12
Renormalization and the continuum limit

Regarding the lattice merely as an ultraviolet cutoff, ultimately we must consider the continuum limit. As when removing any regulator, observable quantities should approach their physical values. On the lattice, however, it is often convenient to measure dimensionful quantities, such as masses, in lattice units. For example, the mass of the first excitation in units of the spacing a gives the correlation length

$$\xi = (ma)^{-1}. \tag{12.1}$$

In the continuum limit m should remain finite while our yardstick of length a becomes singular. Thus we are interested in obtaining a divergent correlation length. In statistical mechanics language, this is the expected behavior at a second order phase transition. For a continuum limit of a field theory defined with a lattice cutoff, we should find the points in the coupling parameter space where the corresponding statistical model exhibits critical behavior. The needed critical phenomenon does not occur in the strong coupling region of lattice gauge theory. From eq. (10.23) we see that the correlation length goes to zero as β becomes small. To take a continuum limit we must search for second order phase transitions at intermediate and small coupling.

As soon as we begin discussing the removal of an ultraviolet cutoff, we must address the question of renormalization. Indeed, quantum field theory is notorious for the plethora of divergences which must be removed in calculations of physical observables. The bare charges and masses which appear in the Lagrangian are in general not well defined and need renormalization. The bare couplings acquire an implicit cutoff dependence chosen in such a manner that physical quantities have a finite limit when the cutoff is removed. For a well-defined renormalizable theory, this procedure should yield unique finite limits for all observables.

In general there are many possible renormalization schemes. In quantum electrodynamics one usually fixes the physical electron mass and the coefficient of the long-range Coulomb force. These parameters of the continuum theory determine the bare mass and charge when a cutoff is

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in place. In a confining theory, such as we want for the strong interactions, the choice is less obvious. One popular selection for non-perturbative studies of the pure gauge theory without fermions is the coefficient K of the Wilson loop area law, which equals the coefficient of the long-distance linear potential between external sources with quark quantum numbers. Another possible choice would be the mass of some physical bound state, such as the lightest glueball.

All of the quantities mentioned in the previous paragraph are defined in terms of long-range effects. This is clear for the long-distance potentials, but it also applies to a particle mass as this parameter determines how the particle propagates over extended distances. It is, however, often convenient to consider physical observables involving only finite length scales. For example, in traditional perturbative renormalization-group discussions one studies vertex functions in momentum space with all legs off-shell at some arbitrarily selected momentum scale μ. Alternatively, one might be interested in some interparticle force at a finite range r. By varying these parameters μ or r, one studies the interrelationships of physics on different length scales.

For now we will restrict our discussion to a theory, such as quarkless gauge theory, which has only one bare dimensionless coupling parameter, g_0. A general physical observable H is a function of the bare coupling as well as the cutoff scale of length a and the scale r on which H is defined

$$H = H(r, a, g_0(a)). \tag{12.2}$$

Here we have explicitly shown the cutoff dependence of the bare coupling $g_0(a)$. The precise form of this dependence depends on the details of the renormalization scheme. For simplicity, we assume that H is dimensionless; if it were not we could simply multiply by the appropriate power of r to make it so. For example, from an interparticle force $F(r)$ construct $H = r^2F$.

As a becomes small and we approach the continuum limit, H should lose cutoff dependence. It should do this while retaining a non-trivial dependence on the scale r. This can only occur at special values of g_0 where critical behavior involving vastly different length scales occurs. To see this more explicitly, consider changing the cutoff by a factor of two. For small cutoff H should not change appreciably if g_0 is appropriately adjusted

$$H(r, \tfrac{1}{2}a, g_0(\tfrac{1}{2}a)) = H(r, a, g_0(a)) + O(a^2). \tag{12.3}$$

In general there are two classes of dimensional parameters which set the scale for the order-a^2 corrections in this equation. First, of course, is the scale r used to define H. In addition, however, we must consider the long-range physical parameters characterizing the continuum theory. In particular, regardless of how large r is, we must expect corrections of order a^2m^2 where m is some typical mass in the physical particle spectrum. The lattice theory should only be expected to approximate continuum physics when the lattice spacing is smaller than both the scale under consideration and the characteristic size of a strongly interacting particle. Of course, if we adopt the renormalization scheme of holding $H(r, a, g_0(a))$ fixed at the given scale, then by definition there are no corrections to eq. (12.3). However, we will now consider varying r in order to compare physics on different length scales and therefore we should remember that these corrections are in principle there.

