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Locally compact products and coproducts in categories of topological groups

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In the category of locally compact groups not all families of groups have a product. Precisely which families do have a product and a description of the product is a corollary of the main theorem proved here. In the category of locally compact abelian groups a family $\{G_j;\ j \in J\}$ has a product if and only if all but a finite number of the G_j are of the form $K_j \times D_j$, where K_j is a compact group and D_j is a discrete torsion free group. Dualizing identifies the families having coproducts in the category of locally compact abelian groups and so answers a question of Z. Semadeni.

It is well known that if C is the category of all topological groups (hereinafter referred to as *TopGps*), of all abelian topological groups, of all compact groups, or of all compact abelian groups with the morphisms being the continuous homomorphisms then, for any family $\{G_j;\ j \in J\}$ of members of C, both the product $\prod^{C}_{j\in J} G_j$ and the coproduct $\coprod^{C}_{j\in J} G_j$ exist. However Semadeni [2, 3] notes that very little is known about coproducts (he could also have said products) in the category *LCA* of

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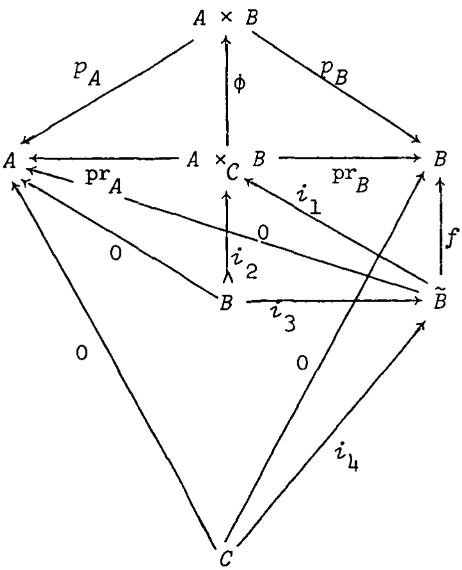
locally compact abelian groups. It is clear that if $\{G_j;\ j \in J\}$ is a family in LCA such that all but a finite number of G_i are discrete, then $\coprod_{j\in J}^{\mathit{LCA}} G_j$ exists. Semadeni [3, Problem 490] asks if there is an infinite family $\{G_j;\ j \in J\}$ of non-discrete members of LCA such that $\coprod_{j\in J}^{\mathit{LCA}} G_j$ exists. We give an affirmative answer and characterize those families $\{G_j;\ j \in J\}$ for which $\coprod_{j\in J}^{\mathit{LCA}} G_j$ exists. This is done by solving the dual problem of describing products in LCA which is a corollary of our main result.

PROPOSITION 1. *If* C *is a full subcategory of* TopGps *which has finite limits, then the finite products in* C *are the finite products in* TopGps .

Proof. It suffices to prove that if A and B are in C , then their product in C , $A \times_C B$, is topologically isomorphic to $A \times B$, their product in TopGps .



Let pr_A and pr_B be the projections of $A \times_C B$ into A and B , respectively. Also let p_A and p_B be the projections of $A \times B$ into A and B respectively. As $A \times B$ is the product in *TopGps*, there exists a continuous homomorphism $\phi : A \times_C B \to A \times B$ such that $\mathrm{pr}_A = p_A\phi$ and $\mathrm{pr}_B = p_B\phi$. Let $\tilde{B}$ be the kernel of pr_A, and i_1 the inclusion of $\tilde{B}$ in $A \times_C B$. Let $f : \tilde{B} \to B$ be $\mathrm{pr}_B i_1$.

The identity map $1 : B \to B$ and the trivial map $0 : B \to A$ imply the existence of a continuous homomorphism i_2 of B into $A \times_C B$ which is a topological embedding. As $\mathrm{pr}_A i_2 = 0$, the definition of $\tilde{B}$ implies the existence of a continuous homomorphism $i_3 : B \to \tilde{B}$ which is a topological embedding.

Let C be the kernel of f. So i_4 is the inclusion map of C in $\tilde{B}$ and fi_4 is the trivial map of C into B. Mapping C into $\tilde{B}$ by i_4 and $\tilde{B}$ into A by 0 yields the trivial map $0 : C \to A$. The pair of maps $0 : C \to A$ and $0 : C \to B$ yield a unique continuous homomorphism $\Theta : C \to A \times_C B$ such that $\mathrm{pr}_A\Theta = 0$ and $\mathrm{pr}_B\Theta = 0$. But there are two maps with this property, $0 : C \to A \times_C B$ and $i_1 i_4 : C \to A \times_C B$. So $i_4 = 0$ and thus C is the trivial group. Hence f is injective. But as $fi_3 = 1$, f is also a retraction and hence f is a topological group isomorphism and $\tilde{B}$ is topologically isomorphic to B.

