ROTATIONAL MODES IN A UNIFORMLY ROTATING STAR

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ABSTRACT

The linear adiabatic oscillations of a uniformly rotating star are examined with respect to a co-rotating frame of reference. It is shown that time-independent displacement fields are allowed and can be represented as toroidal fields. The time-dependent oscillation modes are governed by a system of differential equations of the fourth order in time for axisymmetric perturbations, and of the fifth order for non-axisymmetric perturbations. Therefore, in comparison to a non-rotating spherical star, a rotating star allows a new class of non-axisymmetric oscillation modes with non-zero frequencies. These modes correspond to the r-modes, which originate from purely toroidal displacement fields in a non-rotating spherical star.

For the r-modes of a slowly and uniformly rotating star, a perturbation method has been developed, with inclusion of the perturbation of the gravitational potential. It is seen that both dynamically stable and unstable r-modes do exist.

1. TIME-DEPENDENT AND TIME-INDEPENDENT DISPLACEMENT FIELDS IN A UNIFORMLY ROTATING STAR

Consider a star that is rotating with a uniform angular velocity Ω with respect to an inertial frame of reference. In a co-rotating frame of reference, an orthogonal coordinate system is defined that was first used by Papaloizou et al. (1978). The first coordinate ψ is hold constant on each equipotential surface. The second coordinate χ is defined such that the corresponding coordinate lines are, at each point, orthogonal to the equipotential surface. The third coordinate is the usual azimuthal coordinate φ.

If ϖ denotes the cylindrical polar radius, the transformation from a system of orthogonal Cartesian coordinates defined in the inertial frame of reference can be written as

Space Science Reviews 27 (1980) 653–659. 0038–6308/80/0274–653 $01.05.

$$\left.\begin{aligned} x &= \varpi(\psi, \chi)\cos(\varphi + \Omega t), \\ y &= \varpi(\psi, \chi)\sin(\varphi + \Omega t), \\ z &= z(\psi, \chi). \end{aligned}\right\} \tag{1}$$

The square of the infinitesimal geometrical distance then takes the form

$$ds^2 = g_{\psi\psi}\, d\psi^2 + g_{\chi\chi}\, d\chi^2 + g_{\varphi\varphi}\, d\varphi^2. \tag{2}$$

The equilibrium pressure and density depend only on ψ. Assume that all perturbed quantities are proportional to $\exp[i(\sigma t + m\varphi)]$. The components of the perturbed equation of motion can be written as

$$-\sigma^2\xi_\psi - 2\, i\sigma\Omega \frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial\varpi}{\partial\psi}\xi_\varphi = -\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial}{\partial\psi}(\Phi' + \frac{P'}{\rho}) + Ac^2 \mathrm{div}\,\vec{\xi}, \tag{3}$$

$$-\sigma^2\xi_\chi - 2\, i\sigma\Omega \frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial\varpi}{\partial\chi}\xi_\varphi = -\frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial}{\partial\chi}(\Phi' + \frac{P'}{\rho}), \tag{4}$$

$$-\sigma^2\xi_\varphi + 2\, i\sigma\Omega \left(\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial\varpi}{\partial\psi}\xi_\psi + \frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial\varpi}{\partial\chi}\xi_\chi\right) = -\frac{im}{\sqrt{g_{\varphi\varphi}}}(\Phi' + \frac{P'}{\rho}), \tag{5}$$

where

$$A = \frac{1}{\sqrt{g_{\psi\psi}}}\left(\frac{1}{\rho}\frac{d\rho}{d\psi} - \frac{1}{\rho c^2}\frac{dP}{d\psi}\right), \qquad c^2 = \frac{\Gamma_1 P}{\rho}. \tag{6}$$

In these equations, ξ_ψ, ξ_χ, ξ_φ represent the components of the Lagrangian displacement with respect to the local orthonormal basis.

