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ON THE NON-VANISHING OF A CERTAIN CLASS OF DIRICHLET SERIES

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ABSTRACT. In this paper, we consider Dirichlet series with Euler products of the form $F(s) = \prod_p \left(1 + \frac{a_p}{p^s}\right)$ in $\Re(s) > 1$, and which are regular in $\Re(s) \geq 1$ except for a pole of order m at $s = 1$. We establish criteria for such a Dirichlet series to be non-vanishing on the line of convergence. We also show that our results can be applied to yield non-vanishing results for a subclass of the Selberg class and the Sato-Tate conjecture.

1. **Introduction.** The non-vanishing of L-functions on the line $\Re(s) = 1$ has played a central role in many problems of number theory. The prime number theorem, the Sato-Tate conjecture and the Tchebotarev density theorem are some of the significant consequences of such non-vanishing of L-functions. The Euler product expansion of such L-functions in the half-plane of convergence has always played an important role in establishing their non-vanishing on the line of convergence. In this paper we consider such Euler products and establish criteria under which the Euler product does not vanish on the line of convergence. We also show that our results can be applied to the Selberg class [see S] and the Sato-Tate conjecture.

We shall be considering Dirichlet series which can be written as a product of the form,

$$F(s) = \prod_p \left(1 + \frac{a_p}{p^s}\right) \tag{1}$$

where $a_p \ll 1$ and $F(s)$ is regular in $\Re(s) \geq 1$ except possibly for a pole of order m at $s = 1$.

From (1), we can define in $\Re(s) > 1$,

$$\overline{F(s)} = \prod_p \left(1 + \frac{\overline{a_p}}{p^s}\right). \tag{2}$$

Then it is clear the $\overline{F(s)}$ is also regular in $\Re(s) \geq 1$ except possibly for a pole of order m at $s = 1$. Our goal is to prove

THEOREM 1. *Given, F as in (1), suppose that,*

(a) $F(s)$ *is regular at* $\Re(s) \geq 1$, *and*

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(b) $\prod_p\left(1+\frac{|a_p|^2}{p^s}\right)$ *is regular for* $\Re(s)\geq 1$ *except for a simple pole at* $s=1$,

then $F(1+\mathrm{it})\neq 0$ except possibly for $t=0$.

(c) If (a) and (b) are true in $\Re(s)\geq 1/2$, then we also have, $F(1)\neq 0$.

The hypothesis (c) of Theorem 1 is essential, for consider the example:

$$F(s)=\frac{\zeta(2s)}{\zeta(s)}=\prod_p\left(1+\frac{1}{p^s}\right).$$

In this case, the Euler product in hypothesis (b) has a simple pole at $s=1/2$, and $F(1)=0$.

In our next result we modify condition (c) required in Theorem 1. We prove,

THEOREM 2. *Given, F as in (1), suppose that*

(a) $F(s)$ *is regular for* $\Re(s)\geq 1$, *except for a pole of order m at* $s=1$,

(b) $\prod_p\left(1+\frac{|a_p|^2}{p^s}\right)$ *is regular for* $\Re(s)\geq 1$ *except for a simple pole at* $s=1$ *and,*

(c) $\prod_p\left(1+\frac{{a_p}^2}{p^s}\right)$ *is regular for* $\Re(s)\geq 1$,

then $F(1+\mathrm{it})\neq 0$, $\forall t$ real.

We note that our results are motivated by Rankin's work in [R]. It would be clear in Section 5, that Selberg's conjectures predict that any function F satisfying the above hypothesis, cannot have a simple pole at $s=1$. We consider our results to be a step in that direction.

2. **Lemmas.**

LEMMA 1. *Let f be a function satisfying the following hypothesis:*

(1) f *is holomorphic and non-zero for* $\Re(s)>1$.

(2) On the line $\Re(s)=1$, f *is holomorphic except for a pole of order e at* $s=1$.

(3)

$$\log f(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s},\quad b_n\geq 0.$$

Then any zero of f on the line $\Re(s)=1$ has order $e/2$.

PROOF. For the proof of this lemma we refer the reader to [KM].

LEMMA 2. *Given F as in (1), suppose that* $\prod_p\left(1+\frac{|a_p|^2}{p^s}\right)$ *is regular for* $\Re(s)\geq 1$ *except for a simple pole at* $s=1$. *Then, F has at most a simple pole at* $s=1$.