Similarly, if we put $\tilde{A}$ equal to the kernel of pr_B, then we can show $\mathrm{pr}_A|_{\tilde{A}}$ is a topological isomorphism of $\tilde{A}$ onto A.

We show next that $A \times_C B = \tilde{A}\tilde{B}$ and $\tilde{A} \cap \tilde{B} = \{1\}$;

$$\begin{tikzcd}
 & \tilde{A} \arrow[dl, "0"'] \arrow[d, tail] \arrow[dr, "\mathrm{pr}_A|_{\tilde{A}}"] & \\
B & A \times_C B \arrow[l, "\mathrm{pr}_B"] \arrow[r, "\mathrm{pr}_A"'] & A \\
 & \tilde{B} \arrow[ul, "\mathrm{pr}_B|_{\tilde{B}}"] \arrow[u, tail] \arrow[ur, "0"'] &
\end{tikzcd}$$

Let $g \in \tilde{A} \cap \tilde{B}$. As $g \in \tilde{B}$, $\mathrm{pr}_A(g) = 1$. But $\mathrm{pr}_A|_{\tilde{A}}$ is an isomorphism and so $g = 1$. So $\tilde{A} \cap \tilde{B} = \{1\}$.

Now if $x \in A \times_C B$, then $h = \left(\mathrm{pr}_A|_{\tilde{A}}\right)^{-1}\mathrm{pr}_A(x) \in \tilde{A}$. So $\mathrm{pr}_A(h) = \mathrm{pr}_A(x)$. Thus $\mathrm{pr}_A\left(h^{-1}x\right) = 1$, which implies $h^{-1}x \in \ker\left(\mathrm{pr}_A\right) = \tilde{B}$ Hence $x \in \tilde{A}\tilde{B}$ and $A \times_C B = \tilde{A}\tilde{B}$.

As $\tilde{A}$ and $\tilde{B}$ are normal subgroups of $A \times_C B$ and $A \times_C B = \tilde{A}\tilde{B}$ and $\tilde{A} \cap \tilde{B} = \{1\}$, we have that $A \times_C B$ is algebraically isomorphic to $A \times B$. More precisely ϕ is an algebraic isomorphism of $A \times_C B$ onto $A \times B$.

Now define $\psi : A \times B \to A \times_C B$ by

$$\psi(a, b) = \left[\left(\mathrm{pr}_A|_{\tilde{A}}\right)^{-1}a\right]\left[\left(\mathrm{pr}_B|_{\tilde{B}}\right)^{-1}b\right] .$$

If $g \in A \times_C B$, then $g \in \tilde{A}\tilde{B}$; that is, $g = \tilde{a}\tilde{b}$ with $\tilde{a} \in \tilde{A}$ and $\tilde{b} \in \tilde{B}$. Then $\phi(g) = \left(\mathrm{pr}_A(\tilde{a}), \mathrm{pr}_B(\tilde{b})\right)$. Clearly ϕ and ψ are inverse maps. As ψ is also continuous, ϕ is a topological group isomorphism and $A \times_C B$ is topologically isomorphic to $A \times B$.

REMARKS. It is not necessary to assume C is a full subcategory of *TopGps* in Proposition 1. It suffices to assume that C has identity arrows, 0 arrows, and inclusion arrows.

The reader should be cautioned not to assume *a priori* that, in proofs like the preceding and in the discussions which follow, the products we consider are defined on set theoretical cartesian products.

LEMMA 1. *Let* C *be a full subcategory of* TopGps *which has finite limits. If* F *is a finite subset of a set* J *and* $\overset{C}{\prod\limits_{j\in J}} G_j$ *exists, then* $\overset{C}{\prod\limits_{j\in J\setminus F}} G_j$ *exists and*

$$\overset{C}{\prod_{j\in J}} G_j \simeq \overset{C}{\prod_{j\in F}} G_j \times \overset{C}{\prod_{j\in J\setminus F}} G_j \, .$$

Proof. Let K be the kernel of the canonical map Θ of $\overset{C}{\prod\limits_{j\in J}} G_j$ into $\overset{C}{\prod\limits_{j\in F}} G_j$. We claim that $K = \overset{C}{\prod\limits_{j\in J\setminus F}} G_j$, with the projections of K into G_j being the projections of $\overset{C}{\prod\limits_{j\in J}} G_j$ (for $j \in J\setminus F$) restricted to K .

