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CLASSIFICATION OF CUBIC HOMOGENEOUS POLYNOMIAL MAPS WITH JACOBIAN MATRICES OF RANK TWO

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Abstract

Let K be any field with char $K \neq 2, 3$. We classify all cubic homogeneous polynomial maps H over K whose Jacobian matrix, $\mathcal{J}H$, has $\operatorname{rk} \mathcal{J}H \leq 2$. In particular, we show that, for such an H, if $F = x + H$ is a Keller map, then F is invertible and furthermore F is tame if the dimension $n \neq 4$.

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1. Introduction

Let K be an arbitrary field and $K[x] := K[x_1, x_2, \ldots, x_n]$ the polynomial ring in n variables. For a polynomial map $F = (F_1, F_2, \ldots, F_m) \in K[x]^m$, we denote by $\mathcal{J}F := (\partial F_i/\partial x_j)_{m\times n}$ the Jacobian matrix of F and $\deg F := \max_i \deg F_i$ the degree of F. A polynomial map $H \in K[x]^m$ is called homogeneous of degree d if each H_i is zero or homogeneous of degree d.

A polynomial map $F \in K[x]^n$ is called a Keller map if $\det \mathcal{J}F \in K^*$. The Jacobian conjecture asserts that any Keller map is invertible if char $K = 0$ (see [1, 8]). It is still open for any dimension $n \geq 2$.

Following [14], we call a polynomial automorphism elementary if it is of the form $(x_1, \ldots, x_{i-1}, cx_i + a, x_{i+1}, \ldots, x_n)$, where $c \in K^*$ and $a \in K[x]$ contains no x_i. Furthermore, we call a polynomial automorphism tame if it is a finite composition of elementary ones. The definitions of elementary and tame may be different in other sources, but (as long as K is a generalised Euclidean ring) the definitions of tame are equivalent. The tame generators problem asks if every polynomial automorphism is tame. It has an affirmative answer in dimension two for arbitrary characteristic

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(see [10, 11]) and a negative answer in dimension three for the case of char $K = 0$ (see [14]), and is still open for any $n \geq 4$.

A polynomial map $F = x + H \in K[x]^n$ is called triangular if $H_n \in K$ and $H_i \in K[x_{i+1}, \ldots, x_n]$, for $1 \leq i \leq n-1$. A polynomial map F is called linearly triangularisable if it is linearly conjugate to a triangular map, that is, there exists an invertible linear map $T \in \mathrm{GL}_n(K)$ such that $T^{-1}F(Tx)$ is triangular. A linearly triangularisable map is tame.

Some special polynomial maps have been investigated in the literature. For example, when char $K = 0$, a Keller map $F = x + H \in K[x]^n$ is shown to be linearly triangularisable in the cases: (1) $n = 3$ and H is homogeneous of arbitrary degree d (de Bondt and van den Essen [6]); (2) $n = 4$ and H is quadratic homogeneous (Meisters and Olech [12]); (3) $n = 9$ and F is a quadratic homogeneous quasi-translation (Sun [16]); (4) n is arbitrary and H is quadratic with $\mathrm{rk}\,\mathcal{J}H \leq 2$ (de Bondt and Yan [7]), and to be tame in the case (5) $n = 5$ and H is quadratic homogeneous (de Bondt [2] and Sun [17] independently) and to be invertible in the case (6) $n = 4$ and H is cubic homogeneous (Hubbers [9]). For the case of arbitrary characteristic, de Bondt [5] described the Jacobian matrix $\mathcal{J}H$ of rank two for any quadratic polynomial map H and showed that if $\mathcal{J}H$ is nilpotent then $\mathcal{J}H$ is similar to a triangular map.

In this paper, we study cubic homogeneous polynomial maps H with $\mathrm{rk}\,\mathcal{J}H \leq 2$ for any dimension n when char $K \neq 2, 3$. In Section 2, we classify all such maps (Theorem 2.7). In Section 3, we show that for such an H, if $F = x + H$ is a Keller map, then it is invertible and furthermore it is tame if the dimension $n \neq 4$ (Theorem 3.4).

2. Cubic homogeneous maps H with rk $\mathcal{J}H \leq 2$

For a polynomial map $H \in K[x]^m$, we write $\mathrm{trdeg}_K K(H)$ for the transcendence degree of $K(H)$ over K. It is well known that $\mathrm{rk}\,\mathcal{J}H = \mathrm{trdeg}_K K(H)$ if $K(H) \subseteq K(x)$ is separable and, in particular, if char $K = 0$ (see [8, Proposition 1.2.9]). For arbitrary characteristic, one has $\mathrm{rk}\,\mathcal{J}H \leq \mathrm{trdeg}_K K(H)$ (see [4] or [13]).

It was shown in [5] that when char $K \neq 2$, for any quadratic polynomial map H with $\mathrm{rk}\,\mathcal{J}H \leq 2$, one has $\mathrm{rk}\,\mathcal{J}H = \mathrm{trdeg}_K K(H)$. We will show that when char $K \neq 2, 3$, for any cubic homogeneous polynomial map H with $\mathrm{rk}\,\mathcal{J}H \leq 2$, one has $\mathrm{rk}\,\mathcal{J}H = \mathrm{trdeg}_K K(H)$. The notation $a|_{x=c}$ below means to substitute x by c in a.