Since H is dimensionless, we can scale a factor of two from both r and a in eq. (12.3) to give

$$H(2r, a, g_0(\tfrac{1}{2}a)) = H(r, a, g_0(a)) + O(a^2). \tag{12.4}$$

This equation shows the correlation between the bare coupling for two values of the cutoff and the measured observable at two different length scales. The process leading to this result is now iterated to give the pivotal relation

$$H(2r, a, g_0(a/2^{n+1})) = H(r, a, g_0(a/2^n)) + O(a^2). \tag{12.5}$$

This formula allows us to study the renormalization of g_0 as follows. Assume that for some fixed values of r and a we know the functional dependence of $H(r, a, g_0)$ and $H(2r, a, g_0)$ on the bare coupling. Suppose further that at scale r and in the continuum limit H has the value H_0:

$$\lim_{a\to 0} H(r, a, g_0(a)) = H_0. \tag{12.6}$$

Consider a graph of $H(r, a, g_0)$ as a function of g_0. Neglecting finite cutoff corrections, we find $g_0(a)$ as the value of g_0 where H passes through H_0. Now from $H(2r, a, g_0)$ we find the bare coupling at half this cutoff using eq. (12.4)

$$H(2r, a, g_0(\tfrac{1}{2}a)) = H_0. \tag{12.7}$$

Once we know $g_0(\frac{1}{2}a)$, we define H_1 by

$$H_1 = H(r, a, g_0(\tfrac{1}{2}a)). \tag{12.8}$$

Equation (12.5) now tells us how to find $g_0(\frac{1}{4}a)$:

$$H(2r, a, g_0(\tfrac{1}{4}a)) = H_1. \tag{12.9}$$

Iterating gives

$$H_n = H(r, a, g_0(a/2^n)), \tag{12.10}$$

$$H(2r, a, g_0(a/2^{n+1})) = H_n. \tag{12.11}$$

Graphically, this procedure generates a 'staircase' as illustrated in figure 12.1. This picture is drawn for an asymptotically free theory where $g_0(0) = 0$.

In figure 12.2 we sketch a situation where the functions $H(r, a, g_0)$ and

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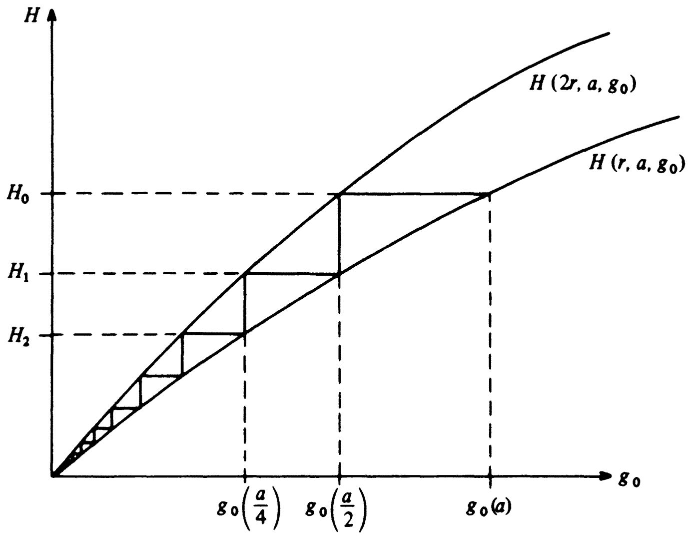

Fig. 12.1. The staircase construction for an asymptotically free theory (Creutz, 1981*a*).