$$\begin{tikzcd}
K \arrow[d, hook] \arrow[rr, "\mathrm{pr}_j|_K"] & & G_j \\
\overset{C}{\prod\limits_{j\in J}} G_j \arrow[rr, "\mathrm{pr}_j"'] \arrow[dd, "\Theta"'] & & G_j \\
& A \arrow[ul, "\phi"'] \arrow[ur, "f_j"'] \arrow[dl, "0"] \arrow[dr, "0"'] & \\
\overset{C}{\prod\limits_{i\in F}} G_i \arrow[rr, "\mathrm{pr}_i"'] & & G_i \, .
\end{tikzcd}$$

Let A be any member of C and f_j , $j \in J\setminus F$, a family of continuous homomorphisms of A into G_j . Define $f_i : A \to G_i$, $i \in F$, to be the trivial homomorphisms. Then there exists a continuous homomorphism ϕ of A into $\overset{C}{\prod\limits_{j\in J}} G_j$ such that $\mathrm{pr}_j\phi = f_j$, $j \in J$. As

$\mathrm{pr}_i\phi = 0$, $i \in F$, $\Theta\phi = 0$, and so $\phi(A) \subseteq K$. Hence $K = \prod^{C}_{j\in J\setminus F} G_j$.

Once we know $\prod^{C}_{j\in J\setminus F} G_j$ exists, the usual categorical argument shows that $\prod^{C}_{j\in J} G_j$ is topologically isomorphic to $\prod^{C}_{j\in F} G_j \times_C \prod^{C}_{j\in J\setminus F} G_j$, which by Proposition 1, is the required result.

PROPOSITION 2. *Let* K *be a full subcategory of* TopGps *which is complete and whose objects are compact. If every compact subgroup of a topological group in* K *is also in* K *, then for any family* $\{G_j;\ j \in J\}$ *in* K , $\prod^{K}_{j\in J} G_j \simeq \prod_{j\in J} G_j$; *that is, the product in* K *and the product in* TopGps *coincide.*

Proof.

$$\begin{tikzcd}
 & \prod^{K}_{j\in J} G_j \arrow[dl, "f"'] \arrow[dr, "\mathrm{pr}_j"] & H \arrow[l, tail, "i"'] \\
\prod_{j\in J} G_j \arrow[rr, "p_j"'] & & G_j
\end{tikzcd}$$.

Let p_j and pr_j be the projections of $\prod_{j\in J} G_j$ into G_j and $\prod^{K}_{j\in J} G_j$ into G_j , respectively. As $\prod_{j\in J} G_j$ is the product in TopGps of $\{G_j;\ j \in J\}$, the maps $\mathrm{pr}_j : \prod^{K}_{j\in J} G_j \to G_j$ induce a continuous homomorphism $f : \prod^{K}_{j\in J} G_j$ into $\prod_{j\in J} G_j$ such that $p_j f = \mathrm{pr}_j$, for each $j \in J$.

Let $x \in \prod^{K}_{j\in J} G_j$ be such that $f(x) = 1$. Then $p_j f(x) = 1$ and so $\mathrm{pr}_j(x) = 1$, for each $j \in J$. Let H be the closed subgroup of $\prod^{K}_{j\in J} G_j$, generated by $\{x\}$. Then $\mathrm{pr}_j(H) = \{1\}$, for all $j \in J$. If $i : H \to \prod_{j\in J}^{K} G_j$ is the inclusion map, then $\mathrm{pr}_j i = 0 = \mathrm{pr}_j 0'$, where 0 is the trivial map of H into G_j and $0'$ is the trivial map of H into $\prod_{j\in J}^{K} G_j$. As $\prod_{j\in J}^{K} G_j$ is the product in K , the uniqueness in the product property implies that $i = 0'$. Thus $H = \{1\}$ and $x = 1$. Hence the map f is injective.

$$\begin{tikzcd}
\prod_{j\in J}^{K} G_j \arrow[rr, "f"] \arrow[dd, "\mathrm{pr}_j"'] & & \prod_{j\in J} G_j \arrow[ddll, "p_j"] \\
 & G_i \arrow[ul, "\Theta_i"'] \arrow[ur, "e_i"] \arrow[dl, "\delta_{ij}"'] & \\
G_j & &
\end{tikzcd}$$

(Here e_i is the injection.)