Theorem 2.1. *Let $s \leq n$. Take $\tilde{x} := (x_1, x_2, \ldots, x_s)$ and $L := K(x_{s+1}, x_{s+2}, \ldots, x_n)$. To prove that for (homogeneous) polynomial maps $H \in K[x]^m$ of degree d,*

$$\mathrm{rk}\,\mathcal{J}H = r \quad \textit{implies} \quad \mathrm{trdeg}_K K(H) = r, \quad \textit{for every } r < s, \tag{2.1}$$

it suffices to show that for (homogeneous) polynomial maps $\tilde{H} \in L[\tilde{x}]^s$ of degree d,

$$\mathrm{trdeg}_L L(\tilde{H}) = s \quad \textit{implies} \quad \mathrm{rk}\,\mathcal{J}_{\tilde{x}}\tilde{H} = s. \tag{2.2}$$

Proof. Suppose that $H \in K[x]^m$ is (homogeneous) of degree d such that (2.1) does not hold. Then there exists an $r < s$ such that $\mathrm{rk}\,\mathcal{J}H = r < \mathrm{trdeg}_K K(H)$. We need to show that (2.2) does not hold.

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Let $s' = \operatorname{trdeg}_K K(H)$. Assume without loss of generality that $H_1, H_2, \ldots, H_{s'}$ are algebraically independent over K and that the components of

$$H' := (H_1, H_2, \ldots, H_{s'}, x_{s'+1}^d, x_{s'+2}^d, \ldots, x_s^d)$$

are algebraically independent over K if $s' < s$. Then

$$\operatorname{rk} \mathcal{J}H' \le r + (s - s') < s = \operatorname{trdeg}_K K(H').$$

For the case of $s' \ge s$, just take $H' = (H_1, H_2, \ldots, H_s)$, again giving $\operatorname{rk} \mathcal{J}H' \le r < s$. Notice that (2.1) is also unsatisfied for H'. So, replacing H by H', we may assume that $H \in K[x]^s$ with $\operatorname{rk} \mathcal{J}H = r < \operatorname{trdeg}_K K(H) = s$.

Observe that $H_1(x_1, x_1x_2, x_1x_3, \ldots, x_1x_n)$ is algebraically independent over K of $x_2, x_3, \ldots, x_n$. On account of the Steinitz–MacLane exchange lemma, we may assume without loss of generality that the components of

$$(H(x_1, x_1x_2, x_1x_3, \ldots, x_1x_n), x_{s+1}, x_{s+2}, \ldots, x_n)$$

are algebraically independent over K. Then the components of

$$H(x_1, x_1x_2, x_1x_3, \ldots, x_1x_n)$$

are algebraically independent over $L := K(x_{s+1}, x_{s+2}, \ldots, x_n)$ and so are the components of

$$\tilde{H} := H(x_1, x_2, \ldots, x_s, x_1x_{s+1}, x_1x_{s+2}, \ldots, x_1x_n) \in L[\tilde{x}]^s,$$

where $\tilde{x} = (x_1, x_2, \ldots, x_s)$. That is, $\operatorname{trdeg}_L L(\tilde{H}) = s$.

Let $G := (x_1, x_2, \ldots, x_s, x_1x_{s+1}, x_1x_{s+2}, \ldots, x_1x_n)$. From the chain rule,

$$\mathcal{J}_{\tilde{x}}\tilde{H} = (\mathcal{J}H)|_{x=G} \cdot \mathcal{J}_{\tilde{x}}G,$$

so $\operatorname{rk} \mathcal{J}_{\tilde{x}}\tilde{H} \le \operatorname{rk}(\mathcal{J}H)|_{x=G} \le \operatorname{rk} \mathcal{J}H < s$. Therefore, (2.2) does not hold for $\tilde{H}$, which completes the proof. □

Lemma 2.2. *Let $H \in K[x]^m$ be a polynomial map of degree d and $r := \operatorname{rk} \mathcal{J}H$. Denote by $|K|$ the cardinality of K.*

(i) *If $|K| > (d-1)r$ and $\mathcal{J}H \cdot x = 0$, then there exist $S \in \mathrm{GL}_m(K)$ and $T \in \mathrm{GL}_n(K)$ such that for $\tilde{H} := SH(Tx)$,*

$$\tilde{H}|_{x=e_{r+1}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

(ii) *If $|K| > (d-1)r + 1$ and $\mathcal{J}H \cdot x \ne 0$, then there exist $S \in \mathrm{GL}_m(K)$ and $T \in \mathrm{GL}_n(K)$ such that for $\tilde{H} := SH(Tx)$,*

$$\tilde{H}|_{x=e_1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, $|K|$ may be one less (that is, at least $(d-1)r$ and $(d-1)r+1$, respectively) if every nonzero component of H is homogeneous.

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Proof. (i) Assume without loss of generality that

$$a_0 := \det \mathcal{J}_{x_1,x_2,\dots,x_r}(H_1, H_2, \dots, H_r) \neq 0.$$

Suppose that $|K| > (d-1)r$. It follows by [3, Lemma 5.1(i)] that there exists a vector $w \in K^n$ such that $a_0(w) \neq 0$. So, $\mathrm{rk}(\mathcal{J}H)|_{x=w} = r$. There exist $n-r$ independent vectors $v_{r+1}, v_{r+2}, \dots, v_n \in K^n$ such that $(\mathcal{J}H)|_{x=w} \cdot v_i = 0$ for $i = r+1, r+2, \dots, n$. We may take $v_{r+1} = w$ since

$$(\mathcal{J}H)|_{x=w} \cdot w = (\mathcal{J}H \cdot x)|_{x=w} = 0.$$

Take $T = (v_1, v_2, \dots, v_n) \in \mathrm{GL}_n(K)$. From the chain rule, we deduce that

$$(\mathcal{J}(H(Tx)))|_{x=e_{r+1}} \cdot e_i = (\mathcal{J}H)|_{x=Te_{r+1}} \cdot Te_i = (\mathcal{J}H)|_{x=w} \cdot v_i \quad (\text{for } 1 \leq i \leq n).$$