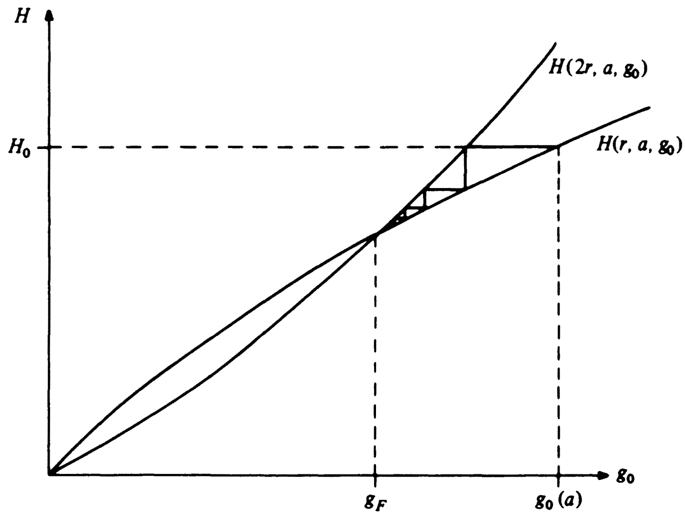

Fig. 12.2. An example of a non-trivial fixed point (Creutz, 1981*a*).

$H(2r, a, g_0)$ cross each other at a non-vanishing coupling. Here the staircase asymptotically approaches this crossing point. At this renormalization-group fixed point g_F, physics becomes scale invariant

$$H(r, a, g_F) = H(2r, a, g_F). \tag{12.12}$$

Note that g_F can be approached either from stronger or weaker coupling.

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As the bare charge at some very small cutoff passes through g_F, the corresponding initial value H_0 drastically changes as we go from a staircase on one side of g_F to the other. Long-distance physics depends non-analytically on the bare coupling and we have a phase transition in the corresponding statistical mechanical system. The critical exponents of the transition are related to the relative slopes of $H(r, a, g_0)$ and $H(2r, a, g_0)$ near the critical point. The absolute slopes of these functions depend on the initial value of a/r used in their definition.

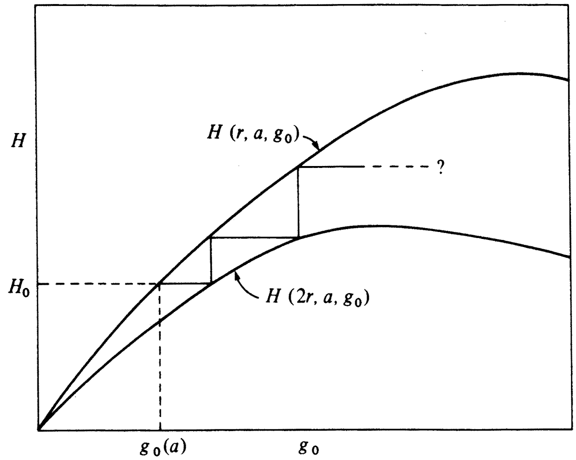

Fig. 12.3. A theory without a non-trivial continuum limit.

The above examples represent conventional ultraviolet attractive fixed points. One could also imagine a situation where at some point g_F eq. (12.12) again holds but

$$(|(\mathrm{d}/\mathrm{d}g)\,H(r,a,g)|-|(\mathrm{d}/\mathrm{d}g)\,H(2r,a,g)|)\,|_{g=g_F} > 0. \qquad (12.13)$$

In this case the staircase construction leads one away from g_F. A continuum limit at such an ultraviolet repulsive fixed point is at best possible only if g_0 is exactly g_F.

Another possible situation is that at some stage in the renormalization process eq. (12.11) has no solution. Such a case is illustrated in figure 12.3. At a certain point in the construction it is no longer possible to maintain H at its desired physical value regardless of what goes to the bare charge. Several authors (Kogut and Wilson, 1974; Baker and Kincaid, 1979; Bender *et al.*, 1981; Freedman, Smolensky and Weingarten, 1982) have suggested that this may be the case for four-dimensional ϕ^4 theory, which may therefore not have a non-trivial continuum limit.

The flows of the coupling illustrated in the above discussion represent a simplified version of the multidimensional flows discussed by Wilson (1971*a*, *b*). In particle physics we are usually interested in the continuum limit of a theory specified by a few renormalizable coupling constants. In statistical mechanics, however, the above rescaling procedure is often discussed in reverse. Starting with a simple model on a lattice of small spacing, one attempts to find an effective theory on a larger lattice spacing but with equivalent physics on long length scales. In general an increasing number of parameters is needed as such a process is iterated.