Now let $\delta_{ij} : G_i \to G_j$ be the identity map if $i = j$ and the trivial homomorphism if $i \neq j$. Then there exist continuous homomorphisms $\Theta_i : G_i \to \prod_{j\in J}^{K} G_j$ such that $\delta_{ij} = \mathrm{pr}_j \Theta_i$ and $\delta_{ij} = p_j e_i$. $j \in J$. By uniqueness in the product property $f\Theta_i = e_i$ So $f\left(\prod_{j\in J}^{K} G_j\right)$ is a compact subgroup of $\prod_{j\in J} G_j$ which contains $e(G_i)$ for each $i \in J$. Thus $f\left(\prod_{j\in J}^{K} G_j\right) = \prod_{j\in J} G_j$ and f is surjective. Hence f is a topological group isomorphism of $\prod_{j\in J}^{K} G_j$ onto $\prod_{j\in J} G_j$.

LEMMA 2. *Let* C *be a full subcategory of TopGps and* K *a full subcategory of* C *which is complete and whose objects are all compact subgroups of topological groups in* C . *If* $K_j \in K$, *for* $j \in J$, *then* $\prod_{j\in J}^{C} K_j$ *exists and is topologically isomorphic to* $\prod_{j\in J}^{K} K_j$.

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Proof. This follows immediately from the fact that $\prod^{\mathsf{K}}_{j\in J} K_j$ is $\prod_{j\in J} K_j$, the product in *TopGps* of $\{K_j : j \in J\}$.

LEMMA 3. *Under the hypotheses of Lemma 2, if* $K_j \leq G_j$, *where* $K_j \in \mathsf{K}$ *and* $G_j \in \mathsf{C}$ *for* $j \in J$ *and* $\prod^{\mathsf{C}}_{j\in J} G_j$ *exists, then the canonical homomorphism of* $\prod^{\mathsf{C}}_{j\in J} K_j$ *into* $\prod^{\mathsf{C}}_{j\in J} G_j$ *is a topological embedding.*

Proof. This follows immediately from the fact that $\prod^{\mathsf{C}}_{j\in J} K_j$ is $\prod_{j\in J} K_j$, and so is compact.

LEMMA 4. *Let* C *be a full subcategory of* *TopGps* *with finite limits and* K *a full subcategory of compact* C*-objects which is complete and contains all compact subgroups of groups in* C. *If* $G_j \in \mathsf{C}$, $j \in J$, *and* $\prod^{\mathsf{C}}_{j\in J} G_j$ *exists and* K *is a compact subgroup of* $\prod^{\mathsf{C}}_{j\in J} G_j$, *then* $K \leq \prod^{\mathsf{C}}_{j\in J} \mathrm{pr}_j(K) \leq \prod^{\mathsf{C}}_{j\in J} G_j$, *where* pr_j *is the canonical projection of* $\prod^{\mathsf{C}}_{j\in J} G_j$ *into* G_j, $j \in J$.

Proof.

$$\begin{tikzcd}
 & K \arrow[dl, "\theta"'] \arrow[dr, "i_1"] & \\
\prod^{\mathsf{C}}_{j\in J} \mathrm{pr}_j(K) \arrow[rr, "i_2"'] \arrow[d, "f_j"'] & & \prod^{\mathsf{C}}_{j\in J} G_j \arrow[d, "\mathrm{pr}_j"] \\
\mathrm{pr}_j(K) \arrow[rr, tail] & & G_j
\end{tikzcd} \quad .$$

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As K is compact, $\mathrm{pr}_j(K)$ is compact and so is in K. Hence $\prod_{j\in J}^{K} \mathrm{pr}_j(K)$ exists and by Lemma 2 is isomorphic to $\prod_{j\in J}^{C} \mathrm{pr}_j(K)$. By Lemma 3, the canonical homomorphism $\prod_{j\in J}^{C} \mathrm{pr}_j(K) \to \prod_{j\in J}^{C} G_j$ is a topological embedding i_2. The maps $\mathrm{pr}_j : K \to \mathrm{pr}_j(K)$ imply the existence of a continuous homomorphism $\theta : K \to \prod_{j\in J}^{C} \mathrm{pr}_j(K)$ such that $f_j\theta = \mathrm{pr}_j$, where f_j are the projections of $\prod_{j\in J}^{C} \mathrm{pr}_j(K)$. Finally observe that $i_2\theta = i_1$, the inclusion of K in $\prod_{j\in J}^{C} G_j$. So θ is a topological embedding, as required.