In particular, $\mathrm{rk}\,\mathcal{J}(H(Tx))|_{x=e_{r+1}} = r$ and the last $n-r$ columns of $(\mathcal{J}(H(Tx)))|_{x=e_{r+1}}$ are zero. There exists $S \in \mathrm{GL}_m(K)$ such that

$$(\mathcal{J}(SH(Tx)))|_{x=e_{r+1}} = S \cdot (\mathcal{J}(H(Tx)))|_{x=e_{r+1}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

(ii) Suppose that $|K| > (d-1)r + 1$. Since $\mathcal{J}H \cdot x \neq 0$, we may assume that

$$\mathrm{rk}(\mathcal{J}H \cdot x, \mathcal{J}_{x_2,x_3,\dots,x_r}H) = r$$

and that

$$a_1 := \det(\mathcal{J}(H_1, H_2, \dots, H_r) \cdot x, \mathcal{J}_{x_2,x_3,\dots,x_r}(H_1, H_2, \dots, H_r)) \neq 0.$$

It follows by [3, Lemma 5.1(i)] that there exists $w \in K^n$ such that $a_1(w) \neq 0$. One may observe that $\mathrm{rk}(\mathcal{J}H)|_{x=w} = r$ and thus there exist independent vectors $v_{r+1}, v_{r+2}, \dots, v_n \in K^n$ such that $(\mathcal{J}H)|_{x=w} \cdot v_i = 0$ for $i = r+1, r+2, \dots, n$. Since $(\mathcal{J}H \cdot x)|_{x=w}$ is the first column of a full column rank matrix,

$$(\mathcal{J}H)|_{x=w} \cdot w = (\mathcal{J}H \cdot x)|_{x=w} \neq 0.$$

So, $v_1 := w$ is independent of $v_{r+1}, v_{r+2}, \dots, v_n$. Take $T = (v_1, v_2, \dots, v_n) \in \mathrm{GL}_n(K)$. Then

$$(\mathcal{J}(H(Tx)))|_{x=e_1} \cdot e_i = (\mathcal{J}H)|_{x=Te_1} \cdot Te_i = (\mathcal{J}H)|_{x=w} \cdot v_i \quad (1 \leq i \leq n).$$

The rest of the proof of (ii) is similar to that of (i).

The last claim follows from [3, Lemma 5.1(ii)], as an improvement to [3, Lemma 5.1(i)]. □

Proposition 2.3. *Assume that* $\mathrm{char}\,K \notin \{1, 2, \dots, d\}$. *Then, for any homogeneous polynomial map* $H \in K[x]^m$ *of degree* d *with* $\mathrm{rk}\,\mathcal{J}H \leq 1$, *the components of* H *are linearly dependent over* K *in pairs and one has* $\mathrm{rk}\,\mathcal{J}H = \mathrm{trdeg}_K K(H)$.

Proof. The case $\mathrm{rk}\,\mathcal{J}H = 0$ is obvious, so let $\mathrm{rk}\,\mathcal{J}H = 1$. On account of Lemma 2.2, we may assume that $\mathcal{J}H|_{x=e_1} = E_{11}$. Let $j \geq 2$. Since $\deg_{x_1} H_j < d$, we infer that either $H_j = 0$ or $\deg_{x_1} \partial H_j/\partial x_1 < \deg_{x_1} \partial H_j/\partial x_i$ for some $i \geq 2$, where $\deg_{x_1} 0 = -\infty$. The latter is impossible due to $\mathrm{rk}\,\mathcal{J}H = 1$, so $H_j = 0$. This holds for all $j \geq 2$, which yields the desired results. □

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LEMMA 2.4. *Let* $H = (h, x_1^2x_2, x_2^2x_3)$ *or* $(h, x_1^2x_3, x_2^2x_3) \in K[x_1, x_2, x_3]^3$, *where* h *is cubic homogeneous, and assume that* $\operatorname{char} K \neq 2, 3$. *Then* $\operatorname{rk} \mathcal{J}H = \operatorname{trdeg}_K K(H)$.

PROOF. It suffices to consider the case of $\operatorname{rk} \mathcal{J}H = 2$. Define a derivation D on $A = K[x_1, x_2, x_3]$ as follows: for any $h \in A$,

$$D(h) = \frac{x_1x_2x_3}{H_2H_3} \det \mathcal{J}H.$$

For $H = (h, x_1^2x_2, x_2^2x_3)$, an easy calculation gives $D = x_1\partial_{x_1} - 2x_2\partial_{x_2} + 4x_3\partial_{x_3}$. It follows that $D(u) = (d_1 - 2d_2 + 4d_3)u$ for any term $u = x_1^{d_1}x_2^{d_2}x_3^{d_3} \in A$. Consequently, $\ker D := \{g \in A \mid D(g) = 0\}$, the kernel of D, is linearly spanned by all terms u with $d_1 - 2d_2 + 4d_3 = 0$. So, the only cubic terms in $\ker D$ are $x_1^2x_2$ and $x_2^2x_3$. Since $\operatorname{rk} \mathcal{J}H = 2$, we have $\det \mathcal{J}H = 0$ and thus $h \in \ker D$, which implies that h is a linear combination of $x_1^2x_2$ and $x_2^2x_3$. Thus, $\operatorname{trdeg}_K K(H) = 2$.

In the case of $H = (h, x_1^2x_3, x_2^2x_3)$, one may verify that $x_1^2x_3$, $x_1x_2x_3$ and $x_2^2x_3$ are the only cubic terms in $\ker D$. The conclusion follows similarly. □

THEOREM 2.5. *Assume that* $\operatorname{char} K \neq 2, 3$. *Then, for any cubic homogeneous polynomial map* $H \in K[x]^m$ *with* $\operatorname{rk} \mathcal{J}H \leq 2$, *one has* $\operatorname{rk} \mathcal{J}H = \operatorname{trdeg}_K K(H)$.

