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Abstract

We give bounds for the Betti numbers of projective algebraic varieties in terms of their classes (degrees of dual varieties of successive hyperplane sections). We also give bounds for classes in terms of ramification volumes (mixed ramification degrees), sectional genus and, eventually, in terms of dimension, codimension and degree. For varieties whose degree is large with respect to codimension, we give sharp bounds for the above invariants and classify the varieties on the boundary, thus obtaining a generalization of Castelnuovo's theory for curves to varieties of higher dimension.

Introduction

Let $X^n \subset \mathbb{P}^N$ be a nondegenerate nonsingular complex projective algebraic variety of dimension n and codimension $a = N - n > 0$. (I am grateful to C. Casagrande for pointing out that the positivity assumption was not explicit in the original version of this paper.) The homology class of X in $\mathbb{P}^N$ is determined by the degree $d = \deg X$. Viewed as an embedded projective variety, X has projective numerical invariants, such as classes (cf. Definition 1.11), degrees of various double and ramification loci etc. Viewed as an abstract variety, X has other important numerical invariants, such as Chern numbers (for example $c_1^n(X) = (-1)^n(K_X^n)$, where K_X is the canonical class of X, $c_n(X) = e(X)$, where $e(X)$ is the Euler–Poincaré characteristic of X and so on), Betti numbers $b_i(X)$, $i = 0, \dots, 2n$, and $b(X) = \sum_{i=0}^{2n} b_i(X)$ and Hodge numbers $h^{p,q}(X)$, $p, q = 0, \dots, n$.

One of the most natural questions to ask is what are the relations between all these invariants of X and, in particular, what are the restrictions imposed on all of them by fixing n, a and d (and possibly some other basic invariants). In the case of curves ($n = 1$), the genus $g(X)$ is the only basic invariant. For $a = 1$, g is determined by d while in general there are a sharp upper bound $g \leqslant \mathcal{C}(d, a)$ and a classification of all curves on the boundary (obtained by Halphen [Hal1882] for $a = 2$ and by Castelnuovo [Cas1889, Cas1893] for any a). Here $\mathcal{C}(d, a) = d^2/2a + \cdots$, where $\cdots$ stand for a term bounded by a linear function in d; cf. (1.4.5) for a precise formula.

Castelnuovo theory for curves (which is still an active topic; it is important to know which genera actually occur for which types of curves) can only serve as a hard to follow model for varieties of higher dimension. One of the main difficulties is that, even for surfaces, there is no single invariant playing a role comparable to that of genus of curves. The most important abstract

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invariants of surfaces are K^2, the geometric genus p_g, the arithmetic genus p_a, the Euler–Poincaré characteristic e and the total Betti number b, and there are some relations between them (b is a linear combination of e, p_g and p_a, p_a is given by the Noether formula). Clearly, the number of important numerical invariants of X grows with dimension. Results comparable to those by Castelnuovo are known only for the geometric genus $p_g(X)=h^{0,n}(X)$ (cf. [Har1981]); however, for higher-dimensional varieties p_g does not play a role comparable to that of genus of curves. In [Zak2012], we showed that bounds for many important numerical invariants of X, such as Chern numbers and (middle) Hodge numbers, are asymptotically proportional to the bounds for (the total) Betti number, and the maximal varieties belong to the same class (cf. Remarks (10) and (11) in § 4). In the present paper, we concentrate on the study of bounds for some of the most important numerical invariants of X, *viz.* Betti numbers (which are topological invariants) and classes and ramification volumes (which are projective invariants).