LEMMA 5. *If H_j and G_j, $j \in J$, are topological groups with $H_j \le G_j$ and $\prod_{j\in J} H_j$ an open subgroup of $\prod_{j\in J} G_j$, then there is a finite subset F of J such that for all $j \in J\backslash F$, $H_j = G_j$.*

Proof. This follows from the definition of the Tychonoff product topology.

LEMMA 6. *Let C be a full subcategory of TopGps which has finite limits. If $G_j \in C$, $j \in J$, and $\prod_{j\in J}^{C} G_j$ exists and is locally compact, then each G_j is locally compact and there exists a finite subset F of J such that for $j \in J\backslash F$, G_j has the component of the identity compact.*

Proof. By Proposition 1 and Lemma 1, each G_j is locally compact. If none of $G_{j_1}, \ldots, G_{j_n}$ has compact component then each contains a copy of R, the additive group of real numbers with the usual topology. Proposition 1 and Lemma 1 then imply that $G_{j_1} \times G_{j_2} \times \ldots \times G_{j_n} \le \prod_{j\in J}^{C} G_j$.

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So $\prod\limits_{j\in J}^{C} G_j$ has a subgroup topologically isomorphic to R^n. But no locally compact group can contain R^n, for arbitrary large n. So all but a finite number of the G_j have compact components.

LEMMA 7. *Let* C *be a full subcategory of* TopGps *with finite limits and* K *a full subcategory of compact* C*-objects which is complete and contains all compact subgroups of groups in* C*, and let* $G_j \in C$, $j \in J$ *be such that each* G_j *has its connected component* $(G_j)_0$ *compact. If* $\prod\limits_{j\in J}^{C} G_j$ *exists then its component* $\left(\prod\limits_{j\in J}^{K} (G_j)\right)_0$ *is topologically isomorphic to* $\prod\limits_{j\in J}^{K} (G_j)_0$.

Proof. By Lemma 3, $\prod\limits_{j\in J}^{K} (G_j)_0 \le \prod\limits_{j\in J}^{C} G_j$, and as $\prod\limits_{j\in J}^{K} (G_j)_0$ is connected, $\prod\limits_{j\in J}^{K} (G_j)_0 \le \left(\prod\limits_{j\in J}^{C} G_j\right)_0$.

$$\begin{tikzcd}
\prod\limits_{j\in J}^{K} (G_j)_0 \arrow[rr, tail, "i_1"] \arrow[dd, "p_j"'] & & \prod\limits_{j\in J}^{C} G_j \arrow[dd, "\mathrm{pr}_j"] \\
& \left(\prod\limits_{j\in J}^{C} G_j\right)_0 \arrow[ul, "\Theta"'] \arrow[ur, tail, "i_3"'] \arrow[dl, "q_j"] & \\
(G_j)_0 \arrow[rr, "i_{2j}"'] & & G_j
\end{tikzcd}$$

Let i_1 be the inclusion map of $\prod\limits_{j\in J}^{K} (G_j)_0$ in $\prod\limits_{j\in J}^{C} G_j$, i_{2j} the inclusion map of $(G_j)_0$ in G_j, pr_j the projection of $\prod\limits_{j\in J}^{C} G_j$ into G_j, i_3 the inclusion map of $\left(\prod\limits_{j\in J}^{C} G_j\right)_0$ in $\prod\limits_{j\in J}^{C} G_j$, q_j the

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restriction of pr_j to $\left(\prod_{j\in J}^{C} G_j\right)_0$ and p_j the projection of $\prod_{j\in J}^{K} (G_j)_0$ into $(G_j)_0$. As $\left(\prod_{j\in J}^{C} G_j\right)_0$ is connected and q_j is continuous, $q_j\left\{\left(\prod_{j\in J}^{C} G_j\right)_0\right\} \leq (G_j)_0$. The family of maps q_j determine a continuous homomorphism $\theta : \left(\prod_{j\in J}^{C} G_j\right)_0 \to \prod_{j\in J}^{K} (G_j)_0$ such that $p_j\theta = q_j$ for each $j \in J$. The family of maps $i_{2j}q_j$ of $\left(\prod_{j\in J}^{C} G_j\right)_0$ into G_j, $j \in J$, determine a unique continuous homomorphism $\delta : \left(\prod_{j\in J}^{C} G_j\right)_0 \to \prod_{j\in J}^{C} G_j$ such that $\mathrm{pr}_j\delta = i_{2j}q_j$. But the maps i_3 and $i_1\theta$ both have this property and thus $i_3 = i_1\theta$. Hence $\left(\prod_{j\in J}^{C} G_j\right)_0 = \prod_{j\in J}^{K} (G_j)_0$.