PROOF. Due to Theorem 2.1, replacing L there by K, we may assume that $H \in K[x_1, x_2, x_3]^3$ and it suffices to show that

$$\operatorname{trdeg}_K K(H) = 3 \quad \text{implies } \operatorname{rk} \mathcal{J}H = 3$$

or, equivalently,

$$\det \mathcal{J}H = 0 \quad \text{implies } \operatorname{trdeg}_K K(H) < 3.$$

So, assume that $\det \mathcal{J}H = 0$. Since we may replace K by an extension field to make it large enough, it follows by Lemma 2.2 that we may assume that $(\mathcal{J}H)|_{x=e_1} = E_{11} + E_{22}$. Then $\mathcal{J}H$ is of the form

$$\begin{pmatrix} x_1^2 + * & * & * \\ * & x_1^2 + * & * \\ * & * & \dfrac{\partial H_3}{\partial x_3} \end{pmatrix},$$

where the x_1-degree of each element $*$ is less than two. By observing the terms with x_1-degree ≥ 5 in $\det \mathcal{J}H$, we see that $(\partial H_3/\partial x_3) \in K[x_2, x_3]$. Now H_2 and H_3 are of the form

$$H_2 = x_1^2x_2 + b_{10}x_1x_3^2 + b_{11}x_1x_2x_3 + b_{12}x_1x_2^2 + b_0(x_2, x_3),$$
$$H_3 = c_{12}x_1x_2^2 + c_{00}x_3^3 + c_{01}x_2x_3^2 + c_{02}x_2^2x_3 + c_{03}x_2^3.$$

We shall show that $x_2^2 \mid H_3$, that is, $c_{00} = c_{01} = 0$.

The part of x_1-degree four of $\det \mathcal{J}H$ is $(\partial H_3/\partial x_3 - (\partial H_2/\partial x_1\partial x_3)(\partial H_3/\partial x_1\partial x_2))x_1^4$ and it follows that $\partial H_3/\partial x_3 - (\partial H_2/\partial x_1\partial x_3)(\partial H_3/\partial x_1\partial x_2) = 0$. Consequently,

$$(3c_{00}x_3^2 + 2c_{01}x_2x_3 + c_{02}x_2^2) = (2b_{10}x_3 + b_{11}x_2)(2c_{12}x_2),$$

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so

$$c_{00} = 0, \quad c_{01} = 2b_{10}c_{12}, \quad c_{02} = 2b_{11}c_{12}.$$

One may observe that the coefficient of $x_1^3x_3^3$ in $\det \mathcal{J}H$ is $2c_{01}b_{10} = 0$, which we can combine with $c_{01} = 2b_{10}c_{12}$ to obtain $c_{01} = 0$. Therefore,

$$H_3 = (c_{12}x_1 + c_{03}x_2 + c_{02}x_3)x_2^2.$$

Moreover, if $c_{12} = 0$, then $c_{02} = 2b_{11}c_{12} = 0$ and thus $H_3 = c_{03}x_2^3$.

We now distinguish two cases.

Case 1: $c_{12} \neq 0$ and $c_{12}x_1 + c_{03}x_2 + c_{02}x_3 \nmid H_i$ for some i.

Then H_3 is the product of two linear forms, of which two are distinct. Hence, we can compose H with invertible linear maps on both sides to obtain a map H' for which $H'_2 = x_1^2x_2$ and $x_2 \nmid H'_1$.

Notice that $H'_1(1, 0, t) \neq 0$. As K has at least five elements, it follows from [3, Lemma 5.1(i)] that there exists a $\lambda \in K$ such that $H'_1(1, 0, \lambda) \neq 0$. Hence, the coefficient of x_1^3 in $H'_1(x_1, x_2, x_3 + \lambda x_1)$ is nonzero and $H'_2(x_1, x_2, x_3 + \lambda x_1) = x_1^2x_2$.

Replacing H' by $H'(x_1, x_2, x_3 + \lambda x_1)$, we may assume that $H'_2 = x_1^2x_2$ and that H'_1 contains x_1^3 as a term. We may even assume that the coefficient of x_1^3 in H'_1 equals 1. Then $\mathcal{J}H'|_{x=e_1}$ is of the form

$$\begin{pmatrix} 1 & * & a \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix}$$

and has rank two. Furthermore, $v_3 = (-a, 0, 1)^t$ belongs to its null space. We may apply the proof of Lemma 2.2 on H' by taking $T = (e_1, e_2, v_3)$ and taking an appropriate $S \in \mathrm{GL}_3(K)$ such that $\tilde{H} := SH'(Tx)$ satisfies $\mathcal{J}\tilde{H}|_{x=e_1} = S\mathcal{J}H'|_{x=Te_1}T = E_{11} + E_{22}$. Since Tx is of the form (L_1, x_2, L_3), and observing the form of $\mathcal{J}H'|_{x=e_1}$, one may also choose Sx to be of the form $(*, x_2, *)$. Then $\tilde{H}_2 = L_1^2x_2$.

So, we can compose $\tilde{H}$ with an invertible linear map on the right to obtain a map $\widetilde{H}'$ for which $\widetilde{H}'_2 = x_1^2x_2$ and $\tilde{H}'_3 = x_2^2L'$ for some linear form L'.

Suppose first that L' is a linear combination of x_1 and x_2. If $\tilde{H}'_1 \in K[x_1, x_2]$, then we are done. Otherwise, we have $\det \mathcal{J}_{x_1,x_2}(\tilde{H}'_2, \tilde{H}'_3) = 0$ and then, by Proposition 2.3, $\mathrm{trdeg}_K K(H'_2, H'_3) < 2$.