PROOF. If F is regular at $s=1$, then there is nothing to prove. So suppose that F has a pole of order m at $s=1$. Let s be real and $s\longmapsto 1^+$. Then

$$\log F(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}\sim m\log\frac{1}{s-1}. \tag{3}$$

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But as $s \longmapsto 1^+$,

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \sum_p \frac{a_p}{p^s} + O(1)$$

so that

$$\sum_p \frac{a_p}{p^s} \sim m \log \frac{1}{s-1}, \quad s \longmapsto 1^+. \tag{4}$$

Similarly, from the hypothesis, we have,

$$\sum_p \frac{|a_p|^2}{p^s} + O(1) \sim \log \frac{1}{s-1}. \tag{5}$$

Using Cauchy's inequality, from (2) and (3) as $s \longmapsto 1^+$ we have,

$$\Big|\sum_p \frac{a_p}{p^s}\Big| \leq \sum_p \frac{|a_p|}{p^s} \leq \Big(\sum_p \frac{|a_p|^2}{p^s}\Big)^{1/2} \Big(\sum_p \frac{1}{p^s}\Big)^{1/2}$$

so that $m = 1$ (since $\sum_p \frac{1}{p^s} \sim \log \frac{1}{s-1}$).

3. **Proof of Theorem 1.** If F has no zeroes on the line $\Re(s) = 1$ there is nothing to prove. So assume without loss of generality that F has a zero at $s = 1$. Consider,

$$G(s) = \zeta(s)F(s)\bar{F}(s) \prod_p \Big(1 + \frac{|a_p|^2}{p^s}\Big).$$

Then it is clear that $G(s)$ satisfies the conditions of the above Lemma 1. Hence, $F(1 + \mathrm{it}) \neq 0$ for all $t \neq 0$. Suppose now that the hypothesis (a) and (b) can be extended to $\Re(s) \geq 1/2$. Then, $G(s)$ is analytic in $\Re(s) \geq 1/2$. Since,

$$\log G(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \quad b_n \geq 0, \tag{6}$$

by Landau's theorem the abscissa of convergence is a real singularity σ_0(say). Since $\zeta(s)$ has zeroes in $\Re(s) \geq 1/2$, $\sigma_0 \geq 1/2$. From (6), we have $\log G(\sigma_0) \geq 0$ for $\sigma \geq \sigma_0$. By continuity, $G(\sigma_0) \geq 1$. However $G(\sigma_0) = 0$. This contradiction proves that $F(1) \neq 0$ and hence $F(1 + \mathrm{it}) \neq 0$ for all $t \neq 0$.

4. **Proof of Theorem 2.** Suppose that $F(1 + i\alpha) = 0$ for some real α. Then let $G(s) = F(s + i\alpha)$. Note that $G(1) = 0$. From (1) and (2) we can write F in the form,

$$F(s) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{r_{p^k} e^{i\theta_{p^k}}}{kp^{ks}}\Big). \tag{7}$$

We then have $\sigma > 1$,

$$G(\sigma) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{r_{p^k} e^{-i\alpha \log p^k + i\theta_{p^k}}}{kp^{k\sigma}}\Big). \tag{8}$$

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Consider the product

$$\prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{r_{p^k}^2 e^{-2i\alpha\log p^k + 2i\theta_{p^k}}}{kp^{k\sigma}}\Big) = H(\sigma) \quad \text{(say)}. \tag{9}$$

Using the inequality:

$$2(1 + r\cos\theta)^2 = 2 + r^2 + 4r\cos\theta + r^2\cos 2\theta \geq 0, \tag{10}$$

it follows from (8), (9) and (10) that for $\sigma > 1$,

$$2\log\zeta(\sigma) + \log\Big(\prod_p \Big(1 + \frac{|a_p|^2}{p^\sigma}\Big)\Big) + 4\Re\log G(\sigma) + \Re\log H(\sigma) \geq 0$$

so that

$$\zeta^2(\sigma)\prod_p\Big(1 + \frac{|a_p|^2}{p^\sigma}\Big)|G^4(\sigma)H(\sigma)| \geq 1. \tag{11}$$

By hypothesis the product in (11) has a zero at $s = 1$ (by hypothesis (c), $H(\sigma)$ is analytic at $s = 1$) so that as $\sigma \longmapsto 1^+$ it is $O(\sigma - 1)$, but this contradicts (4).

5. **Applications.** We first show the application of our results to the Selberg class.