LEMMA 8. *Under the hypotheses of Lemma 7, if* $\prod_{j\in J}^{C} G_j$ *exists and is locally compact, then* $\prod_{j\in J}^{C} G_j$ *has a compact open subgroup* $\prod_{j\in J}^{K} H_j$; *where each* H_j *is an open subgroup of* G_j.

Proof. Lemma 7 says that $\prod_{j\in J}^{C} G_j$ is a locally compact group with compact component and so $\prod_{j\in J}^{C} G_j$ has a compact open subgroup C. Putting $H_j = \mathrm{pr}_j(C)$, where pr_j is the canonical map of $\prod_{j\in J}^{C} G_j$ into G_j, we have that H_j is in K and by Lemma 4, $C \leq \prod_{j\in J}^{K} H_j \leq \prod_{j\in J}^{C} G_j$. So $\prod_{j\in J}^{K} H_j$ is a compact open subgroup of $\prod_{j\in J}^{C} G_j$. By Proposition 1 and Lemma 1, pr_j

is a retraction and thus a quotient mapping. So each H_j is an open subgroup of G_j.

LEMMA 9. *Under the hypotheses of Lemma 7 if* $\prod\limits_{j\in J}^{C} G_j$ *exists and is locally compact, there exists a finite subset* F *of* J *such that for* $j \in J\backslash F$, *the group* G_j *has a compact open normal subgroup* H_j *with* H_j *a maximal compact subgroup of* G_j.

Proof. Let H_j be as in Lemma 8. Let F_1 be the subset of J consisting of those j for which there is a compact group C_j with $H_j \lneq C_j \leq G_j$. For $j \in J\backslash F_1$, set $C_j = H_j$. Then $\prod\limits_{j\in J}^{K} H_j \lneq \prod\limits_{j\in J}^{K} C_j \leq \prod\limits_{j\in J}^{C} G_j$. So $\prod\limits_{j\in J}^{K} H_j$ is an open subgroup of $\prod\limits_{j\in J}^{K} C_j$. By Lemma 5, F_1 is finite.

Now set $F_2 = \{j \in J : H_j \text{ is not normal in } G_j\}$. So for each $j \in F_2$, there exists a $g_j \in G_j$ such that $H_j \neq g_jH_jg_j^{-1}$. For $j \in J\backslash F_2$, set $g_j = 1$. Define an automorphism α of $\prod\limits_{j\in J}^{C} G_j$ by the diagram

$$\begin{tikzcd}
G_j \arrow[d, "I_{g_j}"'] & \prod\limits_{j\in J}^{C} G_j \arrow[l, "\mathrm{pr}_j"'] \arrow[d, "\alpha"] \\
G_j & \prod\limits_{j\in J}^{C} G_j \arrow[l, "\mathrm{pr}_j"]
\end{tikzcd}$$

where $I_{g_j}(x) = g_jxg_j^{-1}$ for $x \in G_j$, and pr_j is the projection. Then $\alpha\left(\prod\limits_{j\in J}^{K} H_j\right)$ is open in $\prod\limits_{j\in J}^{C} G_j$. But

$$\mathrm{pr}_j\alpha\left(\prod_{j\in J}^{K} H_j\right) = I_{g_j}\,\mathrm{pr}_j\left(\prod_{j\in J}^{K} H_j\right) = g_jh_jg_j^{-1} .$$

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So

$$\alpha\left(\prod_{j\in J}^{K} H_j\right) \leq \prod_{j\in J}^{K}\left(g_jH_jg_j^{-1}\right) \leq \prod_{j\in J}^{C} G_j ,$$

and thus $\prod_{j\in J}^{K}\left(g_jH_jg_j^{-1}\right)$ is open in $\prod_{j\in J}^{C} G_j$. Hence

$$\left(\prod_{j\in J}^{K} H_j\right) \cap \left(\prod_{j\in J}^{K} g_jH_jg_j^{-1}\right) = \prod_{j\in J}^{K}\left(H_j \cap g_jH_jg_j^{-1}\right)$$

is open in both $\prod_{j\in J}^{K} H_j$ and $\prod_{j\in J}^{K} g_jH_jg_j^{-1}$. By Lemma 5, $H_j \cap g_jH_jg_j^{-1} = H_j$ for all but a finite number of j and also $H_j \cap g_jH_jg_j^{-1} = g_jH_jg_j^{-1}$ for all but a finite number of j . Hence F_2 is finite. The required set F is $F_1 \cup F_2$.

