ON TOPOLOGICAL STABILITY IN THE GENERAL THREE AND FOUR-BODY PROBLEM

A. Milani and A.M. Nobili

Department of Astronomy, Glasgow University, U.K.
Permanent address: Department of Mathematics,
University of Pisa, Italy.

1. INTRODUCTION

By simple symmetry and change-of-scale considerations the topology of the level manifolds of the classical integrals of the N-body problem is shown to depend only on the value of the integral $z = c^2h$ (total angular momentum squared times total energy). For every hierarchical structure given to the N bodies the problem can be described as a set of N-1 perturbed two-body problems by means of a fitted Jacobian coordinate system; in this setting the Easton inequality, relating potential, momentum of inertia and the z integral, is easily rederived. For N=3 the confinement conditions due to this inequality can be described, in a pulsating synodic reference system, as level lines of a modified potential function on a plane.

For the small parameter $\varepsilon = m_3/m_2$ (mass of the smallest body divided by mass of the secondary body in the main binary) going to zero these level lines reduce to the zero velocity curves of the restricted circular 3-body problem; however, if the two larger masses have an eccentricity $e_2 > 0$, the difference between the actual value of z and its critical value corresponding to the Lagrangian point L_2 contains a "destabilizing" term porportional to e^2_2. By neglecting terms of the order of ε^2 an approximate, and very easy to check, stability criterium is established. Moreover, since it contains a zero order term proportional to e^2_2 it allows also an order-of-magnitude-estimate of the minimum mass m_3 below which no stability at all can be guaranteed on the basis of ten classical integrals only. The minimum mass is given by the reduced mass of the main binary times $e^2_2/2$ and in the Sun-Jupiter-third body system it turns out to be about one half of the mass of the Earth. This means that no stability can be guaranteed in this way for Mercury, Mars, Pluto and, of course, the asteroids.

For $N \geqslant 4$ every hierarchy can be broken: more the hierarchy is strong more easily a close approach of two bodies can be obtained without violating Easton inequality, then the connectedness of the

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collision subset allows any exchange of bodies. However, the time needed to change the z functions of the 3-body subsystems enough to allow such exchange is very long, as can be estimated by a perturbation theory approach using as small parameters not only the mass ratios but also the "scale ratios" among the subsystems.

This paper contains only the statements, with some sketches of the proofs; for a full account the reader should refer to (Milani and Nobili, 1982 and Milani and Nobili, 1983).

2. THE LEVEL MANIFOLDS OF THE CLASSICAL INTEGRALS

The N-body problem, with masses m_i, position $\underline{r}_i$ and velocities $\dot{\underline{r}}_i$ is defined by its kinetic energy T and potential U:

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_i^2 \qquad U=G \sum_{i<j} \frac{m_i m_j}{r_{ij}} \qquad \underline{r}_{ij}=\underline{r}_i-\underline{r}_j. \tag{1}$$

The 10 classical integrals will be denoted by $\underline{\alpha}$ (linear momentum), $\underline{\beta}$ (position of the centre of mass at t=0), $\underline{J}$ (angular momentum), E=T-U (energy). In the 6N-dimensional phase space, if we impose $\underline{\alpha}=\underline{\beta}=0$, $\underline{J}(\underline{r},\dot{\underline{r}})=c$ and $E(\underline{r},\dot{\underline{r}})=h$ we define a manifold $V_{\underline{c},h}$ (generically smooth and 6N-10 dimensional). Then the problem of "topological stability" can be stated in this way: for which N, $\underline{c}$, h does $V_{\underline{c},h}$ have more than 1 connected component (Birkhoff,1927, pp 287-288)? In this case, is the projection of $V_{\underline{c},h}$ on the configuration space also disconnected? A second and more difficult part of the problem is the following: given two open subsets in the same connected component $V_{\underline{c},h}$, is there a solution of the dynamical equations of the N-body problem going from the one to the other? How long does it take?

A major breakthrough occurred in Celestial Mechanics in the seventies with the reply to Birkhoff's old question on the topology of the level manifolds of the classical integrals in the general 3-body problem. The result was that for some values of the energy and of the angular momentum the level manifolds of the classical integrals are topologically disconnected as subsets of the phase-space; moreover, the projections of these disconnected components on the configuration space are also disconnected. Hence forbidden configurations do form a boundary that separates regions of "trapped" motions (Golubev, 1968; Smale 1970a; Marchal 1971; Smale 1970b; Easton, 1971; Tung,1974; Easton, 1975; Marchal and Saari, 1975; Zare,1976 and 1977; Bozis, 1976). Although the relevance of this result for the stability of planetary systems was perceived (Szebehely and Zare,1977; Szebehely and McKenzie,1977; Roy,1979; Walker et al.,1980; Walker and Roy, 1981) there have been some problems in fully exploiting this discovery in assessing the stability of such systems. One of the difficulties is that the proofs of the topological stability criterium use relatively difficult mathematical tools and the computational procedures for actually checking whether it is satisfied or not are long, so that in the process the physical intuition of the meaning of the topological criterium is easily lost.

As it often happens in scientific research, after a new general result has been obtained and it has been definitely assessed, it comes out that there is the possibility to get the same result in an easier way that sometimes gives also some hints for getting further interesting results. That is why we started our work on this subject by giving a new proof of the topological stability criterium for 3-body systems (also in 3 dimensions) that uses only Lagrangian mechanics and elementary calculus (Milani and Nobili, 1982).