The right-hand member of equation (3) is written in the form that was derived by Ledoux et al. (1958, Sect. 75) for the r-component of the equation of motion in the case of a non-rotating spherical star. From equations (3)-(5), it is seen that the perturbed pressure force and the perturbed gravitational force can be represented as the sum of a gradient of a scalar and of a vector that is normal to the equipotential surface, even in a rotating star. Hence, these two forces do not give rise to any normal component of vorticity, as is expected on physical grounds. Any normal component of vorticity is therefore a specific effect of the Coriolis force. This effect is rendered by the ψ-component of the vorticity equation, which is obtained by taking the curl of the equation of motion:

$$\sigma^2(\mathrm{curl}\,\vec{\xi})_\psi = 2\, i\sigma\Omega\frac{1}{\varpi\sqrt{g_{\chi\chi}}}\frac{\partial}{\partial\chi}\left[\varpi\left(\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial\varpi}{\partial\psi}\xi_\psi + \frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial\varpi}{\partial\chi}\xi_\chi\right)\right]$$
$$- 2\, m\sigma\Omega\frac{1}{\varpi}\frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial\varpi}{\partial\chi}\xi_\varphi . \tag{7}$$

In the limiting case of a non-rotating spherical star, equation (7) reduces to

$$\sigma^2 (\mathrm{curl}\ \vec{\xi})_r = 0 \ . \tag{8}$$

Equation (8) has been used as a starting-point for a distinction between two types of displacement fields based on the property that the radial component of vorticity is either zero at all points or non-zero in some regions of the star (see Aizenman et al., 1977). The displacement fields of the first type generally have a non-zero frequency, while the displacement fields of the second type are all time-independent ($\sigma^2 = 0$). Furthermore, when use is made of the decomposition of the displacement field

$$\vec{\xi} = \mathrm{grad}\ U + \mathrm{curl}(T\ \vec{1}_r) + \mathrm{curl}^2(S\ \vec{1}_r) \ , \tag{9}$$

it is seen that the displacement fields of the first type consist of both a longitudinal and a poloidal component (spheroidal modes) and that the displacement fields of the second type are purely toroidal.

In order to extend the previous analysis made for a non-rotating spherical star to a uniformly rotating star, start from equation (7) and use the decomposition of the displacement field

$$\vec{\xi} = \mathrm{grad}\ U + \mathrm{curl}(T\ \vec{1}_\psi) + \mathrm{curl}^2(S\ \vec{1}_\psi) \ . \tag{10}$$

First, note that equation (7) admits of the root

$$\sigma = 0 \ . \tag{11}$$

It is easily verified that the various equations that govern the linear adiabatic oscillations of a uniformly rotating star can be satisfied if solutions are chosen of the form

$$\xi_\psi = 0, \qquad \mathrm{div}\ \vec{\xi} = 0, \qquad P' = 0, \qquad \rho' = 0, \qquad \Phi' = 0. \tag{12}$$

In terms of decomposition (10), the displacement fields defined by (12) can be represented as purely toroidal fields:

$$\vec{\xi} = \mathrm{curl}(T\ \vec{1}_\psi) \ . \tag{13}$$

It may, therefore, be concluded that, in any uniformly rotating star, purely toroidal displacement fields with a zero frequency are allowed.

Second, equation (7) is also satisfied when

$$\frac{\sigma}{\varpi}\left[\frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial}{\partial\chi}(\varpi\ \xi_\varphi) - \frac{\partial\xi_\chi}{\partial\varphi}\right]$$
$$= 2\,i\,\Omega\frac{1}{\varpi\sqrt{g_{\chi\chi}}}\frac{\partial}{\partial\chi}\left[\varpi\left(\frac{1}{\sqrt{g_{\psi\psi}}}\frac{\partial\varpi}{\partial\psi}\xi_\psi + \frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial\varpi}{\partial\chi}\xi_\chi\right)\right]$$
$$- 2\,m\,\Omega\frac{1}{\varpi}\frac{1}{\sqrt{g_{\chi\chi}}}\frac{\partial\varpi}{\partial\chi}\xi_\varphi \ . \tag{14}$$

https://doi.org/10.1017/S0252921100082208 Published online by Cambridge University Press

In the *axisymmetric* case (m = 0), it is easily seen that equation (14) and the other equations that govern the linear adiabatic oscillations of a uniformly rotating star are satisfied by

$$\sigma = 0, \quad \xi_\psi = 0, \quad \xi_\chi = 0, \quad P' = 0, \quad \rho' = 0, \quad \Phi' = 0. \qquad (15)$$

The component ξ_φ is the only non-zero component. Axisymmetric displacement fields of this type can be represented as toroidal fields defined by (13).

