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# ON SELF-INJECTIVE PERFECT RINGS

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ABSTRACT. Let $R$ be a left and right perfect right self-injective ring. It is shown that if the radical of $R$ is countably generated as a left ideal then $R$ is quasi-Frobenius. It is also shown that the same conclusion can be drawn if $r(A \cap B) = r(A) + r(B)$ for all left ideals $A$ and $B$ of $R$.

1. **Introduction.** Let $R$ be a right perfect ring with Jacobson radical $J$. Osofsky in [7, Lemma 11] (see also Faith [3, Lemma 23.19]) showed that if $J/J^2$ is finitely generated as a right or left $R$-module then $R$ is right artinian. If $R$ is also right and left self-injective, then $R$ is quasi-Frobenius (QF). It is an open problem whether a right or left perfect right self-injective ring is QF. Clark and Van Huynh in [2] show that a right and left perfect right self-injective ring is QF if $R/\operatorname{soc}_r(R)$ has a finitely generated right socle. We, in this article, use this result to show that a right and left perfect right self-injective ring is QF if $J/J^2$ is countably generated as a left $R$-module. We also prove that if $R$ is a right self-injective left or right perfect ring for which the condition $r(A \cap B) = r(A) + r(B)$ for all left ideals $A$ and $B$ of $R$ is satisfied, then $R$ is QF. The methods we originally followed were adapted from an argument (basically due to Skornjakov) used by Hajarnavis and Norton in [4], but simpler proofs were given afterwards, upon the suggestions of the second author and the referee.

2. **Main result.** If $A$ is a subset of a ring $R$, then we write $r(A)$ (resp. $\ell(A)$) for the right (resp. left) annihilator of the set $A$. We also write $\operatorname{soc}_r(R)$ (resp. $\operatorname{soc}_\ell(R)$) for the right (resp. left) socle of the ring $R$. The injective envelope of a module $M$ is denoted by $E(M)$ and its dual will be denoted by $M^* = \operatorname{Hom}_R(M, R)$. Following is a list of properties of a right perfect right self-injective ring $R$. These are taken from Utumi [8], Osofsky [7], and Kato [5].

(2.1) $R$ is right pseudo-Frobenius (PF) (that is, $R_R$ is an injective cogenerator of mod-$R$).

(2.2) $\operatorname{soc}_\ell(R) = \operatorname{soc}_r(R)$; we denote the common ideal by $P$.

(2.3) $P$ is essential and finitely generated in $R$, both as a left and right ideal.

(2.4) If $A$ is a right ideal of $R$ then $r\ell(A) = A$.

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(2.5) $\ell(P) = r(P) = J$.
(2.6) If $R$ is also left perfect then $\ell(J) = r(J) = P$.
(2.7) For every simple right (resp. left) module $A$, there exists a primitive idempotent $e$ such that $A \cong eP$ (resp. $A \cong Pe$).

We also consider the following condition, which we shall subsequently refer to as the *right HN condition*: Every homomorphism from a right ideal $I$ into $R$ with finitely generated image is multiplication by an element of $R$.

LEMMA 1. *Let $R$ be a semi-local ring and let $M_R$ be a semi-simple module of dimension $\aleph \geq \aleph_0$. Suppose that $S^* \neq 0$ for every simple right $R$-module $S$. Then $M^*$ is semi-simple of dimension greater than $\aleph$.*

PROOF. Suppose that $M = \otimes_{i \in I} S_i$, where each $S_i$ is simple. Then $M^* \cong \prod_{i \in I} S_i^*$. Because $R$ is semi-local and $S_i^* \neq 0$ for each $i$, there is a simple module $A$ such that $A^I$ embeds in $M^*$. But $A^I$ is semi-simple, so $A^I \cong A^{(J)}$, and it is well-known that this implies $|J| > |I| = \aleph$.

LEMMA 2. *Let $R$ be a ring satisfying the right HN condition, and with finitely generated right socle. If $K \subset I$ is a pair of right ideals such that $I/K$ is semi-simple, then*

$$\ell(K)/\ell(I) \cong \mathrm{Hom}_R(I/K, R).$$

PROOF. Let

$$\varphi\colon \ell(K)/\ell(I) \longrightarrow \mathrm{Hom}_R(I/K, R)$$

be the canonical map given by

$$\varphi\big(r + \ell(I)\big)(x + K) = rx, \qquad r \in \ell(K), x \in I.$$

Then $\varphi$ is injective. To show that $\varphi$ is surjective, let $f \in \mathrm{Hom}_R(I/K, R)$. Then *Im*($f$) is a direct summand of $P$, whence finitely generated. If $\pi\colon I \longrightarrow I/K$ is the canonical map, then by hypothesis, $f \circ \pi\colon I \longrightarrow R$ is given by left multiplication by $r \in R$. Hence $f(x+K) = rx$ for all $x \in I$. But then $r \in \ell(K)$ and easily $\varphi\big(r + \ell(I)\big) = f$, so $\varphi$ is an isomorphism.

LEMMA 3. *Let $R$ be a right self-injective left and right perfect ring $R$. Let*

$$\mathrm{soc}_2(R)/P = \mathrm{soc}_r(R/P).$$

*Then $J^2 \,\mathrm{soc}_2(R) = 0$.*

PROOF. Suppose first that for $x \in R$, $(xR+P)/P$ is simple. Then $(xR+P)/P \cong R/M$, where $M$ is a maximal right ideal of $R$ such that $xM \subseteq P$. Now (2.5) implies that $JxM = 0$, so $Jx \subseteq \ell(M) \subseteq \ell(J) = P$, by (2.6).