If the symmetries of the N-body problem are taken into account it can be shown that the problem of the topological type of $V_{\underline{c},h}$ does not depend really on the 4 parameters $\underline{c}$,h but only on a scalar bifurcation parameter $z = c^2h$. Since T and U are invariant under rotations $R \in SO(3)$ every rotation R maps diffeomorphically $V_{\underline{c},h}$ onto $V_{R\underline{c},h}$ and the topology of $V_{\underline{c},h}$ depends only on c and h. Moreover, since U is a homogeneous function, every change of scale that multiplies all the r_i by a factor $\alpha > 0$ and the time by a factor $\tau > 0$ maps orbits onto orbits provided that α and τ satisfy the "third Kepler law": $\alpha^3/\tau^2 = 1$. (We could also say that the universal constant of gravitation, G, must be invariant under this change of scale). Then $V_{\underline{c},h}$ is mapped diffeomorphically onto $V_{\underline{c}',h'}$ with $c' = c(\alpha^2/\tau)$ and $h' = h/\alpha$. We conclude that there is only one function of $\underline{c}$,h which is invariant under rotations and changes of scale (apart from others functionally dependent) and it is the integral $z = c^2h$.

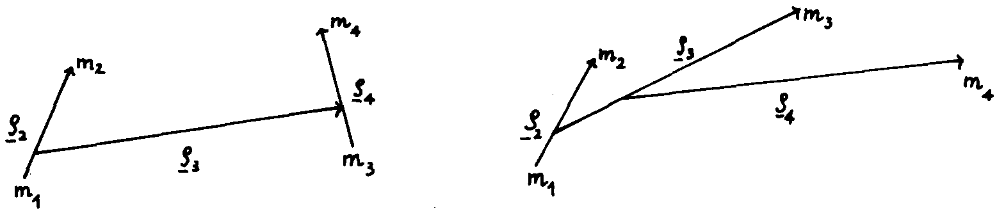
The number of connected components of $V_{\underline{c},h}$ will thus change only when $z = hc^2$ crosses some "critical value"; at the critical value $V_{\underline{c},h}$ is not smooth, the singular points corresponding to the relative equilibrium configurations $\nabla(E - \langle\underline{\omega}, \underline{J}\rangle) = 0$, stationary in a reference frame rotating with angular velocity $\underline{\omega}$ (Smale, 1970b).

3. HIERARCHICAL DYNAMICAL SYSTEMS

The most formal definition of a hierarchy for an N-body system can be given as follows: a hierarchy A is a symbol constructed by using the masses $m_1, m_2, \ldots m_N$, each once and only once, and the operation of forming couples. As an example, for N=4 all the hierarchies are equivalent, by relabelling of the masses or by changing the order in some couples, to one of the following two:

$$B = ((m_1, m_2),(m_3, m_4)) \qquad P = ((m_1, m_2),m_3),m_4) \qquad (2)$$

where B is a double-binary hierarchy and P is a planetary hierarchy (see Figure 1).



B = double-binary hierarchy P = planetary hierarchy

Figure 1

Perhaps a better understanding of the structure of a hierarchy can be obtained by representing it with graphs of the kind introduced by Evans (1968): whenever a couple is formed, an oriented segment is introduced in the graph (Roy, 1982). To write the dynamical equations and the integrals of motion in a way that shows the physical significance of a hierarchy as a set of perturbed 2-body subsystems we need to use Jacobian coordinates (Walker, 1982). They are defined by the reduced masses M_j and by the Jacobian vectors $\underline{\rho}_j$, $j=1,\dots,N$. The $\underline{\rho}_j$ are obtained as linear combinations of the $\underline{r}_i$:

$$\underline{\rho}_j = \sum_{i=1}^{N} a_{ji}\, \underline{r}_i \tag{3}$$

and must satisfy the following four conditions:

(i) the first vector $\underline{\rho}_1$ is the center of mass vector, M_1 is the total mass:

$$M_1 \underline{\rho}_1 = \sum_{i=1}^{N} m_i r_i \qquad M_1 = \sum_{i=1}^{N} m_i \tag{4}$$

(ii) the kinetic energy as a quadratic form in the $\dot{\underline{\rho}}_j$ is diagonal with eigenvalues M_j:

$$2T = \sum_{j=1}^{N} M_j \dot{\underline{\rho}}_j^{\,2} \tag{5}$$

(iii) the product of the reduced masses M_j is equal to the product of the masses m_j:

$$\prod_{i=1}^{N} M_i = \prod_{i=1}^{N} m_i \tag{6}$$

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(iv) the map $\underline{r} \to \underline{\rho}$ preserves orientation; together with (6) this means:

$$\det [a_{ij}] = +1 \tag{7}$$

A Jacobian coordinate system can be chosen in different ways. We will say that a Jacobian system is fitted to a hierarchy A if it is constructed according to the following recursion rule: whenever in A a couple B = (B', B") is formed, the Jacobian vectors of B are in this order: the center of mass of the subsystem B; the Jacobian vectors of B', excluding the first one because it is the center of mass of B'; a new added vector; and the Jacobian vectors of B", excluding the first one again. The same rule applies to the reduced masses, only one of which must be determined. Then the following existence and uniqueness result holds: for every hierarchy A there is one and only one fitted Jacobian coordinate system and it is defined by adding, when the couple (B',B") is formed, a new vector $\underline{\rho}_i$ going from the center of mass of B' to the center of mass of B", and a new reduced mass $M_i = M_i' M_i''/(M_i' + M_i'')$ where M_i', M_i'' are the total masses of B',B".