Suppose next that L' is not a linear combination of x_1 and x_2. Then we may assume that $\tilde{H}'_3 = x_2^2x_3$. By Lemma 2.4(i), $\mathrm{trdeg}_K K(\tilde{H}') < 3$.

Case 2: $c_{12} = 0$ or $c_{12}x_1 + c_{03}x_2 + c_{02}x_3 \mid H_i$ for all i.

Since $x_2^2 \mid H_3$, we can compose H with invertible linear maps on both sides to obtain a map H' for which $H'_1 \in \{x_1^3, x_1^2x_2\}$. After a possible interchange of H'_2 and H'_3, the first two rows of $\mathcal{J}H'$ are independent. Now we may apply the proof of Lemma 2.2 to H'. More precisely, there exist $S, T \in \mathrm{GL}_3(K)$ such that $\tilde{H} := SH'(Tx)$ satisfies $\mathcal{J}\tilde{H}|_{x=e_1} = E_{11} + E_{22}$. If we choose w such that the first two rows of $(\mathcal{J}H')_{x=w}$ are independent, then we can take S such that $Sx = (f_1x_1 + f_2x_2, g_1x_1 + g_2x_2, *)$. By repeating the discussion for $\tilde{H}$ as for H above, we may assume that $\tilde{H}_3 = Lx_2^2$ for some linear form L.

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Let $Tx = (L_1, L_2, L_3)$. Notice that $H_1'(Tx) \in \{L_1^3, L_1^2 L_2\}$ and that $H_1'(Tx)$ is a linear combination of $\tilde{H}_1$ and $\tilde{H}_2$. Hence, we can compose $\tilde{H}$ with a linear map on the left to obtain a map $\tilde{H}'$ for which $\tilde{H}_2' \in \{L_1^3, L_1^2 L_2\}$ and $\tilde{H}_3' = Lx_2^2$.

Suppose first that $\tilde{H}_2' = L_1^2 L_2$. Then $c_{12} \neq 0$, so $c_{12}x_1 + c_{03}x_2 + c_{02}x_3 \mid H_i$ for all i. From this, we infer that $L_2 \mid \tilde{H}_i$ and $L_2 \mid \tilde{H}_i'$ for all i. As $x_2 \nmid \tilde{H}_1$, we deduce that L and L_2 are dependent linear forms, which are independent of x_2. If L and L_2 are linear combinations of L_1 and x_2, then we can reduce to Proposition 2.3 and otherwise we can reduce to Lemma 2.4(ii).

Suppose next that $\tilde{H}_2' = L_1^3$. If L, L_1 and x_2 are linearly dependent over K, then we can reduce to Proposition 2.3. Otherwise, $\tilde{H}$ is as H in the previous case. □

Remark 2.6. Inspired by Lemma 2.4, we investigated maps H of which the components are terms and searched for H with algebraically independent components for which $\det \mathcal{J}H = 0$. One can infer that H has these properties if and only if the matrix with entries $\deg_{x_i} H_j$ has determinant zero over K but not over $\mathbb{Z}$.

We found such a nonhomogeneous H over fields of characteristic five:

$$(x_1^3 x_2, x_1 x_2^2), \quad (x_1^2 x_2, x_1 x_3^2, x_2 x_3)$$

with the following homogenisations:

$$(x_1^3 x_2, x_1 x_2^2 x_3, x_3^4), \quad (x_1^2 x_2, x_1 x_3^2, x_2 x_3 x_4, x_4^3).$$

Besides these homogenisations, we found the following homogeneous H over fields of characteristic five:

$$(x_1^2 x_3^2, x_1 x_2^3, x_2 x_3^3), \quad (x_4 x_1^2, x_1 x_2^2, x_2 x_3^2, x_3 x_4^2).$$

We conclude with a homogeneous H over fields of characteristic seven and a homogeneous H over any characteristic $p \in \{1, 2, \ldots, d\}$, respectively:

$$(x_3 x_1^3, x_1 x_2^3, x_2 x_3^3), \quad (x_1^d, x_1^{d-p} x_2^p).$$

These examples show that the conditions in Proposition 2.3 and Theorem 2.5 cannot be relaxed.

Theorem 2.7. *Suppose that* $\operatorname{char} K \neq 2, 3$ *and let* $H \in K[x]^m$ *be cubic homogeneous. Let* $r := \operatorname{rk} \mathcal{J}H$ *and suppose that* $r \leq 2$. *Then there exist* $S \in \mathrm{GL}_m(K)$ *and* $T \in \mathrm{GL}_n(K)$ *such that, for* $\tilde{H} := SH(Tx)$, *one of the following statements holds:*

(1) $\tilde{H}_{r+1} = \tilde{H}_{r+2} = \cdots = \tilde{H}_m = 0$;

(2) $r = 2$ *and* $\tilde{H} \in K[x_1, x_2]^m$;

(3) $r = 2$ *and* $K\tilde{H}_1 + K\tilde{H}_2 + \cdots + K\tilde{H}_m = Kx_3x_1^2 \oplus Kx_3x_1x_2 \oplus Kx_3x_2^2$.

Furthermore, we may take $S = T^{-1}$ *if* $m = n$.

Proof. By Theorem 2.5, $\operatorname{trdeg}_K K(H) = \operatorname{rk} \mathcal{J}H = r \leq 2$. Since H is homogeneous, we have $\operatorname{trdeg}_K K(tH) = r$ as well, where t is a new variable.

Suppose first that $r \leq 1$. By [4, Theorem 2.7], we may take $\tilde{H}$ as in (1).