Thus, in the axisymmetric case, $\sigma = 0$ is a double root of the equations that govern the linear adiabatic oscillations of a uniformly rotating star. The associated displacement fields are purely toroidal fields. We infer from this that the time-dependent axisymmetric oscillation modes are governed by a system of differential equations of the fourth order in time. Thus, in comparison to the case of a non-rotating spherical star, no essentially new spectra of non-zero oscillation frequencies seem to be expected in a uniformly rotating star when m = 0.

In the *non-axisymmetric* case ($m \neq 0$), equation (14) can no longer be satisfied by a zero frequency and purely toroidal displacement fields. The time-dependent oscillations modes will now be governed by a system of differential equations that is of the fifth order in time. Since the order of the time-dependency is increased by one in comparison to the case of a non-rotating spherical star, a possibility arises here for the existence of a new class of oscillation modes with non-zero frequencies. These modes must originate from toroidal displacement fields -in the sense defined by decomposition (9)- with a zero frequency in a non-rotating spherical star. Their existence was pointed out in the Cowling approximation by Papaloizou et al. (1978), who called them r-modes.

2. AN APPROXIMATION FOR THE r-MODES OF A SLOWLY AND UNIFORMLY ROTATING STAR

A perturbation method has been developed for r-modes of a slowly and uniformly rotating star, with inclusion of the perturbation of the gravitational potential (my colleagues and I are preparing a paper on this matter). A similar perturbation method has been developed in the Cowling approximation by Provost et al. (1980).

In the approximation of a slow rotation, it is convenient to put

$$\Omega = \varepsilon\alpha \,, \qquad (16)$$

where ε is a small dimensionless quantity. It is assumed that the effects of the centrifugal force upon the equilibrium structure are taken into account to terms of order ε^2. The distortion of the equilibrium configuration being small, the use of spherical coordinates (r, θ, φ) is appropriate here. The radial distance to an equipotential surface can be expressed as

$$r = \psi + \varepsilon^2 r_2(\psi, \theta) + O(\varepsilon^4) , \tag{17}$$

with

$$r_2(\psi, \theta) = r_{2,0}(\psi) + r_{2,2}(\psi) P_2(\cos\theta) . \tag{18}$$

Here ψ is defined as the mean radius of an equipotential surface. Expression (17) can be regarded as a relation that defines a mapping of an equipotential surface at radial distance $r(\theta)$ on a spherical surface with radius ψ. By this mapping the eigenvalue problem of the oscillations of a slowly rotating star is defined on the same domain as the eigenvalue problem of the oscillations of a non-rotating spherical star.

The oscillation frequency and the components of the displacement have been expanded as

$$\sigma = \sum_{\kappa=1} \varepsilon^\kappa \sigma_\kappa , \tag{19}$$

and

$$\left.\begin{aligned} \xi_r &= \sum_{\kappa=1} \varepsilon^\kappa \xi^{(\kappa)} , \\ \xi_\theta &= \sum_{\kappa=0} \varepsilon^\kappa \xi_\theta^{(\kappa)} , \\ \xi_\varphi &= \sum_{\kappa=0} \varepsilon^\kappa \xi_\varphi^{(\kappa)} . \end{aligned}\right\} \tag{20}$$

