}(e)$ imply that

$$\dim \mathfrak{c}_\mathfrak{p}(e) - \dim \mathfrak{c}_\mathfrak{u}(e) \geqslant \dim \mathfrak{c}_\mathfrak{p}(\tilde{h}, e) > \dim \mathfrak{c}_\mathfrak{p}(\tilde{h},\tilde{e}),$$

so

$$\dim \mathcal{C}(e) = \dim P - \dim \mathfrak{c}_p(e) + \dim \mathfrak{c}_u(e) < \dim P - \dim \mathfrak{c}_\mathfrak{p}(\tilde{h},\tilde{e}).$$

Thus, $\mathcal{C}(e)$ is not an irreducible component of $\mathcal{C}(\mathfrak{u})$.

If $[\mathfrak{c}_u(e),\tilde{h}] \neq \mathfrak{c}_\mathfrak{u}(e)$, then $\mathfrak{c}_\mathfrak{u}(e) \cap \mathfrak{p}(0;\tilde{h}) \neq \{0\}$. Therefore, $\mathfrak{c}_\mathfrak{u}(e) \not\subseteq \overline{P \cdot e} \subseteq \bigoplus_{j \geqslant 1} \mathfrak{p}(j;\tilde{h})$, so $\mathcal{C}(e)$ is not an irreducible component of $\mathcal{C}(\mathfrak{u})$, by Lemma 2.4. □

Corollary 2.8 yields the following strategy to determine the irreducible components of $\mathcal{C}(\mathfrak{u})$.

Strategy 2.9.

(1) For each $i = 1, \ldots, s$, check whether e_i is distinguished. If so, then $\mathcal{C}(e_i)$ is an irreducible component, by Lemma 2.5.

(2) Determine all the admissible $h \in \mathfrak{z}(\mathfrak{l})$. For each $i = 1, \ldots, s$, check whether e_i is in $\mathfrak{p}(1, h)$ for some admissible h such that e_i is not linked to h, so that $\mathcal{C}(e_i)$ is not an irreducible component of $\mathcal{C}(\mathfrak{u})$, by Corollary 2.8.

(3) For the remaining i not dealt with in steps (1) and (2), use ad hoc methods to determine whether $\mathcal{C}(e_i)$ is an irreducible component or not.

Remark 2.10. Although we made the assumption that P acts on $\mathfrak{u}$ with finitely many orbits above, the theory still applies with suitable adaptations when the P-orbits can be parametrized nicely, as explained below.

A *family of representatives of P-orbits in $\mathfrak{u}$ over an irreducible variety X* is given by a subset $e(X) = \{e(t) \mid t \in X\}$ of $\mathfrak{u}$ such that the map $t \mapsto e(t)$ is an isomorphism from X onto its image in $\mathfrak{u}$, and such that, for $t, t' \in X$ distinct, we have $P \cdot e(t) \neq P \cdot e(t')$ but $\dim P \cdot e(t) = \dim P \cdot e(t')$.

Suppose that the P-orbits in $\mathfrak{u}$ can be parametrized by a finite number of families $e_1(X_1), \ldots, e_s(X_s)$. All of the above theory then has a suitable adaption, when we replace the single orbits e_i by the families $e_i(X_i)$. For example, we can define irreducible varieties $\mathcal{C}(e_i(X_i))$, and the irreducible components of $\mathcal{C}(\mathfrak{u})$ are of this form. For the notion of a family $e(X)$ being linked to an admissible $h \in \mathfrak{z}(\mathfrak{l})$, we require that $[h, e(t)] = e(t)$, for all $t \in X$, and $P \cdot e(X)$ to be dense in $\mathfrak{p}(1; h)$, and the subsequent results have similar adaptations. Therefore, with this assumption on the action of P on $\mathfrak{u}$, there is a version of Strategy 2.9 to determine the irreducible components of $\mathcal{C}(\mathfrak{u})$. We note that this assumption does hold for the action of a Borel subgroup on the Lie algebra of its unipotent radical, as explained in [**3**, § 2].