In the Jacobian coordinates angular momentum and moment of inertia with respect to an axis $\underline{e}$ are of the same form as in the usual coordinates:

$$\underline{c} = \sum_{i=1}^{N} m_i \underline{r}_i \times \dot{\underline{r}}_i = \sum_{i=1}^{N} M_i \underline{\rho}_i \times \dot{\underline{\rho}}_i \tag{8}$$

$$I_{\underline{e}} = \sum_{i=1}^{N} m_i |\underline{r}_i \times \underline{e}|^2 = \sum_{i=1}^{N} M_i |\underline{\rho}_i \times \underline{e}|^2 \tag{9}$$

Let us suppose that the cyclic coordinates $\underline{\rho}_1$ are ignored, or that $\underline{\rho}_1 = \dot{\underline{\rho}}_1 = 0$; the energy integral is:

$$h = T - U(\underline{\rho}) \tag{10}$$

where U is considered as a function of the configuration $\underline{\rho} = (\underline{\rho}_2, \dots, \underline{\rho}_N)$. The angular momentum c (referred to the center of mass) satisfies the inequality

$$c^2 \leqslant I_{\underline{e}} \cdot 2T \tag{11}$$

involving the kinetic energy T and the moment of inertia $I_{\underline{e}}$ referred to the unit angular momentum vector $\underline{e} = \underline{c}/c$.

By using the property of U of being a homogeneous function of degree (-1) we can combine (10) and (11) in an inequality to be satisfied by the configuration $\underline{\rho}$ with given h and $\underline{c}$. Let $\lambda = \sqrt{I_e}$ be the "scale" of the configuration $\underline{\rho}$ and $\underline{u} = \underline{\rho}/\lambda$ the configuration independent from scale; since $U(\underline{u}) = \lambda\, U(\underline{\rho})$ we have

$$2h\lambda^2 + 2U(\underline{u})\,\lambda - c^2 \geq 0 \qquad (12)$$

and the reality condition for the scale λ gives the Easton (1971) inequality

$$U^2(\underline{u}) + 2hc^2 \geq 0 \qquad (13)$$

4. TRAPPING MECHANISMS IN THE GENERAL AND IN THE RESTRICTED THREE-BODY PROBLEM.

Easton inequality (13) reduces the bifurcation problem to the "constant scale configuration manifold" $I_{\underline{e}} = 1$. Since by the projection π on the invariable plane $U(\pi(\underline{\rho})) \geq U(\underline{\rho})$; and the relative equilibria are planar, the planar case always gives all the relevant information on the connected components of the level manifolds. Moreover we can choose a "pulsating synodic" reference system in the invariable plane such that $\pi(\underline{\rho}_2) = (-1,0)$; then the sphere $I_{\underline{e}} = 1$ is parametrized by $\pi(\underline{\rho}_3) = (x,y)$ and the potential U is:

$$U(x,y) = G(M_2 + r^2 M_3)^{\frac{1}{2}} \left[m_1 m_2 + m_3\left(\frac{m_2}{r_2} + \frac{m_1}{r_1}\right) \right] \qquad (14)$$

where $r = (x^2 + y^2)^{\frac{1}{2}}$, $r_1 = ((x-\mu)^2 + y^2)^{\frac{1}{2}}$ and $r_2 = ((x-\mu+1)^2 + y^2)^{\frac{1}{2}}$ are the distances of the projection of m_3 from, respectively, the origin, the projection of m_1 and the projection of m_2 and μ is, as usual, $m_2/(m_1+m_2)$. $M_2 = m_1 m_2/(m_1+m_2)$ and $M_3 = m_3(m_1+m_2)/(m_1+m_2+m_3)$ are the reduced masses. Hence the computation of the number of connected components of $U(x,y) \geq (-2z)^{1/2}$ = constant can be done with the same techniques used to study the zero-velocity curves in the restricted 3-body problem (see Figure 2).

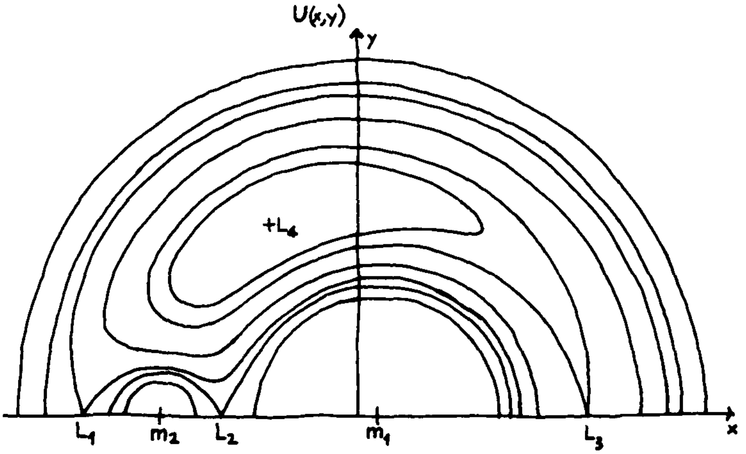

Figure 2

This similarity can be better understood by expanding $U(x,y)$ in power series of the small parameter $\varepsilon = m_3/m_2$, where m_3 is the smallest mass in the system and m_2 the mass of the secondary body in the main binary. We get

$$U(x,y) = A\left[1 + \frac{\varepsilon}{1-\mu}\ \Omega(x,y) + O(\varepsilon^2) \right] \tag{15}$$

where $\Omega(x,y) = r^2/2 + \mu/r_2 + (1-\mu)/r_1$ is the well known function defining the zero-velocity curves in the restricted case and A is a constant, $A = GM_2^3/2\,(m_1 + m_2)$.