Our approach to bounding the Betti numbers in terms of n, a and d consists in first bounding them in terms of important projective invariants called *classes*. If X is a projective variety as above, then the nth class or simply the *class* $\mu_n=\mu_n(X)$ is defined as the degree d^* of the dual variety $X^*\subset\mathbb{P}^{N^*}$ (called *codegree*) provided that X^* is a hypersurface in $\mathbb{P}^{N^*}$ and zero otherwise (we recall that X^* is the locus of tangent hyperplanes to X, that is, the hyperplanes containing the embedded tangent space $T_{X,x}$ at some point $x\in X$). For the purpose of this introduction, one can define the *ith class* $\mu_i(X)$, $0\leqslant i\leqslant n$, as the class of the intersection of X with a general linear subspace of codimension $n-i$ in $\mathbb{P}^N$. In particular, μ_0 is equal to $d=\deg X$ and μ_1 is the class of a general curve section C of X, which, by the Riemann–Hurwitz formula, equals $2\pi+2d-2$, where $\pi=g_C$ is the sectional genus of X. Using Lefschetz theory, in § 2 we show that $b_i=b_{2n-i}\leqslant\mu_i+\mu_{i-2}+\mu_{i-4}+\cdots$, $i\leqslant n$, $b=b(X)\leqslant\mu_n+2\mu_{n-1}+3\mu_{n-2}+\cdots$ and $e=e(X)=(n+1)\mu_0-n\mu_1+(n-1)\mu_2-\cdots$ (cf. Theorem 2.9). Thus, the problem of bounding Betti numbers is reduced to the problem of bounding classes.

Bounding classes which is more or less equivalent to bounding the codegree is an interesting and important problem in itself. For example, while varieties of codegree one and two are, respectively, linear spaces and quadrics, classification of varieties of codegree three is already deep and nontrivial (cf. [Zak1993, ch. IV, Theorem 5.2]) and classification of varieties of codegree four has not yet been completed. In [Zak2004], we proved various sharp lower bounds for d^* (the case of surfaces was dealt with in [Zak1973]) and classified the varieties on the boundary. However, to bound the Betti numbers from above we need *upper* bounds for classes.

To get such bounds, we consider the ramification divisor $R\subset X$ of a general linear projection $p:X\to\mathbb{P}^n$, which is a handy tool to explore the geometry of X, often more convenient than the canonical class K (one has $R\sim K+(n+1)H$, where H is a hyperplane section) because it is ample [Zak1993, ch. I, Corollary 2.14] and even very ample [Ein1982]. In § 1, we bound classes in terms of *ramification volumes* r_i. To wit, put $r_i=\deg R^i=(R^iH^{n-i})$, $i=0,\dots,n$. In Theorem 1.12 we show that, for all i, $\mu_i\leqslant r_i$, and thus our problem is reduced to bounding the numbers r_i.

Using the Hodge index theorem, one can show that the subsequent quotients r_i/r_{i-1} form a nonincreasing sequence, that is, $r_1/r_0\geqslant r_2/r_1\geqslant\cdots\geqslant r_n/r_{n-1}$, and so $r_i\leqslant r_1^i/r_0^{i-1}=r_1^i/d^{i-1}$. Clearly, for $n>1$ the number r_1 is stable under passing to a general hyperplane section, and thus is bounded by Castelnuovo's theorem for curves: $\mu_1=r_1=2\pi+2d-2\leqslant 2\mathcal{C}(d,a)+2d-2$. Thus, Castelnuovo theory for curves yields bounds for the numbers r_i (hence also for classes and Betti numbers) in terms of dimension, codimension and degree (cf. respectively Corollaries 1.5, 1.13 and 2.14). We also obtain universal sharp bounds for classes and Betti numbers in terms of dimension and degree; these bounds do not involve codimension and are attained only by hypersurfaces (cf. Theorems 1.18 and 2.16 and Example 2.6). An easy consequence is that one always has $b(X) < d^{n+1}$ (Theorem 2.18).

In the case when, instead of fixing the degree, one considers varieties of given dimension n and codimension a defined by equations of degree not exceeding d (it is this setup that was studied in [Mil1964, Ole1951, Tho1965]), we get a sharp bound for classes in Theorem 1.21; this bound is attained if and only if X is a complete intersection of a hypersurfaces of degree d. We also obtain a bound for the Betti numbers of varieties from this class; cf. Theorem 2.20 and Corollary 2.22.

The next questions to ask are whether the bounds for the classes and Betti numbers given in §§ 1 and 2 are good (that is, asymptotically sharp in d), how to improve them and to find sharp bounds and where to look for varieties on the boundary. These questions are dealt with in § 3. In Theorem 3.1, we show that if some class, ramification volume or the (total or middle) Betti number of X is large enough, then $X \subset V \subset \mathbb{P}^N$ is a codimension-one subvariety in a variety V^{n+1} of (minimal) degree a in $\mathbb{P}^N$. We proceed with giving sharper bounds for our invariants for subvarieties of codimension one in varieties of minimal degree and studying which varieties of minimal degree contain smooth codimension-one subvarieties. This allows us to prove nice general bounds for classes and Betti numbers. For example, we show that $b(X) < d^{n+1}/a^n$ provided that $d \geqslant 2(a+1)^2$ (cf. Theorem 3.16(ii)).