3. The case of a finite number of B-orbits

This section is devoted to the proof of Theorem 1.1. So, in this section $P = B$ is a Borel subgroup of a simple algebraic group G, and U is the unipotent radical of B. Furthermore, we assume that B acts on $\mathfrak{u}$ with a finite number of orbits. As mentioned in § 1, this means that G is of type A_n, for $n \leqslant 4$, or of type B_2. We proceed on a case by case basis using Strategy 2.9 to determine the irreducible components of $\mathcal{C}(\mathfrak{u})$, and observe that we obtain the description as given in Theorem 1.1.

In each case we give a list of representatives of the B-orbits in $\mathfrak{u}$. We calculated these using an adaptation of the computer program explained in [**3**], which gives the same representatives as in [**2**, Table 2] and as previously calculated in [**5**]. The notation used for these representatives is as follows. We fix an enumeration $\{\beta_1, \dots, \beta_N\}$ of the roots of $\mathfrak{b}$ with respect to a maximal torus T of B, and for each β_i we fix a generator e_{β_i} for the corresponding root space. This enumeration of the roots is listed, where the roots are given as vectors with respect to the simple roots as labelled in [**1**, Planches I–IX]. Each of the representatives of the B-orbits in $\mathfrak{u}$ is of the form $\sum_{i \in I} e_{\beta_i}$, where $I \subseteq \{1, \dots, N\}$, and we represent this element as the coefficient vector with respect to the e_{β_i}.

We briefly explain the meaning of an admissible element h in the present setting. Such an h belongs to a maximal toral subalgebra of $\mathfrak{b}$, and $\mathfrak{q} = \bigoplus_{j \geqslant 0} \mathfrak{g}(j; h)$ is a parabolic subalgebra of $\mathfrak{g}$ such that $\bigoplus_{j>0} \mathfrak{g}(j; h) \subseteq \mathfrak{b} \subseteq \mathfrak{q}$. So, in this case, Corollary 2.8 states that if a representative e of a B-orbit in $\mathfrak{u}$ lies in $\mathfrak{q}(1; h) = \mathfrak{g}(1; h)$ for such a $\mathfrak{q}$, and e is not in the dense $C_B(h)$-orbit in $\mathfrak{q}(1; h)$, then $\mathcal{C}(e)$ is not an irreducible component of $\mathcal{C}(\mathfrak{u})$.

3.1. G is of type A_1

There is just one root of $\mathfrak{b}$ and there are two B-orbits in $\mathfrak{u}$: the regular and the zero orbit. Here, $\mathfrak{u}$ is abelian and $\mathcal{C}(\mathfrak{u}) = \mathfrak{u} \times \mathfrak{u}$ is irreducible and equal to $\mathcal{C}(e)$ where e lies in the regular orbit.

3.2. G is of type A_2

The roots of $\mathfrak{b}$ are given by

$$\beta_1 \colon 10, \qquad \beta_2 \colon 01, \qquad \beta_3 \colon 11.$$

There are five B-orbits in $\mathfrak{u}$ with representatives

$$e_1 \colon 110, \qquad e_2 \colon 100, \qquad e_3 \colon 010, \qquad e_4 \colon 001, \qquad e_5 \colon 000.$$

Apart from e_1, each of the e_i lies in $\mathfrak{b}(1; h)$ for some admissible h, for which e_i is not linked to h. Therefore, using Strategy 2.9, we get that $\mathcal{C}(\mathfrak{u}) = \mathcal{C}(e_1)$ is irreducible.

3.3. G is of type A_3

The roots of $\mathfrak{b}$ are given by

$$\beta_1 \colon 100, \qquad \beta_2 \colon 010, \qquad \beta_3 \colon 001, \qquad \beta_4 \colon 110, \qquad \beta_5 \colon 011, \qquad \beta_6 \colon 111.$$

There are 16 B-orbits in $\mathfrak{u}$ with representatives

$$\begin{array}{llll} e_1\colon 111000, & e_2\colon 110000, & e_3\colon 101010, & e_4\colon 101000,\\ e_5\colon 100010, & e_6\colon 100000, & e_7\colon 011000, & e_8\colon 010001,\\ e_9\colon 010000, & e_{10}\colon 001100, & e_{11}\colon 001000, & e_{12}\colon 000110,\\ e_{13}\colon 000100, & e_{14}\colon 000010, & e_{15}\colon 000001, & e_{16}\colon 000000. \end{array}$$

All of the e_i except for e_1, e_3, e_8 are in $\mathfrak{b}(1;h)$ for some admissible h not linked to e_i. We see that e_1 and e_3 are distinguished, so $\mathcal{C}(e_1)$ and $\mathcal{C}(e_3)$ are irreducible components. Below, we verify by direct calculation that $\mathcal{C}(e_8)$ is not an irreducible component.