NOTATION. Let K_j be a compact open normal subgroup of a topological group G_j , for each $j \in J$. Let $\prod_{j\in J} K_j$ be the product of the K_j in *TopGps* . We denote by $\prod_{j\in J}\langle G_j ;\, K_j\rangle$ the topological group which has as its underlying group the cartesian product $\prod_{j\in J} G_j$ and whose topology is determined by letting $\prod_{j\in J} K_j$, with its Tychonoff product topology, be an open subgroup. So $\prod_{j\in J}\langle G_j ;\, K_j\rangle$ is a locally compact group.

LEMMA 10. *Under the hypotheses of Lemma 7, if each* K_j *is a compact open normal subgroup of* G_j *such that* G_j/K_j *is torsion free and* $\prod_{j\in J}\langle G_j ;\, K_j\rangle$ *is in* C *, then* $\prod_{j\in J}\langle G_j ;\, K_j\rangle$ *is the product of* $\{G_j ;\, j \in J\}$ *in* L *, the full subcategory of* C *whose objects are the locally compact* C*-objects.*

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Proof.

$$\begin{tikzcd}
 & \overset{K}{\prod_{j\in J}} K_j \arrow[rr, tail] \arrow[d, "\mathrm{pr}_j"'] & & \prod_{j\in J}\langle G_j;\, K_j\rangle \arrow[d, "\mathrm{pr}_j"'] \\
(G_j)_0 \arrow[r, tail] & K \arrow[rr, tail] & & G_j \\
L_0 \arrow[u] \arrow[r] & U \arrow[u, "f_j'"] \arrow[uu, bend right, "\Theta"'] \arrow[rr] & & L . \arrow[u, "f_j"] \arrow[uu, bend right, "\delta"']
\end{tikzcd}$$

Let L be any locally compact group in C and f_j a family of continuous homomorphisms of L into G_j. Let L_0 be the component of l and $(G_j)_0$ the component of G_j. As K_j is open in G_j, $(G_j)_0 \le K_j$. Clearly $f_j(L_0) \le (G_j)_0$. Let U be an open subgroup of L such that U/L_0 is compact.

Consider the diagram

$$\begin{tikzcd}
U \arrow[r, "f_j"] \arrow[d, "q_3"'] & G_j \arrow[d, "q_1"] \\
U/L_0 \arrow[r, "\phi"'] & G_j/(G_j)_0 \arrow[d, "q_2"] \\
 & G_j/K_j ,
\end{tikzcd}$$

where q_1, q_2, and q_3 are the natural quotient maps. As $f_j(L_0) \le (G_j)_0$, there is an induced continuous homomorphism $\phi : U/L_0 \to G_j/(G_j)_0$. Since U/L_0 is compact and G_j/K_j is torsion free and discrete, $q_2\phi(U/L_0) = \{1\}$. Hence $f_j(U) \le K_j$.

Let f_j' be the restriction of f_j to U. The family of maps

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$f'_j : U \to K_j$ determine a continuous homomorphism $\Theta : U \to \prod_{j\in J}^{K} K_j$ such that $\mathrm{pr}_j\Theta = f'_j$, for each $j \in J$. The maps f_j of L into G_j determine a (not necessarily continuous) homomorphism $\delta : L \to \prod_{j\in J} \langle G_j;\ K_j\rangle$ such that $\mathrm{pr}_j\delta = f_j$, $j \in J$. Noting that $\delta|_U = \Theta$ and that Θ is continuous on the open subgroup U of L, we have that δ is continuous, and the proof of the lemma is complete.

THEOREM. *Let* C *be a full subcategory of* TopGps *with finite limits and* K *a full subcategory of* C *which is complete and has as its objects all compact subgroups of topological groups in* C .

(i) Let $\{G_j;\ j \in J\}$ *be a family in* C *such that* $\prod_{j\in J}^{\mathsf{C}} G_j$ *exists and is locally compact. Then there exists a finite subset* F *of* J *such that for each* $j \in J\backslash F$, *the group* G_j *has a compact open normal subgroup* K_j *with* G_j/K_j *torsion free such that*

$$\prod_{j\in J}^{\mathsf{C}} G_j \simeq \prod_{j\in F} G_j \times \prod_{j\in J\backslash F} \langle G_j;\ K_j\rangle$$

where $\prod_{j\in F}$ *and* $\times$ *denote the usual product in* TopGps .