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Suppose next that $r = 2$. By [4, Theorem 2.7], H is of the form $g \cdot h(p, q)$ such that g, h and (p, q) are homogeneous and $\deg g + \deg h \cdot \deg(p, q) = 3$.

If $\deg h \leq 1$, then every triple of components of h is linearly dependent over K and thus we may take $\tilde{H}$ as in (1). If $\deg h = 3$, then $\deg(p, q) = 1$ and $\deg g = 0$, whence we may take $\tilde{H}$ as in (2).

So, assume that $\deg h = 2$. Then $\deg(p, q) = 1$ and $\deg g = 1$. If g is a linear combination of p and q, then we may take $\tilde{H}$ as in (2). If g is not a linear combination of p and q, then we may take $\tilde{H}$ as in (3) or (1).

Finally, if $m = n$ and $\tilde{H} = SH(Tx)$ is as in (1), then $SH(S^{-1}x) = \tilde{H}(T^{-1}S^{-1}x)$ is still as in (1). So, we may take $S = T^{-1}$. If $m = n$ and $\tilde{H} = SH(Tx)$ is as in (2) or (3), then $T^{-1}H(Tx) = T^{-1}S^{-1}\tilde{H}$ is still as in (2) or (3), whence we may also take $S = T^{-1}$. □

3. Cubic homogeneous Keller maps $x + H$ with $\operatorname{rk} JH \leq 2$

For two matrices $M, N \in \operatorname{Mat}_n(K[x])$, we say that M is similar over K to N if there exists $T \in \operatorname{GL}_n(K)$ such that $N = T^{-1}MT$.

Theorem 3.1. *Let $F = x + H \in K[x]^n$ be a Keller map with* $\operatorname{trdeg}_K K(H) = 1$. *Then $\mathcal{J}H$ is similar over K to a triangular matrix and the following statements are equivalent:*

(1) $\det \mathcal{J}F = 1$;

(2) *$\mathcal{J}H$ is nilpotent;*

(3) $(\mathcal{J}H) \cdot (\mathcal{J}H)|_{x=y} = 0$, *where $y = (y_1, y_2, \ldots, y_n)$ are n new variables.*

Proof. Since $\operatorname{trdeg}_K K(H) = 1$, by [4, Corollary 3.2] there exists a polynomial $p \in K[x]$ such that $H_i \in K[p]$ for each i, say, $H_i = h_i(p)$, where $h_i \in K[t]$ for each i. Write $h_i' = \partial h_i/\partial t$. Then

$$\mathcal{J}H = h'(p) \cdot \mathcal{J}p. \tag{3.1}$$

Assume without loss of generality that

$$h_1' = h_2' = \cdots = h_s' = 0$$

and that

$$0 \leq \deg h_{s+1}' < \deg h_{s+2}' < \cdots < \deg h_n'.$$

For $s < i < n$,

$$\deg h_i'(p) = \deg h_i' \cdot \deg p \leq (\deg h_{i+1}' - 1) \cdot \deg p = \deg h_{i+1}'(p) - \deg p.$$

Since the degrees of the entries of $\mathcal{J}p$ are less than $\deg p$, we deduce from (3.1) that the nonzero entries on the diagonal of $\mathcal{J}H$ have different degrees in increasing order. Furthermore, the nonzero entries beyond the $(s+1)$th entry on the diagonal of $\mathcal{J}H$ have positive degrees.

By (3.1), $\operatorname{rk}(-\mathcal{J}H) \leq 1$ and thus $n - 1$ eigenvalues of $-\mathcal{J}H$ are zero. It follows that the trailing degree of the characteristic polynomial of $-\mathcal{J}H$ is at least $n - 1$. More precisely,

$$\det(tI_n + \mathcal{J}H) = t^n - \operatorname{tr}(-\mathcal{J}H) \cdot t^{n-1}$$

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and thus

$$\det \mathcal{J}F = (t^n - \operatorname{tr}(-\mathcal{J}H) \cdot t^{n-1})|_{t=1} = 1 + \operatorname{tr} \mathcal{J}H.$$

Observe that the diagonal of $\mathcal{J}H$ is totally zero, except maybe the $(s+1)$th entry, which is a constant.

Thus, $\partial p/\partial x_i = 0$ for all $i > s+1$ and $\mathcal{J}H$ is lower triangular. If the $(s+1)$th entry on the diagonal of $\mathcal{J}H$ is nonzero, then (1), (2) and (3) do not hold. If the $(s+1)$th entry on the diagonal of $\mathcal{J}H$ is zero, then $\partial p/\partial x_i = 0$ for all $i > s$, whence (1), (2) and (3) hold. □

Let $H \in K[x]^n$ be homogeneous of degree $d \geq 2$. Then $x + H$ is a Keller map if and only if $\mathcal{J}H$ is nilpotent (see, for example, [8, Lemma 6.2.11]). So, we first investigate nilpotent matrices over $K[x]$.

Lemma 3.2. *Suppose that $N \in \mathrm{Mat}_2(K[x])$ is nilpotent. Then there exist $a, b, c \in K[x]$ such that*

$$N = c\begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix}.$$

Furthermore, N is similar over K to a triangular matrix if and only if a and b are linearly dependent over K.

Proof. Since $\det N = 0$, we may write N in the form

$$N = c \cdot \begin{pmatrix} b \\ a \end{pmatrix} \cdot (a \ \ -\tilde{b}),$$

where $a, b \in K[x]$ and $\tilde{b}, c \in K(x)$. Since $\operatorname{tr} N = 0$, we have $\tilde{b} = b$. If we choose a and b to be relatively prime, then $c \in K[x]$ as well.