The topological stability criterium for a general 3-body problem, completely similar to the Hill stability criterium for the restricted problem, requires the computation of the difference Δz between the actual value z of the c^2h integral and its value z_2 at the Lagrangian point L_2. By expanding z in power series of ε we obtain

$$\frac{z}{M_2^3} = G^2(m_1+m_2)^2\frac{e_2^2-1}{2} + \varepsilon\,\frac{G(m_1+m_2)a_2}{1-\mu}\,J + O(\varepsilon^2)+O(\varepsilon^2 e_2^2) \tag{16}$$

where a_2, e_2, n_2 are the osculating semimajor axis, eccentricity and mean motion of the main binary (obviously changing in time) and J is the"Jacobi" function defined as

$$J = \bar{h}_3 - n_2 \frac{\langle \underline{c}_3 \cdot \underline{c}_2 \rangle}{e_2} \tag{17}$$

$\bar{h}_3$ and $\underline{c}_3$ being the energy and angular momentum of M_3, and $\underline{c}_2$ the angular momentum of M_2 (per unit mass) in the usual Jacobian coordinates. If we expand also z_2 in power series of ε the resulting formula is

$$\frac{z_2}{M_2^3} = -\frac{1}{2}\,G^2(m_1+m_2)^2 + \varepsilon\frac{G(m_1+m_2)a_2}{1-\mu}\ J_2 + O(\varepsilon^2) \tag{18}$$

where J_2 is the Jacobi constant computed at the equilibrium point L_2 of the corresponding restricted problem with the same masses m_1, m_2 and a distance between them equal to a_2. We stress that, with this definition of J_2, formula (18) is correct because it can be proved (Milani and Nobili, 1982) that for a given general 3-body problem the Jacobi function J_2 corresponding to the Lagrangian point L_2 is equal to the Jacobi integral computed at the equilibrium point L_2 of the corresponding restricted 3-body problem apart from terms of the order of ε.

From (16) and (18) we compute now $\Delta z = z - z_2$. By using the usual units of the restricted problem (such that $G = 1$, $m_1 + m_2 = 1$, $a_2 = 1$) we have:

$$\frac{\Delta z}{\mu^3(1-\mu)^3} = \frac{e_2^2}{2} + \varepsilon\frac{\Delta J}{1-\mu} + O(\varepsilon^2) + O(\varepsilon\, e_2^2) \tag{19}$$

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where $\Delta J = J - J_2$ must be less than zero in order to guarantee the stability of the restricted problem according to Hill's criterium. The analogous topological stability criterium in the general 3-body problem requires $\Delta z < 0$. Neglecting terms of the order of ε^2 and terms of the order of εe_2^2 we can give an approximate stability criterium in the general 3-body problem requiring that

$$\frac{\Delta z}{\mu^3(1-\mu)^3} = \frac{e_2^2}{2} + \varepsilon \frac{\Delta J}{1-\mu} < 0. \tag{20}$$

According to the approximate criterium (20) a Hill "unstable" 3-body system (i.e. $\Delta J > 0$) will still be "unstable"–in the sense that we cannot guarantee its stability–in the general case; on the other hand, a Hill stable one (i.e. J of the order of -1 in these units) can be stable in the general case only if the mass of the smallest body satisfies the inequality

$$m_3 \gtrsim \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{e_2^2}{2} \tag{21}$$

(see also Marchal and Bozis, 1982) where the destabilizing effect of the eccentricity of the binary is quantified. For the Sun-Jupiter-third body system the (21) means: $m_3 \gtrsim 0.4\ m_E$, m_E being the mass of the Earth, so that no stability at all can be guaranteed, on the basis of the ten classical integrals only, for small objects like Mercury, Mars, Pluto and the asteroids even if it can be easily shown that they are all stable according to Hill's criterium (see Table 1). One way of understanding why no tiny body can be proved to be topologically stable in the general 3-body problem is that in this case the bifurcation parameter is c^2h, i.e. total angular momentum squared times total energy, and a tiny body contributes very little to it. In other words, Jupiter does not care very much where Mercury, Pluto, Mars or any asteroid is. This does not mean, of course, that they will actually be so much perturbed by Jupiter to cross its orbit. We simply cannot guarantee their stability by using a criterium based on the classical integrals. On the contrary, in the restricted problem the bifurcation parameter is the Jacobi integral, which contains energy and angular momentum of the third body only (per unit mass) so that Hill's criterium is meaningful no matter how small the third body is.

We note that the meaning of the connected components is different for the general and for the restricted 3-body problem. In the restricted problem a zero velocity curve enclosing a bounded region of allowed motion means that the test particle cannot escape; as an example, in the restricted case all the asteroids up to Thule cannot cross the critical 8-shaped Hill's curve, which is fixed in the rotating frame, so that they can never escape (see also Farinella and Nobili, 1978). On the other hand, even if a Hill stable 3-body system can be proved to be stable in the general case too, this does not exclude the escape of one of the three bodies. The reason is that the (x,y) plot must be multiplied by a variable scale factor because it is drawn in a pulsating

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synodic reference system. But if the topological criterium is satisfied (i.e. $\Delta z < 0$) the hierarchy will never be broken, e.g. in the sense that the distance of m_3 from the primary is constrained forever to be smaller than the distance between the primary and the secondary body in the binary.