The above discussion of the dependence of the bare coupling on cutoff is often formulated in differential form. If our renormalization prescription is to set H at scale r to H_0 for all values of the cutoff a, then we have the equation

$$a(\mathrm{d}/\mathrm{d}a)\,H(r,a,g_0(a)) = 0$$
$$= a(\partial/\partial a)\,H(r,a,g_0)+\gamma(g_0)\,(\partial/\partial g_0)H(r,a,g_0). \quad (12.14)$$

This is a form of the renormalization group equation (Gell-Mann and Low, 1954; Petermann and Stueckelberg, 1953). The renormalization group function $\gamma(g_0)$ is defined

$$\gamma(g_0) = a(\mathrm{d}/\mathrm{d}a)\,g_0(a). \quad (12.15)$$

Knowledge of $\gamma(g_0)$ determines the cutoff dependence of g_0 up to an integration constant. Notice that once a renormalization prescription has been selected, then g_0 and a are no longer independent variables. We can freely trade off cutoff dependences for dependence on g_0 and vice versa. This interplay between dimensionful and dimensionless parameters forms the basis of the phenomenon of dimensional transmutation, the subject of the next chapter.

Zeros in the renormalization group function $\gamma(g_0)$ correspond to the scale-invariant crossing points discussed earlier in this chapter. As the lattice spacing becomes small, the bare coupling approaches a fixed point

$$\lim_{a\to 0} g_0(a) = g_F. \quad (12.16)$$

Equation (12.15) then implies

$$\gamma(g_F) = 0. \quad (12.17)$$

Note furthermore that for g_0 near g_F we have

$$\gamma(g_0) = a(\mathrm{d}/\mathrm{d}a)\,g_0(a) \begin{Bmatrix} > 0, g_0 > g_F \\ < 0, g_0 < g_F. \end{Bmatrix} \quad (12.18)$$

Thus for an ultraviolet attractive fixed point, such as being considered here, the first non-vanishing derivative of γ must be positive.

In general the precise form of γ will vary with the details of the renormalization scheme. In particular, γ depends on the choice of physical observable H and the scale r on which it is measured. Nevertheless, those zeros of γ representing ultraviolet-attractive fixed points must be universal if the continuum limit is to be unique. The scheme dependence of the renormalization group function appears already in the strong coupling limit. First consider the renormalization prescription of holding the string tension fixed. Application of eq. (10.9) to $SU(3)$ gives the strong coupling expression

$$K = a^{-2}\log(3g_0^2(a)) + O(g_0^{-2}). \tag{12.19}$$

If K is independent of a, a derivative gives

$$0 = a(\mathrm{d}/\mathrm{d}a)\,K = -2K + (2/(a^2 g_0))\,\gamma(g_0) + \dots \tag{12.20}$$

Using eq. (12.19) to eliminate a^2 in favor of g_0, we have

$$\gamma(g_0) = g_0 \log(3g_0^2) + \dots \tag{12.21}$$

Note that this does not vanish in strong coupling; therefore, one must look elsewhere for a continuum theory.

Now suppose that, instead of using the string tension, we renormalize by holding the mass gap fixed. Equation (10.23) gives

$$m_g = a^{-1} 4 \log(3g_0^2) + O(g_0^{-2}). \tag{12.22}$$

Proceeding in analogy with eqs (12.19–21), we find

$$0 = a(\mathrm{d}/\mathrm{d}a)\,m_g = -m_g + (8/(a g_0))\,\gamma(g_0) + \dots \tag{12.23}$$

$$\gamma(g_0) = \tfrac{1}{8} g_0 \log(3g_0^2) + \dots \tag{12.24}$$

Note the change in the normalization between eqs (12.21) and (12.24). Away from a zero of $\gamma(g_0)$ the lattice spacing is not small. This influences the relationships among observables and can appear as a scheme dependence of the renormalization-group function.

Problems

1. What does it physically mean to change the initial value H_0 in eq. (12.7)?

2. Suppose near a fixed point g_F that the renormalization-group function behaves as $\gamma(g_0) = (g_0 - g_F)\,\lambda + O((g_0 - g_F)^2)$. Show that the correlation length diverges at g_F as

$$\xi \propto (g_0 - g_F)^{-1/\lambda}.$$