Furthermore, a and b are linearly dependent over K if and only if the rows of N are linearly dependent over K, if and only if N is similar over K to a triangular matrix. □

Lemma 3.3. *Let $H \in K[x]^2$ be cubic homogeneous such that $\mathcal{J}_{x_1,x_2}H$ is nilpotent. Then there exists $T \in \mathrm{GL}_2(K)$ such that for $\tilde{H} := T^{-1}H(T(x_1, x_2), x_3, x_4, \ldots, x_n)$, one of the following statements holds:*

(1) *$\mathcal{J}_{x_1,x_2}\tilde{H}$ is a triangular matrix;*

(2) *there are independent linear forms $a, b \in K[x]$ such that*

$$\mathcal{J}_{x_1,x_2}\tilde{H} = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \quad \text{and} \quad \mathcal{J}_{x_1,x_2}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

(3) $\operatorname{char} K = 3$ *and there are independent linear forms $a, b \in K[x]$ such that*

$$\mathcal{J}_{x_1,x_2}\tilde{H} = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \quad \text{and} \quad \mathcal{J}_{x_1,x_2}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Proof. Suppose that (1) does not hold. By Lemma 3.2, there are $a, b, c \in K[x]$ such that

$$\mathcal{J}_{x_1,x_2}H = c\begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix},$$

where a and b are linearly independent over K. As H is cubic homogeneous, the entries of $\mathcal{J}_{x_1,x_2}H$ are quadratic homogeneous, so $c \in K$ and a and b are independent linear forms.

If we take

$$T = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \quad \text{then } \mathcal{J}_{x_1,x_2}\tilde{H} = \begin{pmatrix} \tilde{a}\tilde{b} & -\tilde{b}^2 \\ \tilde{a}^2 & -\tilde{a}\tilde{b} \end{pmatrix},$$

where $\tilde{a} = c \cdot a|_{x_1=cx_1}$ and $\tilde{b} = c^{-1} \cdot b|_{x_1=cx_1}$.

We claim that the coefficient k_2 of x_2 in $\tilde{b}$ is 0. Suppose conversely that $k_2 \neq 0$. Then the coefficient of x_2^3 in

$$3\tilde{H}_1 = \mathcal{J}_{x_1,x_2}\tilde{H}_1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tilde{b}(x_1\tilde{a} - x_2\tilde{b})$$

is nonzero. In particular, char $K \neq 3$. One may verify that

$$\mathcal{J}_{x_1,x_2}(\tilde{H}_1 + \tfrac{1}{3}k_2^{-1}\tilde{b}^3) = (\tilde{c}\tilde{b}, 0),$$

where $\tilde{c} := \tilde{a} + k_2^{-1}\tilde{b}(\partial\tilde{b}/\partial x_1)$. As a consequence, $\partial(\tilde{c}\tilde{b})/\partial x_2 = \partial 0/\partial x_1 = 0$. Furthermore, $\tilde{c}$ and $\tilde{b}$ are independent, and so also are $\tilde{a}$ and $\tilde{b}$. From $\partial(\tilde{c}\tilde{b})/\partial x_2 = 0$, we have $\tilde{c}\tilde{b} \in K[x_1, x_3, x_4, \ldots, x_n]$ if char $K \neq 2$. Since $\tilde{c}$ and $\tilde{b}$ are independent, we deduce that if char $K = 2$, then $\tilde{c}\tilde{b} \in K[x_1, x_3, x_4, \ldots, x_n]$ as well. Since the coefficient λ of x_2 in $\tilde{b}$ is nonzero, we have $\tilde{c} = 0$, which is a contradiction.

So, the coefficient of x_2 in $\tilde{b}$ is 0. Similarly, the coefficient of x_1 in $\tilde{a}$ is 0. Consequently,

$$\mathcal{J}_{x_1,x_2}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix},$$

where $\lambda, \mu \in K$. Therefore,

$$\mathcal{J}_{x_1,x_2}\tilde{H} = \begin{pmatrix} (\lambda x_2 + \cdots)(\mu x_1 + \cdots) & -(\mu x_1 + \cdots)^2 \\ (\lambda x_2 + \cdots)^2 & -(\lambda x_2 + \cdots)(\mu x_1 + \cdots) \end{pmatrix}.$$

So, the coefficient of $x_1^2 x_2$ in $2\tilde{H}_1$ is equal to both $\lambda\mu$ and $-2\mu^2$. Similarly, the coefficient of $x_1 x_2^2$ in $2\tilde{H}_2$ is equal to both $\lambda\mu$ and $-2\lambda^2$. It follows that either $\lambda = \mu = 0$ or $0 \neq \lambda = -2\mu = 4\lambda$. In the former case, $\tilde{H}$ satisfies (2). In the latter case, char $K = 3$ and $\lambda = \mu$. Replacing $\tilde{H}$ by $\lambda\tilde{H}(\lambda^{-1}(x_1, x_2), x_3, x_4, \ldots, x_n)$, we see that $\tilde{H}$ satisfies (3). □

Theorem 3.4. *Suppose that* char $K \neq 2, 3$. *Let* $H \in K[x]^n$ *be cubic homogeneous such that* $x + H$ *is a Keller map, that is,* $\mathcal{J}H$ *is nilpotent.*

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(i) *If* $\operatorname{rk}\mathcal{J}H = 1$*, then there exists* $T \in \mathrm{GL}_n(K)$ *such that for* $\tilde{H} := T^{-1}H(Tx)$*,*

$$\tilde{H}_1 \in K[x_2, x_3, x_4, \ldots, x_n],$$
$$\tilde{H}_2 = \tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0.$$