Consider the pairs of strictly upper triangular matrices $(x(\alpha,\lambda), y(\alpha,\lambda,a,b,c))$ for $\alpha,\lambda \in \Bbbk^\times$, $a,b,c \in \Bbbk$, with entries above the diagonal given by

$$\begin{pmatrix} \lambda & 0 & 1 & \lambda a & b & c\\ & 1 & 0\ , & & a & \alpha b\\ & & \alpha\lambda & & & \alpha\lambda a \end{pmatrix}.$$

It is straightforward to check that $(x(\alpha,\lambda), y(\alpha,\lambda,a,b,c)) \in \mathcal{C}(\mathfrak{u})$ and that $x(\alpha,\lambda) \in B\cdot e_1$. Therefore, $(x(\alpha,\lambda), y(\alpha,\lambda,a,b,c)) \in \mathcal{C}(e_1)$ for all $\alpha,\lambda \in \Bbbk^\times$ and $a,b,c \in \Bbbk$. Letting $\lambda \to 0$, we see that $(x(\alpha,0), y(\alpha,0,a,b,c)) \in \mathcal{C}(e_1)$ for all $\alpha \in \Bbbk^\times$. We have that $x(\alpha,0) = e_8$ and via a calculation we see that $\{y(\alpha,0,a,b,c) \mid \alpha \in \Bbbk^\times,\ a,b,c \in \Bbbk\}$ is a dense subset of

$$\mathfrak{c}_{\mathfrak{u}}(e_8) = \left\{ \begin{matrix} 0 & a & b\\ & c & d\\ & & 0 \end{matrix} \;\middle|\; a,b,c,d \in \Bbbk \right\}.$$

Therefore, $(e_8, \mathfrak{c}_{\mathfrak{u}}(e_8)) \subseteq \mathcal{C}(e_1)$, and hence $\mathcal{C}(e_8) \subseteq \mathcal{C}(e_1)$.

Putting this all together, we get that $\mathcal{C}(\mathfrak{u}) = \mathcal{C}(e_1) \cup \mathcal{C}(e_3)$.

3.4. G is of type A_4

The roots of $\mathfrak{b}$ are given by

$$\begin{array}{lllll} \beta_1\colon 1000, & \beta_2\colon 0100, & \beta_3\colon 0010, & \beta_4\colon 0001, & \beta_5\colon 1100,\\ \beta_6\colon 0110, & \beta_7\colon 0011, & \beta_8\colon 1110, & \beta_9\colon 0111, & \beta_{10}\colon 1111. \end{array}$$

There are 61 B-orbits in $\mathfrak{u}$ with representatives

$$\begin{array}{llll} e_1\colon 1111000000, & e_2\colon 1110000000, & e_3\colon 1101000000, & e_4\colon 1100001000,\\ e_5\colon 1100001000, & e_6\colon 1100000000, & e_7\colon 1011010000, & e_8\colon 1011000000,\\ e_9\colon 1010010010, & e_{10}\colon 1010010000, & e_{11}\colon 1010000010, & e_{12}\colon 1010000000,\\ e_{13}\colon 1001010000, & e_{14}\colon 1001000010, & e_{15}\colon 1001000000, & e_{16}\colon 1000011000,\\ e_{17}\colon 1000010000, & e_{18}\colon 1000001010, & e_{19}\colon 1000001000, & e_{20}\colon 1000000010,\\ e_{21}\colon 1000000000, & e_{22}\colon 0111000000, & e_{23}\colon 0110000001, & e_{24}\colon 0110000000, \end{array}$$

https://doi.org/10.1017/S0013091513000746 Published online by Cambridge University Press