(ii) Let L *be the full subcategory of* C *having as its objects all locally compact groups in* C . *If* $\{G_j;\ j \in J\}$ *is a family of topological groups in* L *such that for each* $j \in J\backslash F$, *where* F *is a finite set,* G_j *has a compact open normal subgroup* K_j *with* G_j/K_j *torsion free, and if* $\prod_{j\in J\backslash F} \langle G_j;\ K_j\rangle$ *is in* L *then*

$$\prod_{j\in F} G_j \times \prod_{j\in J\backslash F} \langle G_j\,;\ K_j\rangle$$

is the product in L *of* $\{G_j;\ j \in J\}$.

COROLLARY 1. *Let* LC *be the category consisting of all locally compact groups and all continuous homomorphisms between them. Let*

$\{G_j;\ j \in J\}$ *be a family of topological groups in* LC . *Then* $\prod^{LC}_{j\in J} G_j$ *exists if and only if* J *has a finite subset* F *such that for each* $j \in J\setminus F$, G_j *has a compact open normal subgroup* K_j *such that* G_j/K_j *is torsion free.*

COROLLARY 2. *Let* LCA *be the category consisting of all locally compact abelian groups and all continuous homomorphisms between them. Let* $\{G_j;\ j \in J\}$ *be a family of topological groups in* LCA . *Then* $\prod^{LCA}_{j\in J} G_j$ *exists if and only if* J *has a finite subset* F , *such that for each* $j \in J\setminus F$, *the group* G_j *is topologically isomorphic to* $K_j \times D_j$, *where* K_j *is a compact group and* D_j *is a discrete torsion free group. Further if* $\prod^{LCA}_{j\in J} G_j$ *exists, then it is topologically isomorphic to* $\prod_{j\in F} G_j \times \prod_{j\in J\setminus F} K_j \times \prod^{D}_{j\in J\setminus F} D_j$, *where* $\prod$ *denotes the usual product in* TopGps *and* $\prod^{D}_{j\in J\setminus F} D_j$ *denotes the cartesian product group* $\prod_{j\in J\setminus F} D_j$ *with the discrete topology.*

By dualizing Corollary 2 we obtain

COROLLARY 3. *Let* LCA *be the category consisting of all locally compact abelian groups and all continuous homomorphisms between them. Let* $\{G_j;\ j \in J\}$ *be a family of topological groups in* LCA . *Then the coproduct* $\coprod^{LCA}_{j\in J} G_j$ *exists if and only if* J *has a finite subset* F *such that for each* $j \in J\setminus F$, *the group* G_j *is topologically isomorphic to* $A_j \times B_j$, *where* A_j *is a discrete group and* B_j *is a compact connected group. Further if* $\coprod^{LCA}_{j\in J} G_j$ *exists, then it is topologically isomorphic to* $\prod_{j\in J} G_j \times \sum^{D}_{j\in J\setminus F} A_j \times \coprod^{K}_{j\in J\setminus F} B_j$ *where* $\sum^{D}_{j\in J\setminus F} A_j$ *denotes the restricted* *direct sum of the* A_j *with the discrete topology and* $\coprod^{K}_{j\in J\setminus F} B_j$ *the coproduct in the category of compact abelian groups of the* B_j *; that is,* $\coprod^{K}_{j\in J\setminus F} B_j$ *is* $\left(\prod^{D}_{j\in J\setminus F} \hat{B}_j\right)^{\wedge}$ *, where* $\wedge$ *denotes the dual group.*

LEMMA 11. *Let* G *be a closed subgroup of* $K \times D$ *, where* K *is a compact abelian group and* D *is a discrete torsion free abelian group. Then* G *is topologically isomorphic to* $K_1 \times D_1$ *, where* K_1 *is a compact group and* D_1 *is a torsion free group.*

Proof. By duality $\hat{G}$ is a quotient group of $\hat{K} \times \hat{D}$, where $\hat{K}$ is a discrete group and $\hat{D}$ is a compact connected group. So $\hat{G}$ is a quotient group of $F \times \hat{D}$, where F is a (suitable) discrete free abelian group. As $\hat{D}$ is divisible, Proposition 5 of [1] implies that $\hat{G}$ is topologically isomorphic to the product of a compact connected group and a discrete group. Dualizing again yields the required result.

COROLLARY 4. *The largest full subcategory of* LCA *which contains the category of all compact abelian groups and has products (and coproducts) and kernels is the category whose objects are all those topological groups of the form* $K \times D$ *, where* K *is a compact abelian group and* D *is a discrete torsion free abelian group.*

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