At the various orders, the components of the displacement have been further decomposed as

$$\left.\begin{aligned} \xi_r^{(\kappa)} &= \sum_{\ell,m} a_{\ell,m}^{(\kappa)}(\psi)\, Y_\ell^m(\theta, \varphi) , \\ \xi_\theta^{(\kappa)} &= \sum_{\ell,m} \left(\frac{b_{\ell,m}^{(\kappa)}(\psi)}{\psi} \frac{\partial Y_\ell^m}{\partial\theta} + \frac{T_{\ell,m}^{(\kappa)}(\psi)}{\psi \sin\theta} \frac{\partial Y_\ell^m}{\partial\varphi} \right) , \\ \xi_\varphi^{(\kappa)} &= \sum_{\ell,m} \left(\frac{b_{\ell,m}^{(\kappa)}(\psi)}{\psi \sin\theta} \frac{\partial Y_\ell^m}{\partial\varphi} - \frac{T_{\ell,m}^{(\kappa)}(\psi)}{\psi} \frac{\partial Y_\ell^m}{\partial\theta} \right) . \end{aligned}\right\} \tag{21}$$

For the zeroth and first orders of the frequency, one obtains

$$\sigma_1 = \frac{2\, m\alpha}{\ell(\ell+1)} , \tag{22}$$

$$\sigma_2 = 0 . \tag{23}$$

Expression (22) agrees with the expression obtained by Papaloizou et al.

(1978) and by Provost et al. (1980). This expression reflects the effect of the Coriolis force. It is therefore obtained correctly even when an approximation is made for the perturbed gravitational force.

σ_3 appears as a complex eigenvalue parameter in a system of homogeneous differential equations, which is of the sixth order in the radial coordinate ψ. From these equations it follows that the complex conjugate of an eigenvalue is also an eigenvalue. Consequently, both dynamically stable and unstable modes will be found. In the Cowling approximation, the system of equations reduces to a second order system. As noted by Provost et al., the eigenvalue problem is of the Sturm-Liouville type, so that the eigenvalues all become real.

σ_4 appears as an eigenvalue parameter in a non-homogeneous term of a sixth order system of non-homogeneous differential equations in the radial coordinate ψ. σ_4 is also complex.

The displacement field can be determined to the first order in ε as

$$\left.\begin{aligned} \xi_r &= 0 , \\ \xi_\theta &= \left[T^{(0)}_{\ell,m}(\psi) + \varepsilon T^{(1)}_{\ell,m}(\psi)\right] \frac{1}{\psi \sin\theta} \frac{\partial Y^m_\ell}{\partial\varphi} , \\ \xi_\varphi &= -\left[T^{(0)}_{\ell,m}(\psi) + \varepsilon T^{(1)}_{\ell,m}(\psi)\right] \frac{1}{\psi} \frac{\partial Y^m_\ell}{\partial\theta} . \end{aligned}\right\} \qquad (24)$$

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DISCUSSION

J. COX: I believe r-modes were first actually discovered by Ledoux way back in 1940. They were then rediscovered.

SMEYERS: He notes in the Handbuch article that there was an established solution. In fact, he noted that this equation was of fifth order in time.

EDWARDS: Would you remind us as to what the Cowling approximation is?

SMEYERS: It is that approximation in which you neglect the perturbation of the gravitational potential on the basis that if you represent several modes in the star, the perturbation is given by an integral expression over the perturbation of density. If you have many perturbations in density, you can expect that the integral will tend to zero.

ROBINSON: Do the third and fourth order terms merely displace the r-mode frequencies or do they split them?

SMEYERS: Apparently what happens is that for the first order approximation, you have the same frequency for all of the r-modes. In this particular case, the analysis of $\Delta\rho$ and $\Delta\mu$ suggests that you get an infinity of values σ_3 for each common value σ_1. In the Cowling approximation, it appears that we will get an infinity of r-modes. For each value of σ_3, you get one value of σ_4. It is not surprising that, from a mathematical point of view, as you increase the order of the system with respect to time, you get a new spectrum. The point is that this is the only new spectrum that you can expect in the rotating case. Personally, I expected that we would perhaps get one more, but that is apparently not true.

KNOELKER: That may happen with differential rotation.

SMEYERS: Yes, you may have that with differential rotation but not in uniform rotation.