(ii) *If* $\operatorname{rk}\mathcal{J}H = 2$*, then either* H *is linearly triangularisable or there exists* $T \in \mathrm{GL}_n(K)$ *such that for* $\tilde{H} := T^{-1}H(Tx)$*,*

$$\tilde{H}_1 - (x_1x_3x_4 - x_2x_4^2) \in K[x_3, x_4, \ldots, x_n],$$
$$\tilde{H}_2 - (x_1x_3^2 - x_2x_3x_4) \in K[x_3, x_4, \ldots, x_n],$$
$$\tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0.$$

Furthermore, $x + tH$ *is invertible over* $K[t]$ *if* $\operatorname{rk}\mathcal{J}H \le 2$*, where* t *is a new variable. Moreover,* $x + tH$ *is tame over* $K[t]$ *if either* $\operatorname{rk}\mathcal{J}H = 1$ *or* $\operatorname{rk}\mathcal{J}H = 2$ *and* $n \ne 4$*. In particular,* $x + \lambda H$ *is invertible and tame under the above condition respectively for every* $\lambda \in K$*.*

Proof. We may take $\tilde{H}$ as in (1), (2) or (3) of Theorem 2.7. If $\operatorname{rk}\mathcal{J}H = 1$, then $\tilde{H}$ is as in (1) of Theorem 2.7, that is, $\tilde{H}_i = 0, 2 \le i \le n$, whence (i) holds because $\operatorname{tr}\mathcal{J}\tilde{H} = 0$. So, assume that $\operatorname{rk}\mathcal{J}H = 2$. Notice that $\mathcal{J}H$ is nilpotent.

If $\tilde{H}$ is as in (1) or (2) of Theorem 2.7, that is, $\tilde{H}_i = 0, 3 \le i \le n$ or $\tilde{H} \in K[x_1, x_2]^n$, then $\mathcal{J}_{x_1,x_2}(\tilde{H}_1, \tilde{H}_2)$ is nilpotent.

If $\tilde{H}$ is as in (3) of Theorem 2.7, that is,

$$K\tilde{H}_1 + K\tilde{H}_2 + \cdots + K\tilde{H}_n = Kx_3x_1^2 \oplus Kx_3x_1x_2 \oplus Kx_3x_2^2,$$

then $\tilde{H}_3 = 0$, because $x_3^{-1}\tilde{H}_3$ is the constant part with respect to x_3 of $\operatorname{tr}\mathcal{J}\tilde{H} = 0$. So, $\mathcal{J}_{x_1,x_2}(\tilde{H}_1, \tilde{H}_2)$ is nilpotent in any case.

One may observe that, in all the cases (1), (2) and (3) of Theorem 2.7, if $\mathcal{J}_{x_1,x_2}(\tilde{H}_1, \tilde{H}_2)$ is similar over K to a triangular matrix, then $\mathcal{J}\tilde{H}$ is similar over K to a triangular matrix, and so is $\mathcal{J}H$, and thus H is linearly triangularisable.

Now suppose that $\mathcal{J}_{x_1,x_2}(\tilde{H}_1, \tilde{H}_2)$ is not similar over K to a triangular matrix. Noticing that $\operatorname{char} K \ne 2, 3$, $\mathcal{J}_{x_1,x_2}(\tilde{H}_1, \tilde{H}_2)$ must be as in (2) of Lemma 3.3, that is,

$$\mathcal{J}_{x_1,x_2}\tilde{H} = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \quad \text{and} \quad \mathcal{J}_{x_1,x_2}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where a, b are linearly independent linear forms.

If $\tilde{H}_1 \in K[x_1, x_2, x_3]$, then $a, b \in k[x_3]$, which is a contradiction. So, $\tilde{H}$ is not as in (2) or (3) of Theorem 2.7 and thus is as in (1) of Theorem 2.7, that is, $\tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0$. Consequently, by a linear coordinate transformation, we may take $\tilde{H}$ such that $a = x_3$ and $b = x_4$. So, (ii) holds.

For the last claim, when $\operatorname{rk}\mathcal{J}H = 1$, $\tilde{H}$ is of the form in (i), whence $x + t\tilde{H}$ is elementary and thus tame. When $\operatorname{rk}\mathcal{J}H = 2$, $\tilde{H}$ is of the form in (ii) and it suffices to show that the automorphism

$$F = (x_1 + tx_4(x_3x_1 - x_4x_2), x_2 + tx_3(x_3x_1 - x_4x_2), x_3, x_4, x_5)$$

is tame over $K[t]$.

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For that purpose, let $w = t(x_3x_1 - x_4x_2)$ and let $D := x_4\partial_{x_1} + x_3\partial_{x_2}$ be a derivation of $K[t][x_1, x_2, x_3, x_4]$. Observe that D is triangular and $w \in \ker D$, and hence that $F = (\exp(wD), x_5)$. Therefore, F is tame over $K[t]$ from Lemma 3.5 below. □

Recall that a derivation D of $K[x]$ is called locally nilpotent if for every $f \in K[x]$ there exists an m such that $D^m(f) = 0$. For such a derivation, $\exp D := \sum_{i=0}^{\infty} (1/i!)D^i$ is a polynomial automorphism of $K[x]$. A derivation D of $K[x]$ is called triangular if $D(x_i) \in K[x_{i+1}, \dots, x_n]$ for $i = 1, 2, \dots, n - 1$ and $D(x_n) \in K$. A triangular derivation is locally nilpotent.

Lemma 3.5. *Let D be a triangular derivation of $K[t][x]$ and $w \in \ker D$, that is, $D(w) = 0$. Then $(\exp(wD), x_{n+1})$ is tame over $K[t]$.*

Proof. From [15, Corollary], there exists a k such that $(\exp(wD), x_{n+1}, x_{n+2}, \dots, x_{n+k})$ is tame over $K(t)$. Inspecting the proof of [15, Corollary] yields that $(\exp(wD), x_{n+1})$ is tame over $K[t]$. □

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