Finally, in § 4 we discuss some generalizations and open problems.

1. Bounds for ramification and classes

Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n, and let R be an ample (Cartier) divisor on X. Denote by H a hyperplane section of X, and let $d = (H^n) = \deg X$ and $r_i = (R^i H^{n-i}) = \deg R^i$ (so that, in particular, $r_0 = d$, $r_1 = \deg R$ and $r_n = r = (R^n)$). Then the subsequent quotients r_i/r_{i-1} form a nonincreasing sequence, that is, $r_1/r_0 \geqslant r_2/r_1 \geqslant \cdots \geqslant r_n/r_{n-1}$, and so $r = r_n \leqslant r_{n-1}^2/r_{n-2} \leqslant r_{n-2}^3/r_{n-3}^2 \leqslant \cdots \leqslant r_i^{n-i+1}/r_{i-1}^{n-i} \leqslant \cdots \leqslant r_1^n/r_0^{n-1} = r_1^n/d^{n-1}$. In particular, $r_i \leqslant r_1^i/d^{i-1}$, $i = 1, \ldots, n$, and $(R^n) \leqslant (\deg R)^n/(\deg X)^{n-1}$.*

Proof. Theorem 1.1 is a special case of the Concavity theorem [Laz2004, Example 1.6.4] for $\alpha = R$, $\beta = H$, $s_i = r_i$, which, in its turn, is an easy consequence of the Hodge index theorem for surfaces (cf. Remark 1.2(v) for a thorough historic discussion). □

Remarks 1.2. (i) Let X_k be the section of X by a general linear subspace $\mathbb{P}^{a+k} \subset \mathbb{P}^N$, $0 \leqslant k \leqslant n$. Then it is clear that $r_i(X) = r_i(X_k)$, $0 \leqslant i \leqslant k$.

(ii) The bounds in Theorem 1.1 are sharp. In fact, it is clear that if $R \sim \alpha H$ for some $\alpha > 0$, then $r_i = \alpha^i d$, $i = 0, \ldots, n$, and all the inequalities in Theorem 1.1 turn into equalities. It is also easy to show the converse; cf. [BFJ2009, Theorem D] for an analogous result for big and nef line bundles.

(iii) The bound in Theorem 1.1 is much better than a general one given in [Ful1998, Example 8.4.7] (*viz.* $(R^n) \leqslant (\deg R)^n$), but, unlike Fulton's bound, it fails if we replace n copies of R by n divisors $R_1, \ldots, R_n$ that are not equivalent to each other. To see this, it suffices to consider the nonsingular quadric $X \subset \mathbb{P}^3$ and two ample divisors $R_1 \sim \ell_1 + 2\ell_2$ and $R_2 \sim 2\ell_1 + \ell_2$, $d_i = \deg R_i = 3$, $i = 1, 2$, where ℓ_1 and ℓ_2, $(\ell_1 \cdot \ell_2) = 1$ are two generators. Then $d(R_1 \cdot R_2) = 10 > d_1 d_2 = 9$.

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(iv) Theorem 1.1 actually holds for arbitrary nef (numerically effective) line bundles; cf. [Laz2004, 1.6] or [Dem1993, Proposition 5.2] for a complex analytic analogue.

(v) An easy induction argument reduces Theorem 1.1 to the case of smooth surfaces, where it is an easy consequence of the Hodge index theorem (cf. for example [Voi2002, Theorem 6.32]). Here follows (a brief and incomplete) account of the history of Theorem 1.1 and its predecessor, the Hodge index theorem.