Table 1 summarizes the results of the exact and the approximate criterium applied to 3-body subsystems of the Solar System and shows the usefulness of the simple approximate criterium.

Table 1: Exact and Approximate Computations of the Stability Parameter in the General 3-Body Problem

3-Body Subsystem: Sun+	Δz	δz	ε	ΔJ	"Stable"
Jupiter-Mercury	+0.000192	+0.000190	1.7×10^{-4}	-5.463	No
Jupiter-Venus	-0.005127	-0.005142	2.6×10^{-3}	-2.448	Yes
Jupiter-Earth	-0.003634	-0.003646	3.1×10^{-3}	-1.520	Yes
Jupiter-EM Center of Mass	-0.003691	-0.003703	3.2×10^{-3}	-1.519	Yes
Jupiter-Mars	+0.000896	+0.000895	3.4×10^{-4}	-0.726	No
Jupiter-Saturn	-0.040128	-0.030410	3.0×10^{-1}	-0.105	Yes
Jupiter-Uranus	-0.025013	-0.023389	4.6×10^{-2}	-0.535	Yes
Jupiter-Neptune	-0.056640	-0.051579	5.4×10^{-2}	-0.971	Yes
Jupiter-Pluto	+0.000752	+0.000751	3.5×10^{-4}	-1.115	No
Earth-Moon	+0.000142	+0.000138	1.2×10^{-2}	+0.0002	No

As far as the Sun-Jupiter-exterior planet case is concerned, the relevant critical value of the c^2h integral is z_1, corresponding to the L_4 relative equilibrium point (see Figure 2). But whenever $m_1 \gg m_2 \gg m_3$ the computation of the critical value z_1 is not needed if the approximate criterium (20) is used because the difference $z_1 - z_2$ turns out to be zero apart from terms of the order of $\varepsilon\mu$ or terms of the order of ε^2.

5. BREAKING FOUR-BODY HIERARCHIES

We will show now that, as stated by Marchal (1971), no topological stability criterium based on the ten classical integrals only can be formulated for a 4-body system, i.e. every 4-body hierarchy could in

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principle be broken. For a fixed invariable plane, i.e. for a fixed $\underline{e}$, and for fixed values of h,c a given configuration $\underline{\rho}(0)$ can be changed along a continuous path $\underline{\rho}(s)$ to a new configuration $\underline{\rho}(1)$ provided that the continuous function $K(s) = I_{\underline{e}} U^2(\underline{\rho}(s))$ never falls below its initial value $K(0) \geqslant -2hc^2$, so that inequality (13) is always fulfilled. This does not necessarily mean that there is a solution of the dynamical equations connecting $\underline{\rho}(0)$ to $\underline{\rho}(1)$; it simply means that the ten classical integrals do not exclude the existence of such a solution.

Let us define $' \equiv d/ds$ and the Jacobian vectors $\underline{\rho}_2$ $\underline{\rho}_3$ and $\underline{\rho}_4$ as in Figure 1. Let us then keep $\underline{\rho}_4$ constant and change the length of $\underline{\rho}_2$ and $\underline{\rho}_3$ in such a way that the moment of inertia remains constant, i.e. $\dot{I}_{\underline{e}} = 0$. Now the question is: can we have $U'(\underline{\rho}(s)) \geqslant 0$ in such a way that Easton inequality (13) is always fulfilled and nevertheless the hierarchy of the 4-body system is finally broken? $I_{\underline{e}} = 0$ means:

$$\rho_3' = -\frac{M_2}{M_3}\,\frac{\rho_2}{\rho_3}\,\rho_2' \tag{22}$$

and we require

$$U'(\underline{\rho}(s)) = \langle\underline{\rho}_2', \nabla_2 U\rangle + \langle\underline{\rho}_3', \nabla_3 U\rangle > 0 \ . \tag{23}$$

In a 4-body system with a double-binary B hierarchy given by (2) (see also Figure 1), the potential U is easily computed as a multipole expansion of the gravitationsl effect of each binary on the center of mass of the other; the mixed terms, that means terms containing both ρ_2/ρ_3 and ρ_4/ρ_3 , do not appear until the fourth order in the ratios ρ_2/ρ_3 and ρ_4/ρ_3 is reached (Milani and Nobili, 1983):

$$\begin{aligned} U = {} & \frac{G\, m_1 m_2}{\rho_2} + \frac{G\, m_3 m_4}{\rho_4} + \\ & + \frac{G(m_1+m_2)(m_3+m_4)}{\rho_3}\Bigg[1+\mu_2(1-\mu_2)\frac{\rho_2^2}{\rho_3^2} P_2(\cos\theta_{23}) + \mu_4(1-\mu_4)\frac{\rho_4^2}{\rho_3^2} P_2(\cos\theta_{43}) + \\ & + \mu_2(1-\mu_2)(1-2\mu_2)\frac{\rho_2^3}{\rho_3^3} P_3(\cos\theta_{23}) + \mu_4(1-\mu_4)(1-2\mu_4)\frac{\rho_4^3}{\rho_3^3} P_3(\cos(\pi-\theta_{43})) + \\ & + \text{4th order terms}\Bigg] \end{aligned} \tag{24}$$