e_{25}: 0101000110,	e_{26}: 0101000100,	e_{27}: 0101000010,	e_{28}: 0101000000,
e_{29}: 0100001100,	e_{30}: 0100001000,	e_{31}: 0100000100,	e_{32}: 0100000001,
e_{33}: 0100000000,	e_{34}: 0011100000,	e_{35}: 0011000000,	e_{36}: 0010100010,
e_{37}: 0010100000,	e_{38}: 0010000010,	e_{39}: 0010000001,	e_{40}: 0010000000,
e_{41}: 0001110000,	e_{42}: 0001100100,	e_{43}: 0001100000,	e_{44}: 0001010000,
e_{45}: 0001000100,	e_{46}: 0001000000,	e_{47}: 0000111000,	e_{48}: 0000110000,
e_{49}: 0000101000,	e_{50}: 0000100010,	e_{51}: 0000100000,	e_{52}: 0000011000,
e_{53}: 0000010001,	e_{54}: 0000010000,	e_{55}: 0000001100,	e_{56}: 0000001000,
e_{57}: 0000000110,	e_{58}: 0000000100,	e_{59}: 0000000010,	e_{60}: 0000000001,
e_{61}: 0000000000.			

Except for e_1, e_3, e_7, e_9, e_{14}, e_{23} and e_{25}, we can check that each e_i lies in $\mathfrak{b}(1;h)$ for some admissible h not linked to e_i. The representatives e_1, e_3, e_7, e_9 and e_{25} are distinguished, so the corresponding $\mathcal{C}(e_i)$ are irreducible components of $\mathcal{C}(\mathfrak{u})$. Below, we verify by direct calculation that $\mathcal{C}(e_{23})$ and $\mathcal{C}(e_{14})$ are not irreducible components.

We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & a & c & e & f \\ & \alpha\lambda & \lambda & 1 & & \alpha\lambda a & \lambda a & a+e \\ & & 0 & 0\,, & & & 0 & \alpha^{-1}(a-b) \\ & & & 1 & & & & b \end{pmatrix} \in \mathcal{C}(e_3)$$

for all $\alpha, \lambda \in \Bbbk^\times$ and $a, b, c, e, f \in \Bbbk$, and

$$\mathfrak{c}_{\mathfrak{u}}(e_{14}) = \left\{ \begin{matrix} a & c & e & f \\ & 0 & 0 & a+e \\ & & 0 & d \\ & & & b \end{matrix} \;\middle|\; a, b, c, d, e, f \in \Bbbk \right\}.$$

Letting $\lambda \to 0$, we see that $\mathcal{C}(e_{14}) \subseteq \mathcal{C}(e_3)$.

Similarly, we have

$$\begin{pmatrix} \lambda & 0 & 0 & 1 & \lambda a & \lambda b & c & e \\ & 1 & 0 & 0 & & a & b & \alpha c \\ & & 1 & 0\,, & & & a & \lambda\alpha b \\ & & & \alpha\lambda & & & & \lambda\alpha a \end{pmatrix} \in \mathcal{C}(e_1)$$

for all $\alpha, \lambda \in \Bbbk^\times$ and $a, b, c, e \in \Bbbk$, and

$$\mathfrak{c}_{\mathfrak{u}}(e_{23}) = \left\{ \begin{matrix} 0 & 0 & c & e \\ & a & b & d \\ & & a & 0 \\ & & & 0 \end{matrix} \;\middle|\; a, b, c, d, e \in \Bbbk \right\}.$$

Letting $\lambda \to 0$, we see that $\mathcal{C}(e_{23}) \subseteq \mathcal{C}(e_1)$.