Now if $x \in \mathrm{soc}_2(R)$, then $x = x_1 + \cdots + x_n + y$ where each $(x_iR + P)/P$ is simple and $y \in P$. Thus, by the previous paragraph, $Jx \subseteq P$ and so $J\,\mathrm{soc}_2(R) \subseteq P$. This implies, by (2.5) again, that $J^2 \subseteq \mathrm{soc}_2(R) = 0$.

THEOREM 1. *Let R be a right self-injective left and right perfect ring. If $J/J^2$ is a countably generated left R-module then R is quasi-Frobenius.*

PROOF. It is sufficient, by [2], to prove that $R/P$ has finite dimensional right socle. Condition (2.3) stipulates that $P_R$ is finitely generated, and condition (2.7) implies that $S^* \neq 0$ for every simple right $R$-module $S$. Clearly, $R$ is right HN. Thus we can use Lemma 1 and Lemma 2 to deduce that if $\mathrm{soc}_2(R)/P$ is infinite dimensional, then $\ell(P)/\ell\big(\mathrm{soc}_2(R)\big)$ is of uncountable dimension as a left $R/J$-module. However, using Lemma 3, $J^2 \subseteq \ell\big(\mathrm{soc}_2(R)\big) \subseteq \ell(P) = J$, so $J/J^2$ has uncountable dimension as a left $R/J$-module.

The proof of the following Lemma can be found in [4, Proposition 2].

LEMMA 4. *Let R be a ring for which $r\ell(xR) = xR$ for every $x \in R$. If also $r(A \cap B) = r(A) + r(B)$ for every pair of left ideals A and B, then R is left HN.*

The following Lemma is an immediate consequence of Camillo [1].

LEMMA 5. *Let M be a right R-module over a right perfect ring R. If every quotient module of M has finite Goldie dimension, then M is nötherian.*

PROOF. It follows from [1] that for every submodule $N$ of $M$, there exists a finitely generated submodule $T$ of $N$ such that $N/T$ has no maximal submodules. This can not occur with non-zero right modules over right perfect rings; see, *e.g.*, Faith[3]. Hence $N = T$ is finitely generated.

THEOREM 2. *Let R be a right self-injective left and right perfect ring. If R is also left HN, then R is quasi-Frobenius.*

PROOF. We have to show that $R$ is right nötherian. Since $R$ is right perfect, Lemma 5 makes it clear that it is sufficient to show that $R/K$ has finite Goldie dimension for every right ideal $K$ of $R$. Since $R$ is left perfect, it suffices to show that $R/K$ has finitely generated socle $I/K$. Suppose, then, that $I/K$ has dimension $\aleph \geq \aleph_0$. Then Lemma 1 and Lemma 2 imply that $\ell(K)/\ell(I)$ has dimension greater than $\aleph$. Now $R$ is a left HN ring and (2.2) and (2.3) imply that $\mathrm{soc}_l(R)$ is finitely generated. Hence we can use the left-sided version of Lemma 1 and Lemma 2 to deduce that $r\ell(I)/r\ell(K) = I/K$ (by (2.4)) has dimension greater than that of $\ell(K)/\ell(I)$. This contradiction concludes the proof of the theorem.

COROLLARY. *Let R be a right self-injective left and right perfect ring. If also $r(A \cap B) = r(A) + r(B)$ for every pair of left ideals A and B, then R is quasi-Frobenius.*

3. **Remarks and a question.** Morita and Tachikawa consider in [6] the following condition imposed on a module $M$.

(3.1) If $A$ and $B$ are submodules of $M$ such that $M/A \cong M/B$ then $A \cong B$,

and its converse

(3.2) If $A$ and $B$ are submodules of $M$ such that $A \cong B$ then $M/A \cong M/B$.

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Morita and Tachikawa in [6, Theorem 3.1] show that $R^{(n)}$ satisfies condition (3.1) for artinian rings. We generalize this for semi-local rings in the following Proposition, whose proof is simple and is therefore omitted.

PROPOSITION. *Let M be a quasi-projective module for which every epimorphism is an isomorphism. Then M satisfies (3.1).*

COROLLARY. *If R is a semi-local ring then $R^{(n)}$ satisfies (3.1).*

Morita and Tachikawa prove in [6, Theorems 4.1 and 4.4] that a right and left artinian ring is QF if and only if $R^{(n)}$ satisfies (3.2) both as a left and as a right $R$-module. We show that if $R$ is a right self-injective semi-local ring then $R^{(n)}$ satisfies (3.2) as a right $R$-module. This follows as a corollary of the following

PROPOSITION. *Let M be a quasi-injective module for which every monomorphism is an isomorphism. Then M satisfies (3.2).*

PROOF. Let $A$ and $B$ be submodules of $M$ such that $\alpha: A \cong B$ is an isomorphism. Then $\alpha$ induces an isomorphism $\bar{\alpha}: E(A) \longrightarrow E(B)$, and this in turn induces an isomorphism $\bar{\bar{\alpha}}: E(M) \longrightarrow E(M)$. Since $M$ is quasi-injective, $\bar{\bar{\alpha}}$ induces a monomorphism $\beta: M \longrightarrow M$. The hypothesis implies that $\beta$ is an isomorphism, and this induces an isomorphism $M/A \longrightarrow M/B$.

COROLLARY. *If R is a semi-local right self-injective ring then the right R-module $R^{(n)}$ satisfies condition (3.2).*

Osofsky gives in [7] an example of a commutative semi-perfect self-injective ring which is not QF. Thus Theorem 4.4 of Morita and Tachikawa referred to above cannot in general hold true for non-artinian rings. We end this note with the following question: *Suppose that R is a right or left perfect ring such that $R^{(n)}$ satisfies (3.2) on both sides. Is R QF?*

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