Proceedings of the Edinburgh Mathematical Society (1985) **28**, 9–11 ©

ON IDEALLY FINITE LIE ALGEBRAS WHICH ARE LOWER SEMI-MODULAR

by DAVID TOWERS

(Received 19th October 1983)

The purpose of this paper is twofold: first to correct the statement of Theorem 1 in [4], and secondly to consider related problems in the class of ideally finite Lie algebras.

Throughout, L will denote a Lie algebra over a field K, $F(L)$ will be its Frattini subalgebra and $\phi(L)$ its Frattini ideal. We will denote by $\mathfrak{X}$ the class of Lie algebras all of whose maximal subalgebras have codimension 1 in L. The Lie algebra with basis $\{u_{-1}, u_0, u_1\}$ and multiplication $u_{-1}u_0 = u_{-1}$, $u_{-1}u_1 = u_0$, $u_0u_1 = u_1$ will be labelled $L_1(0)$.

Theorem 1 of [4] claimed that a necessary and sufficient condition for L to belong to $\mathfrak{X}$ is that $L/\phi(L) = S \oplus R$, where S is a simple ideal isomorphic to $L_1(0)$, or is $\{0\}$, and R is a supersoluble ideal of $L/\phi(L)$ (possibly $\{0\}$). The necessity is correct, but not the sufficiency. The only problem is that $L_1(0)$ may not belong to $\mathfrak{X}$; when it does is described in the following lemma.

Lemma 0. *$L_1(0) \in \mathfrak{X}$ if and only if K has characteristic two, or $\sqrt{K} = \{\sqrt{k} : k \in K\} \subseteq K$.*

Proof. If K has characteristic two, then $F(L_1(0))$ is spanned by u_0 and so all maximal subalgebras of $L_1(0)$ are two dimensional.

Let $S(\lambda, \mu, \nu)$ be the 1-dimensional subalgebra of $L_1(0)$ spanned by $\lambda u_{-1} + \mu u_0 + \nu u_1$ $(\lambda, \mu, \nu \in K)$. Then any 1-dimensional subalgebra of $L_1(0)$ is of the form $S(\lambda, \mu, 0)$ or $S(\lambda, \mu, 1)$. If $\sqrt{K} \subseteq K$ then $S(\lambda, \mu, 0)$ is contained in the subalgebra spanned by u_0 and u_{-1}, and $S(\lambda, \mu, 1)$ is contained in that spanned by $\lambda u_{-1} + \mu u_0 + u_1$ and $\alpha u_1 - u_{-1}$ where $\alpha^2\lambda^2 + 2\alpha(\lambda - \mu^2) + 1 = 0$.

If $\sqrt{K} \nsubseteq K$, let $\alpha \in \sqrt{K}$, $\alpha \notin K$. Then, when the characteristic of K is different from two, the subalgebra of $L_1(0)$ spanned by $(\alpha^2/2)u_1 - u_{-1}$ is maximal.

Using the above lemma and the fact that L is supersoluble whenever $L/\phi(L)$ is supersoluble ([1], Theorem 6), we can correct Theorem 1 of [4] as follows.

Theorem 1. *Let L be a finite-dimensional Lie algebra.*

(i) *If $\sqrt{K} \nsubseteq K$ and K has characteristic different from 2, then $L \in \mathfrak{X}$ if and only if L is supersoluble.*

(ii) *If $\sqrt{K} \subseteq K$ or K has characteristic two, then $L \in \mathfrak{X}$ if and only if $L/\phi(L) = S \oplus R$ where S is a simple ideal of $L/\phi(L)$ isomorphic to $L_1(0)$, or is $\{0\}$, and R is a supersoluble ideal of $L/\phi(L)$ (possibly $\{0\}$).*

https://doi.org/10.1017/S0013091500003151 Published online by Cambridge University Press

The Lie algebra L is *lower semi-modular* if, whenever U, V are distinct subalgebras of L both of which are maximal in the subalgebra W of L, then $U \cap V$ is maximal in both U and V. Recall that L is *ideally finite* if every element of L lies in a finite-dimensional ideal of L. The reader is referred to [3] for any results on ideally finite Lie algebras which are used. Also following Stewart in [3] we call L *hypercyclic* if it has an ascending series of ideals $(L_\alpha)_{\alpha \leqq \sigma}$ such that $\dim L_{\alpha+1}/L_\alpha = 1$ for all $\alpha < \sigma$.