where $\mu_2 = m_2/(m_1+m_2) = \mu$, $\mu_4 = m_4/(m_3+m_4)$; θ_{23} is the angle between $\underline{\rho}_2$ and $\underline{\rho}_3$, θ_{43} the angle between $\underline{\rho}_4$ and $\underline{\rho}_3$; P_2, P_3 are the usual Legendre polynomials. Inequality (23) becomes:

$$U' = \langle \underline{\rho}_2' , \frac{-Gm_1 m_2}{\rho_2^3} \underline{\rho}_2 \rangle \left\{ 1+O\left(\frac{m_3+m_4}{m_1+m_2} \frac{\rho_2^3}{\rho_3^3}\right) - \frac{\rho_2^3}{\rho_3^3} \cdot \frac{m_1+m_2+m_3+m_4}{m_1+m_3} \left[1 + \right.\right.$$

$$\left.\left. +O\left(\mu_2(1-\mu_2)\frac{\rho_2^2}{\rho_3^2}\right) +O\left(\mu_4(1-\mu_4)\frac{\rho_4^2}{\rho_3^2}\right)\right]\right\} > 0 \qquad (25)$$

so that for any "sufficiently" hierarchical 4-body system all the terms inside the curly brackets are small compared to the first one and this inequality can be satisfied by shortening ρ_2 (that means, because of (22), by lengthening ρ_3). As far as a 4-body system with a planetary hierarchy P is considered, the gravitational potential expansion is given by Walker et al (1980) and an inequality similar to (25) can be written, containing different small parameters depending on the different hierarchy.

When ρ_2 is so small that

$$\frac{Gm_1 m_2}{\rho_2} > U(\underline{\rho}(0)) \qquad (26)$$

$U' > 0$ is no more required provided $\rho_2' = 0$; $\underline{\rho}_3, \underline{\rho}_4$ can be rotated at will, with $\rho_2 =$ constant; ρ_3 and ρ_4 can be changed with the condition $M_3\rho_3\rho_3' + M_4\rho_4\rho_4' \geqslant 0$ (i.e. $I' = 0$ with $\rho_2' = 0$) until when

$$\frac{Gm_4 m_2}{r_{24}} > U(\underline{\rho}(0)) ;$$

then, with m_2 and m_4 fixed, m_1 and m_3 can be moved and the hierarchy is definitely broken.

This hierarchy-breaking procedure can be easily understood by considering that the set of collisions on I_e = constant $\neq 0$ is connected for $N > 3$; therefore the sets $U >$ (very large constant) are also connected.

6. SECULAR PERTURBATIONS ON THE c^2h INTEGRAL DUE TO THE FOURTH BODY

Breaking a 4-body hierarchy requires breaking also the hierarchy of a 3-body subsystem stable according to the topological stability criterium. This means that the c^2h function of the subsystem, which is no more an integral of the motion because of the fourth body perturbation, does change by a significant amount. However, if the hierarchy is very strong, the perturbations will surely act very **slowly**. After all, the Solar System is a hierarchical system and 70% of all the observed 3 and 4-body multiple stellar systems are a close pair with a distant companion or two close pairs at a large distance (Voigt, 1974), i.e. they are strongly hierarchical dynamical systems.

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So the relevant question is: how slowly do the perturbations of a fourth body act on a topologically stable 3-body subsystem? Can we estimate the lifetime of the 3-body system against the perturbations of a given fourth body?

Let us consider the case of a double-binary B hierarchy given by (2) and let us restrict for simplicity to the planar case (computations in the planetary P case are similar, although more involved). We want to study the secular time variation $\dot{z}_{23\,sec}$ of the c^2h "integral" z_{23} of the 3-body subsystem $((m_1, m_2), m_3+m_4)^{23}$ (see Figure 3) due to the fact that (m_3+m_4) is not actually a point-mass but a binary.

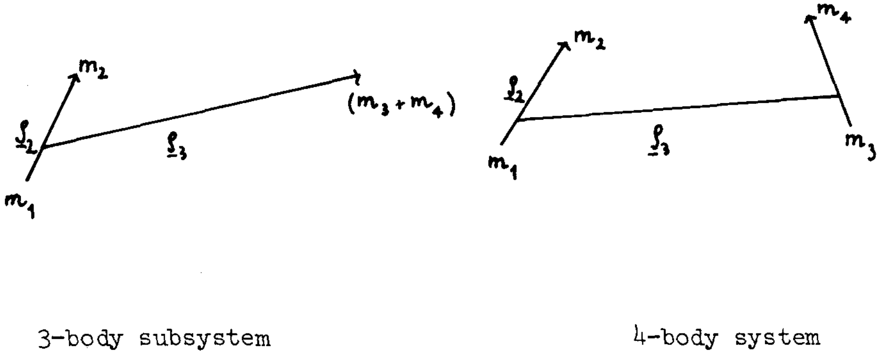

Figure 3

$z_{23} = c_{23}^2\, h_{23}$ can be computed from the angular momentum and energy of the 3-body subsystem, which are given by:

$$h_{23} = M_2 h_2 + M_3 h_3 - R_{23} \qquad c_{23} = M_2 c_2 + M_3 c_3 \tag{27}$$