Combining the above, the decomposition of $\mathcal{C}(\mathfrak{u})$ into irreducible components is given by $\mathcal{C}(\mathfrak{u}) = \mathcal{C}(e_1) \cup \mathcal{C}(e_3) \cup \mathcal{C}(e_7) \cup \mathcal{C}(e_9) \cup \mathcal{C}(e_{25})$.

https://doi.org/10.1017/S0013091513000746 Published online by Cambridge University Press

3.5. G is of type B_2

The roots of $\mathfrak{b}$ are given by

$$\beta_1 : 10, \qquad \beta_2 : 01, \qquad \beta_3 : 11, \qquad \beta_4 : 12.$$

There are seven B-orbits in $\mathfrak{u}$, with representatives

$$\begin{array}{llll} e_1 : 1100, & e_2 : 1001, & e_3 : 1000, & e_4 : 0100, \\ e_5 : 0010, & e_6 : 0001, & e_7 : 0000. & \end{array}$$

The two orbit representatives e_1 and e_2 are distinguished. Each of the other orbit representatives e_i lies in $\mathfrak{b}(1; h)$ for some h that is not linked to e_i. So, using Strategy 2.9, we have that $\mathcal{C}(\mathfrak{u}) = \mathcal{C}(e_1) \cup \mathcal{C}(e_2)$.

Remark 3.1. All of the material in § 2 is valid when P is a parabolic subgroup of a reductive algebraic group G and U is the unipotent radical of P. We note, however, that in contrast to Theorem 1.1, $\mathcal{C}(\mathfrak{u})$ is not equidimensional in general when there are finitely many P-orbits in $\mathfrak{u}$. In fact, the difference in the dimensions of irreducible components can be arbitrarily large, as shown in the example below.

Let $m \geqslant 2$ be an integer and let P be the parabolic subgroup of $\mathrm{GL}_{m+2}(\Bbbk)$, which is the stabilizer of a flag of subspaces $\Bbbk \subseteq \Bbbk^2 \subseteq \Bbbk^{m+2}$ in $\Bbbk^{m+2}$. Then, P admits only a finite number of orbits on the Lie algebra of its unipotent radical $\mathfrak{u}$ (see [**4**]). However, one can calculate that $\mathcal{C}(\mathfrak{u})$ has two irreducible components of dimensions $4m + 1$ and $3m + 2$.

4. The case of an infinite number of B-orbits

We continue using the notation from the last section, but we remove the assumption that B acts on $\mathfrak{u}$ with a finite number of orbits. We also use the notation for families of B-orbits $e(X)$ in $\mathfrak{u}$, as explained in Remark 2.10.

We begin by observing that the analogue of Theorem 1.1 does not hold when there are infinitely many B-orbits in $\mathfrak{u}$.

Lemma 4.1. *Suppose that B acts on $\mathfrak{u}$ with an infinite number of orbits. Then $\mathcal{C}(\mathfrak{u})$ is not equidimensional.*

Proof. Let $e \in \mathfrak{u}$ be in the regular nilpotent orbit. By Lemma 2.3 (ii), we then have that $\mathcal{C}(e)$ is an irreducible component of $\mathcal{C}(\mathfrak{u})$ of dimension $\dim B$.

Let G be one of the minimal cases where B acts on $\mathfrak{u}$ with an infinite number of orbits. We can then see from the calculations below that there is a family of B-orbits $e(X)$ parametrized by some irreducible variety X of positive dimension such that $\mathfrak{c}_{\mathfrak{b}}(e(t)) = \mathfrak{c}_{\mathfrak{u}}(e(t))$ for all $t \in X$; in fact, we can take $X = \Bbbk^{\times}$. For any G we can find a Levi subgroup H of G containing T, which is one of the groups for the minimal infinite cases. Let $e_H(X)$ be a family of $B \cap H$-orbits as above. Then define $e(X)$ to be the sum of $e_H(X)$ and the simple root vectors e_α with $e_\alpha \notin \mathfrak{h}$. It is then straightforward to check that the elements of $e(X)$ are pairwise in different B-orbits (if they were in the same B-orbit, then they would have to be in the same $B \cap H$-orbit) and $\mathfrak{c}_{\mathfrak{b}}(e(t)) = \mathfrak{c}_{\mathfrak{u}}(e(t))$

https://doi.org/10.1017/S0013091513000746 Published online by Cambridge University Press

for all $t \in X$. Thus, there is a family of B-orbits $e(X)$ parametrized by some irreducible variety X of positive dimension such that $\mathfrak{c}_\mathfrak{b}(e(t)) = \mathfrak{c}_\mathfrak{u}(e(t))$ for all $t \in X$.