Our main result is the following.

Theorem 2. *Let L be an ideally finite Lie algebra. Then the following are equivalent.*

(i) *L is locally supersoluble.*
(ii) *$L \in \mathfrak{X}$ and is locally soluble.*
(iii) *L is hypercyclic.*
(iv) *L is locally soluble and lower semi-modular.*

Proof. (i)⇒(ii): Suppose that L is supersoluble; then L is clearly locally soluble. Let M be a maximal subalgebra of L and pick $x \notin M$. Then there is an ideal I of L with $x \in I$ and $\dim I < \infty$. Now $L = M + I$, so M has finite codimension in L. Put $C = C_L(I) = \{x \in L: xI = 0\}$, so that $\dim L/C < \infty$. If $C \subseteq M$, then M/C is a maximal subalgebra of L/C, which is supersoluble, and so M has codimension 1 in L. So suppose that $C \nsubseteq M$, and hence that $L = C + M$. But $C \cap M$ is an ideal of L, and so $L/C = (C+M)/C \cong M/C \cap M$. It follows that $C \cap M$ has finite codimension in M and hence in L. Thus $M/C \cap M$ is a maximal subalgebra of $L/C \cap M$, which is soluble, and again M has codimension 1 in L.

(ii)⇒(iii): Suppose that $L \in \mathfrak{X}$ and is locally soluble. We need only prove that the minimal ideals of L are 1-dimensional. Clearly we may assume that the centre of L is trivial, and hence that L is residually finite. Let A be a minimal ideal of L. Then A is finite dimensional and so, by Lemma 3.4 of [3], there is an ideal K of L with $\dim L/K < \infty$ and $K \cap A = \{0\}$. Now all maximal subalgebras of L/K have codimension 1 in L/K, and L/K is soluble. Hence L/K is supersoluble. But $A \cong (A+K)/K$, which is a minimal ideal of L/K, and so is 1-dimensional.

(iii)⇒(i): Let L be hypercyclic and let U be a finitely generated subalgebra of L. Then $U \subseteq I$ where I is a finite-dimensional ideal of L. Clearly I, and hence A, is supersoluble.

(ii)⇒(iv): Let $L \in \mathfrak{X}$ be locally soluble, and let U, V be distinct subalgebras of L, both of which are maximal in the subalgebra W of L. Since (ii) is equivalent to (i), and hypothesis (i) is subalgebra closed, we may assume that $W = L$. Then U, V have codimension 1 in L, so $L = U + V$. Clearly, $U \cap V$ has codimension 1 in both U and V, and so L is lower semi-modular.

(iv)⇒(i). Let L be locally soluble and lower semi-modular, and let U be a finitely generated subalgebra of L. Then U is a finite-dimensional soluble Lie algebra which is lower semi-modular. It follows from Lemma 5 of [2] that U is supersoluble.

We can deduce from the above result the following analogue of Theorem 6 of [1].

Corollary 3. *Let L be an ideally finite Lie algebra, let A be an ideal of L with $A \subseteq \phi(L)$ (the Frattini ideal of L), and suppose that L/A is hypercyclic. Then L is hypercyclic.*

Proof. We have that L/A is locally soluble, and A is locally nilpotent (see [3]), so L

is locally soluble. Furthermore, all maximal subalgebras of L contain A, and so have codimension 1 in L, by Theorem 2. It follows from Theorem 2 again that L is hypercyclic.

REFERENCES

1. D. W. BARNES, On the cohomology of soluble Lie algebras, *Math. Z.* **101** (1967) 343–349.

2. A. G. GEIN, Semimodular Lie algebras, *Siberian Math. J.* **17** (1976), 189–193.

3. I. N. STEWART, *Lie algebras generated by finite-dimensional ideals* (Pitman 1975).

4. D. A. TOWERS, Lie algebras, all of whose maximal subalgebras have codimension 1, *Proc. Edinburgh Math. Soc.* **24** (1981), 217–219.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LANCASTER
LANCASTER
LA1 4YL