where h_2, c_2 and h_3, c_3 are the energy and angular momentum (per unit mass) of the two binaries (m_1, m_2) with Jacobian vector $\underline{\rho_2}$ and $(m_1 + m_2, m_3 + m_4)$ with Jacobian vector $\underline{\rho_3}$ assumed as unperturbing each other, while R_{23} is the interaction potential:

$$R_{23} = \frac{G(m_1+m_2)(m_3+m_4)}{\rho_3}\ \mu_2(1-\mu_2)\left[\frac{\rho_2^2}{\rho_3^2} P_2(\cos\theta_{23}) + O\left(\frac{\rho_2^3}{\rho_3^3}\right)\right] \tag{28}$$

so that the relationship between h_3, R_{23} and $\bar{h}_3$ as defined in section 4 is simply

$$\bar{h}_3 = h_3 + R_{23} \tag{29}$$

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To compute $\dot{z}_{23}$ we would like to apply the usual techniques of perturbation theory, but one more difficulty arises because the small parameters with respect to which the perturbing functions can be developed as in (28) should be independent from the dynamical variables $\underline{\rho}_i$, $\dot{\underline{\rho}}_i$. To overcome this difficulty we consider an auxiliary de-hierarchized system in which the initial $\underline{\rho}_3$ has been shortened by a factor $\lambda (0 < \lambda < 1)$ and the masses have been changed in such a way that the ratios $\mu_2(1-\mu_2)$ and $\mu_4(1-\mu_4)$ are divided by a factor $\sigma(0 < \sigma < 1)$. If we assume that the initial 4-body system (i.e. without shortening) is the actual hierarchical dynamical system whose stability we are investigating, and that the de-hierarchized system has a ratio between its Jacobian vectors of the order of 1 and very large μ (i.e. $\mu_2 = \mu_4 \simeq 1/2$ in the de-hierarchized system), then λ is of the order of $\text{Max}(\rho_2/\rho_3, \rho_4/\rho_3)$ and σ is of the order of $\text{Max}\ (\mu_2(1-\mu_2), \mu_4(1-\mu_4))$, ρ_2, ρ_3, ρ_4 and μ_2, μ_4 being the actual ones. Let us call S_{j3} (j=2 or 4) the perturbing functions of the two de-hierarchized 3-body systems with Jacobian radius vectors $\underline{\rho}_2$, $\lambda\,\underline{\rho}_3$ and $\underline{\rho}_4$ $\lambda\underline{\rho}_3$. Let us also call R_{j3} (j=2 or 4) the perturbing functions of the corresponding 3-body systems where $\underline{\rho}_3$ has not been shortened; R_{23} is given by (28) and R_{43} is obviously analogous. Then, the relationship between R_{j3} and S_{j3} is simply

$$R_{j3} = \lambda^3\ \sigma\ S_{j3} + O(\lambda^4\,\sigma) \tag{30}$$

The potential function U of the 4-body system, given by (24), contains also a "mixed term", i.e.:

$$U = \frac{G\,m_1 m_2}{\rho_2} + \frac{G\,m_3 m_4}{\rho_4} + \frac{G(m_1+m_2)(m_3+m_4)}{\rho_3} + R_{23} + R_{43} + R_{24} \tag{31}$$

where R_{24} contains the mixed terms but is of higher order, as can be seen from (24), that is, comparing with the de-hierarchized system:

$$R_{24} = \sigma^2\lambda^5\ S_{24} + O(\sigma^2\ \lambda^6)\quad . \tag{32}$$

We now compute the time derivative of z_{23}, using e.g. a Poisson bracket formalism with the Hamiltonian of the full 4-body system:

$$h = M_2h_2 + M_3h_3 + M_4h_4 - R_{23} - R_{34} - R_{24}$$

where h_4 is the energy, per unit mass, of the binary (m_3, m_4) with Jacobian radius vector $\underline{\rho}_4$ and reduced mass $M_4 = m_3m_4/(m_3+m_4)$. For $\dot{z}_{23}$ (in the planar case) we get:

$$\begin{aligned}\dot{z}_{23} &= \{z_{23},h\} = c_{23}^2\ \{h_{23},h\} + 2c_{23}h_{23}\,\{c_{23},h\} = \\ &= c_{23}^2\left\{ n_3\ \frac{\partial(R_{43}+R_{24})}{\partial \ell_3} + n_2\ \frac{\partial R_{24}}{\partial \ell_2} + \{R_{23}, R_{43}+R_{24}\}\right\} - \\ &-2h_{23}\ c_{23}\ \frac{\partial\ (R_{43}+R_{24})}{\partial g_4}\end{aligned} \tag{33}$$

where ℓ_j, g_j (j=2,3,4) are the usual angular variables of the binary system with Jacobian radius vector $\underline{\rho}_j$, i.e. ℓ_j are the mean anomalies and g_j the arguments of the pericenters.