We then have that $\dim \mathcal{C}(e(X)) = \dim B + \dim X$. Thus, there must be an irreducible component of $\mathcal{C}(\mathfrak{u})$ of dimension strictly larger than $\dim B$. $\square$

We move on to describe the irreducible components of $\mathcal{C}(\mathfrak{u})$ for the cases where $\mathfrak{g}$ is of type A_5, B_3, C_3, D_4 and G_2. These are the minimal cases in which there is an infinite number of B-orbits in $\mathfrak{u}$. We have determined the irreducible components using the adaptation of Strategy 2.9, as discussed in Remark 2.10. The calculations are very similar in spirit to those discussed in § 3, so we omit the details. We use a parametrization of orbits given by the programme from [**3**]; most of this information can also be extracted from [**2**].

From the descriptions given below, we see that the structure of $\mathcal{C}(\mathfrak{u})$ is already rather complicated, and there does not appear to be a nice way to parametrize the irreducible components already in these minimal infinite cases. We have investigated the possibility of doing this in terms of a suitable notion of *distinguished families* of B-orbits in $\mathfrak{u}$. However, the natural candidates do not give the irreducible components as desired.

4.1. *G* is of type A_5

The roots of $\mathfrak{b}$ are given by

$$\begin{array}{lllll} \beta_1: 10000, & \beta_2: 01000, & \beta_3: 00100, & \beta_4: 00010, & \beta_5: 00001, \\ \beta_6: 11000, & \beta_7: 01100, & \beta_8: 00110, & \beta_9: 00011, & \beta_{10}: 11100, \\ \beta_{11}: 01110, & \beta_{12}: 00111, & \beta_{13}: 11110, & \beta_{14}: 01111, & \beta_{15}: 11111. \end{array}$$

The B-orbits in $\mathfrak{u}$ are given by a one-dimensional family $e_{29}(\Bbbk^\times)$, given by $t \mapsto 1010101010t0000$, and 274 other orbits. We have that $\mathcal{C}(e_{29}(\Bbbk^\times))$ is an irreducible component of dimension 21 and there are 12 irreducible components of dimension 20 given by $\mathcal{C}(e_i)$, where e_i is one of the following:

$$\begin{array}{lll} e_1: 111110000000000, & e_3: 111010001000000, & e_7: 110110010000000, \\ e_8: 110110000001000, & e_{10}: 110100010001000, & e_{23}: 101110100000000, \\ e_{25}: 101100100000010, & e_{53}: 100100000011010, & e_{94}: 011010001000100, \\ e_{103}: 010110010100000, & e_{107}: 010100010101000, & e_{119}: 010010000101100. \end{array}$$

4.2. *G* is of type B_3

The roots of $\mathfrak{b}$ are given by

$$\begin{array}{lllll} \beta_1: 100, & \beta_2: 010, & \beta_3: 001, & \beta_4: 110, & \beta_5: 011, \\ \beta_6: 111, & \beta_7: 012, & \beta_8: 112, & \beta_9: 122. & \end{array}$$

The B-orbits in $\mathfrak{u}$ are given by a one-dimensional family $e_{12}(\Bbbk^\times)$, given by $t \mapsto 0100011t0$, and 34 other orbits. We have that $\mathcal{C}(e_{12}(\Bbbk^\times))$ is an irreducible component of dimension 13

https://doi.org/10.1017/S0013091513000746 Published online by Cambridge University Press

and there are four irreducible components of dimension 12 given by $\mathcal{C}(e_i)$, where e_i is one of the following:

$$e_1\colon 111000000,\qquad e_2\colon 110000100,\qquad e_4\colon 101010000,\qquad e_5\colon 101000001.$$

4.3. *G* is of type C_3

The roots of $\mathfrak{b}$ are given by

$$\beta_1\colon 100,\quad \beta_2\colon 010,\quad \beta_3\colon 001,\quad \beta_4\colon 110,\quad \beta_5\colon 011,$$
$$\beta_6\colon 111,\quad \beta_7\colon 021,\quad \beta_8\colon 121,\quad \beta_9\colon 221.$$