The Selberg class S consists of functions of complex variable s satisfying the following properties:

(i) (Dirichlet Series) For $\Re(s) > 1$,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $a_1 = 1$ and we shall write $a_n(F) = a_n$ for the coefficients of Dirichlet Series of F;

(ii) (Analytic Continuation) $F(s)$ extends to a meromorphic function so that $(s - 1)^m F(s)$ is an entire function of finite order for some integer $m \geq 0$;

(iii) (Functional Equation) $\exists$ numbers $Q > 0$, $\alpha_i \geq 0$ and $\beta_i \in C$ with $\Re(\beta_i) \geq 0$ such that the function

$$\phi(s) = \epsilon Q^s \prod_{i=1}^{d} \Gamma(\alpha_i s + \beta_i)F(s)$$

satisfies the functional equation

$$\phi(s) = \bar{\phi}(1 - s)$$

where $\bar{\phi}(s) = \overline{\phi(1 - \bar{s})}$ and ϵ is a complex number of absolute value 1;

(iv) (Euler Product)

$$F(s) = \prod_p F_p(s)$$

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where

$$\log F_p(s) = \sum_{n=1}^{\infty} \frac{b_{p^k}(F)}{p^{ks}}$$

where $b_{p^k}(F) = O(p^{k\theta})$ for some $\theta < 1/2$, p denotes a prime number (here and throughout this paper);

(v) (Ramanujan Hypothesis) $a_n = O(n^\epsilon)$ for any fixed $\epsilon > 0$.

For further details on the Selberg class the reader may refer to [C-G], [RM] and [S]. It has been shown that Selberg's conjectures imply the non-vanishing of functions in S on the line $\Re(s) = 1$.

We shall prove that the non-vanishing property for a certain subclass of S follows from Theorem 1 and Theorem 2.

Given F and G in S we define

$$F \otimes G(s) = \prod_p H_p(s)$$

where

$$H_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{c_{p^k}(F)c_{p^k}(G)}{kp^{ks}}\right)$$

and

$$b_{p^k}(F) = \frac{c_{p^k}(F)}{k}, b_{p^k}(G) = \frac{c_{p^k}(G)}{k}.$$

It is clear that $F \otimes G(s)$ converges absolutely for $\Re(s) > 1$.

We shall call $F \otimes G(s)$ as the tensor product of F and G.

Then, from Theorem 1 we obtain,

THEOREM 3. *If $F \in S$ is entire, and $F \otimes \bar{F}$ is regular in $\Re(s) \geq 1/2$ except for a simple pole at $s = 1$, then $F(1 + \mathrm{it}) \neq 0$, for all t real.*

Similarly from Theorem 2 we obtain

THEOREM 4. *If $F \in S$ has a pole of order m at $s = 1$ and*

(a) $F \otimes \bar{F}$ is regular in $\Re(s) \geq 1$ except for a simple pole at $s = 1$, and

(b) $F \otimes F$ has analytic continuation to line $\Re(s) = 1$, then $F(1 + \mathrm{it}) \neq 0$ for all t real.

It is clear that the non-vanishing property of the Riemann-zeta function, the Dirichlet L-functions attached to a primitive character χ and the Artin L-functions attached to irreducible characters of a Galois extension over Q can all be deduced from Theorems 3 and 4.

We now show that Theorem 1 can be applied to the Sate-Tate conjecture. We first, describe the Sato-Tate conjecture. Let E be an elliptic curve defined over Q. For each prime p, we consider the reduction E_p of E modulo p. Let $a_p = p + 1 - |E_p(F_p)|$ where $|E_p(F_p)|$ is the cardinality of set $E_p(F_p)$ of projective solutions over F_p, the finite field of p-elements (see [2] p. 297). We then have Hasse's inequality, $|a_p| \leq 2\sqrt{p}$. Let us write

$$a_p = \sqrt{p}(e^{i\theta_p} + e^{-i\theta_p}) = 2\sqrt{p}\cos\theta_p.$$

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Sato and Tate conjectured (independently) that if the elliptic curve is not of CM type, then the θ_p's are uniformly distributed with respect to the measure

$$\frac{2}{\pi}\sin^2\theta\,d\theta.$$

Serre (see [5]) reformulated the above conjecture as follows. Let $\alpha_p = e^{i\theta_p}$ and $\beta_p = e^{-i\theta_p}$. For each m, define the L-series

$$L_m(s) = \prod_p \prod_{j=0}^{m}\left(1 - \frac{\alpha_p^{m-j}\beta_p^j}{p^s}\right).$$

Each $L_m(s)$ converges for $\Re(s) > 1$. Serre [Se] showed that if each $L_m(s)$ extends to an entire function and $L_m(1 + it) \neq 0$ for all real t then θ_p's are uniformly distributed with respect to the (Sato-Tate) measure $\frac{2}{\pi}\sin^2\theta\,d\theta$. Hence we have the following corollary of Theorem 1.

COROLLARY 1. *If each $L_m(s)$ has an analytic continuation to $\Re(s) \geq 1/2$ then θ_p's are uniformly distributed with respect to measure $\frac{2}{\pi}\sin^2\theta\,d\theta$.*

Note that K. Murty in his paper [KM] proves a stronger result namely that the analytic continuation of each $L_m(s)$ to $\Re(s) = 1$ alone suffices to imply the Sato-Tate conjecture.

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