Before exploiting (30) and (32) to estimate the orders of magnitude, let us remember that we are interested only in secular effects because we assumed that the initial configuration is such that z_{23} is smaller than the critical value , and therefore short term perturbations do not affect stability (close approaches are impossible before a large enough change in z_{23} takes place). We can therefore apply the usual technique of averaging so that terms containing $\partial/\partial\ell_2$ give zero when averaged over ℓ_2 and terms containing $\partial/\partial\ell_3$ give zero when averaged with respect to ℓ_3 (this is the so-called "Lagrange theorem on the stability of the Solar System"), and therefore only second order terms are left in the long-term evolution of z_{23} . Also the $\partial/\partial g_4$ terms average out whenever $e_4 = 0$ or $e_2 = e_2 = 0$ (e_j, j=2,3,4, being the osculating eccentricity of the binary with Jacobian radius vector $\underline{\rho}_j$), i.e. the long term evolution of z_{23} can be described as

$$\dot{z}_{23} \text{ (long term)} \simeq \text{2nd order terms} + O(e_2e_4) + O(e_3e_4) \qquad (34)$$

By using the estimates (30) and (32) of the perturbing functions we finally get (see Milani and Nobili, 1983):

$$\dot{z}_{23} \text{ (long term)} \simeq O(\lambda^6\sigma^2) + O(\lambda^3\sigma e_3e_4) + O(\lambda^5\sigma^2e_2e_4). \qquad (35)$$

We can comment formula (35) by saying that it provides a significant order-of-magnitude upper estimate of the time needed to break the stability of the $((m_1,m_2),m_3 + m_4)$ subsystem:

$$\Delta t \gtrsim -\Delta z_{23}/\dot{z}_{23}\text{(long term)} \qquad (36)$$

(where Δz_{23} is z_{23} minus the critical value corresponding to the L_2 equilibrium for the same masses m_1, m_2 and $m_3 + m_4$), provided that also the analogous z_{43} for the $(m_1+ m_2,(m_3, m_4))$ system is controlled in a similar way. However, as usual in perturbation theory, the order of magnitude estimates that depend upon the "principle of the averages" as (34) and (35) hold only in the assumption that no significant reasonance occurs between the three mean motions n_2, n_3 and n_4 .

REFERENCES:

Birkhoff, G.D. : 1927, "Dynamical Systems", AMS, Providence.
Bozis, G.: 1976, "Zero Velocity Surfaces for the General Planar 3-Body Problem", Astroph. and Space Sci., 43, 355-368.
Easton, R.: 1971, "Some Topology of the 3-Body Problem", Journal of Diff. Equations, 10, 371-377.
Easton, R.: 1975, "Some Topology of the n-Body Problems", Journal of Diff. Equations, 19, 258-269.
Evans, D.S.: 1968, "Stars of Higher Multiplicity", Quarterly J.Roy. Astr. Soc. 9, 388.

Farinella, P. and Nobili, A.M. : 1978, "A Simple Explanation of Some Characteristics of the Asteroidal Belt Based on the Restricted 3-Body Problem". Moon and Planets 18, 241-250.
Golubev, V.G. : 1968, "Hill Stability in the Unrestricted 3-Body Problem" Soviet Phys. Dokl. 13, 373.
Marchal, C. : 1974, "Qualitative Study of a N-Body System: a New Condition of Complete Scattering". Astron. Astrophys. 10, 278-289.
Marchal, C. and Saari, D.G.: 1975, "Hill Regions for the General 3-Body Problem", Cel. Mech. 12, 115-129.
Marchal, C. and Bozis, G.: 1982, "Hill Stability and Distance Curves for the General 3-Body Problem". Cel. Mech. 26, 311-333.
Milani, A. and Nobili, A.M.: 1982, "On Topological Stability in the General 3-Body Problem". Cel. Mech., in press.
Milani, A. and Nobili, A.M.: 1983, "On Topological Stability in the 4-Body Systems". In preparation.
Roy, A.E.: 1979, "Empirical Stability Criteria in the Many-Body Problem." In "Instabilities in Dynamical Systems". V. Szebehely, Ed., Reidel, Dordrecht, pp.177-210.
Roy, A.E.: 1982, "The Stability of N-Body Hierarchical Dynamical Systems." In "Applications of Modern Dynamics to Celestial Mechanics and Astrodynamics". V. Szebehely, Ed., Reidel, Dordrecht, pp.103-130.
Smale, S.: 1970a, "Topology and Mechanics, I." Inventiones Math. 10, 305-331.
Smale, S.: 1970b, "Topology and Mechanics, II. "Inventiones Math. 11, 45-64.
Szebehely, V. and Zare, K. : 1977, "Stability of Classical Triplets and of their Hierarchy". Astron. Astrophys. 58, 145-152.
Szebehely, V. and McKenzie, R.: 1977, "Stability of Planetary Systems with Bifurcation Theory". Astronomical Journal 82, 79-83
Tung, C.C.: 1974, "Some Properties of the Classical Integrals of the General Problem of three Bodies". Scientia Sinica, 17, 306-330.
Voigt, H.H. 1974, "Outline of Astronomy" Vol.II, Noordhoff International Publishing, Leyden.
Walker, I.W., Emslie G.A., and Roy, A.E.: 1980, "Stability Criteria in Many Body Systems, I." Celestial Mechanics, 22, 371-402.
Walker, I.W. and Roy, A.E.: 1981, "Stability Criteria in Many Body Systems, II". Celestial Mechanics, 24, 195-225.
Walker, I.W.: 1983, "Stability Criteria in Many Body Systems, IV", Celestical Mechanics, 29, 149.
Zare, K.: 1976, "The Effects of Integrals on the Totality of Solutions of Dynamical Systems". Celestial Mechanics, 14, 73-83
Zare, K.: 1977, "Bifurcation Points in the Planar Problem of Three Bodies". Celestial Mechanics, 16, 35-38.