The B-orbits in $\mathfrak{u}$ are given by a one-dimensional family $e_4(\Bbbk^\times)$, given by $t \mapsto 101010t00$, and 34 other orbits. We have that $\mathcal{C}(e_4(\Bbbk^\times))$ is an irreducible component of dimension 13 and there are three irreducible components of dimension 12 given by $\mathcal{C}(e_i)$, where e_i is one of the following:

$$e_1\colon 111000000,\qquad e_2\colon 110000100,\qquad e_{12}\colon 011000001.$$

4.4. *G* is of type D_4

The roots of $\mathfrak{b}$ are given by

$$\beta_1\colon 1000,\quad \beta_2\colon 0100,\quad \beta_3\colon 0010,\quad \beta_4\colon 0001,\quad \beta_5\colon 1100,\quad \beta_6\colon 0110,$$
$$\beta_7\colon 0101,\quad \beta_8\colon 1110,\quad \beta_9\colon 1101,\quad \beta_{10}\colon 0111,\quad \beta_{11}\colon 1111,\quad \beta_{12}\colon 1211.$$

The B-orbits in $\mathfrak{u}$ are given by two one-dimensional families $e_8(\Bbbk^\times)$, given by $t \mapsto 101101t00000$, and $e_{37}(\Bbbk^\times)$, given by $t \mapsto 0100000111t0$, and 98 other orbits. We have that $\mathcal{C}(e_8(\Bbbk^\times))$ and $\mathcal{C}(e_{37}(\Bbbk^\times))$ are irreducible components of dimension 17 and there are four irreducible components of dimension 16 given by $\mathcal{C}(e_i)$, where e_i is one of the following:

$$e_1\colon 111100000000,\qquad e_2\colon 111000000100,$$
$$e_4\colon 110100000100,\qquad e_{31}\colon 011100001000.$$

4.5. *G* is of type G_2

The roots of $\mathfrak{b}$ are given by

$$\beta_1\colon 10,\quad \beta_2\colon 01,\quad \beta_3\colon 11,\quad \beta_4\colon 21,\quad \beta_5\colon 31,\quad \beta_6\colon 32.$$

The B-orbits in $\mathfrak{u}$ are given by a one-dimensional family $e_4(\Bbbk^\times)$, given by $t \mapsto 0101t0$, and 11 other orbits. We have that $\mathcal{C}(e_4(\Bbbk^\times))$ is an irreducible component of dimension 9 and there are two irreducible components of dimension 8 given by $\mathcal{C}(e_i)$, where e_i is one of the following:

$$e_1\colon 110000,\qquad e_2\colon 100001.$$

Acknowledgements. Part of the research for this paper was carried out while both authors were staying at the Mathematical Research Institute Oberwolfach supported by the 'Research in Pairs' programme.

https://doi.org/10.1017/S0013091513000746 Published online by Cambridge University Press

References

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chapters 4–6 (Hermann, Paris, 1975).
2. H. Bürgstein and W. H. Hesselink, Algorithmic orbit classification for some Borel group actions, *Compositio Math.* **61**(1) (1987), 3–41.
3. S. M. Goodwin and G. Röhrle, Calculating conjugacy classes in Sylow p-subgroups of finite Chevalley groups, *J. Alg.* **321**(11) (2009), 3321–3334.
4. L. Hille and G. Röhrle, A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical, *Transform. Groups* **4**(1) (1998), 317–337.
5. V. V. Kashin, Orbits of an adjoint and co-adjoint action of Borel subgroups of a semisimple algebraic group, in *Problems in group theory and homological algebra*, pp. 141–158 (Yaroslavl State University, 1990).
6. A. G. Keeton, Commuting varieties associated with symmetric pairs, PhD thesis, University of California (1995).
7. H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics: D, Volume 1 (Vieweg, Braunschweig, 1984).
8. P. Levy, Commuting varieties of Lie algebras over fields of prime characteristic, *J. Alg.* **250**(2) (2002), 473–484.
9. A. Premet, Nilpotent commuting varieties of reductive Lie algebras, *Invent. Math.* **154**(3) (2003), 653–683.
10. R. W. Richardson, Commuting varieties of semisimple Lie algebras and algebraic groups, *Compositio Math.* **38**(3) (1979), 311–327.