Around 1933, Hodge proved a theorem to the effect that if S is a nonsingular complex projective algebraic surface of geometric genus p_g and Q is the (real) quadratic form corresponding to the intersection pairing on the (classes of) 2-cycles of the underlying topological 4-manifold, then the number of positive terms in the signature of Q is equal to $2p_g+1$. At about the same time, Du Val observed that the intersection matrix of the components of the exceptional divisor of the resolution of a 'nice' surface singularity is negative definite. Thus, it was only natural that Du Val proposed to Hodge to show that the signature of the intersection form on the (classes of) *algebraic* 2-cycles has only one positive term, *viz.* its restriction on the orthogonal complement of a hyperplane section H is negative definite. In [Hod1937], Hodge succeeded in doing that by using the (transcendental) method of harmonic integrals developed in his previous papers. Soon after that, Bronowski [Bro1937] and Segre [SeB1937] gave simple geometric proofs of this fact, which later acquired the name of the *Hodge index theorem*; another proof was given by Grothendieck [Gr1958] some twenty years later. In fact, both Bronowski [Bro1937, $n^0 4$, Theorem (I,II)] and Segre [SeB1937, $n^0 4$] have actually proved Theorem 1.1 in the case of surfaces. It should be noted that Theorem 1.1 for a variety X of arbitrary dimension is easily reduced to the case of surfaces by replacing X by intersections of general divisors from the linear systems $|H|$ and $|kR|$, $k \gg 0$.

Almost simultaneously with [Hod1937], A. D. Alexandrov published a series of papers in Mat. Sbornik devoted to the theory of mixed volumes (in the sense of Minkowski). In particular, he gave two different proofs of what is now known as the *Alexandrov–Fenchel inequality* for mixed volumes generalizing the famous isoperimetric inequality. One of these proofs is based on a linear algebra lemma resembling the Hodge index theorem (cf. [AA1938, § 3] and [Gro1990, § 2]). Around 1979, Khovanskiĭ [Kho1979] and Teissier [Tei1979] (again independently and almost simultaneously) derived the Alexandrov–Fenchel inequality and its analogues in algebraic geometry, including Theorem 1.1, from the Hodge index theorem. This approach revealing amazing connections between classical and algebraic geometry was further explained and developed in other papers by Khovanskiĭ and Teissier and also in [BFJ2009, DN2006, Gro1990, KKh2012, Tim1999] (the author is grateful to the referee for pointing out some of these references). However, in spite of its beauty, this method does not contribute much to understanding Theorem 1.1 and its generalizations since it is ultimately based on the Hodge index theorem of which Theorem 1.1 is a direct consequence.

(vi) Theorem 1.1 and its generalizations, and local analogues, are thoroughly discussed in [Laz2004, 1.6]. Variants of this theorem have been repeatedly rediscovered and reproved, usually as auxiliary results, by various authors under different guises; cf. for example [BBS1989, Lemma 0.15.1] and [Dem1993, Proposition 5.2]. However, the Hodge index theorem is still the main ingredient of most of the proofs.[1] Thus, it seems desirable to find out what makes

[1] The only exception I know of is [Dem1993], where the proof of the Kähler version of [Laz2004, Theorem 1.6.1] is based on the Aubin–Calabi–Yau theorem [Dem1993, Lemma 5.1] and the inequality between the arithmetic and geometric means. This last inequality is actually equivalent to the claim that an n-cube has the largest volume

the Hodge index theorem tick. Peskine showed that the Hodge index theorem can be deduced from Castelnuovo's bound for (possibly singular and nonreduced) curves (cf. Example 1.4 below), and so the relationship between this theorem and bounds for codegree exploited in this section might turn out to be deeper than it looks. In fact, the author recently found that both (a general form of) the Hodge index theorem and the Castelnuovo inequality for the genus of curves are consequences of a general theory yielding an upper bound for the dimension of the ambient space of primitive families of intersecting linear subspaces.

The following example illustrates Theorem 1.1 in a very simple case.

Example 1.3. Let $X=Q\subset\mathbb{P}^3$ be a nonsingular quadric, let ℓ_1 and ℓ_2 be its generators and let $R\sim\alpha_1\ell_1+\alpha_2\ell_2$. Then $(R^2)=2\alpha_1\alpha_2\leqslant(\alpha_1+\alpha_2)^2/2=(\deg R)^2/\deg\ X$ with equality holding if and only if $\alpha_1=\alpha_2=\alpha$, that is, $R\sim\alpha H$.

The next example plays a crucial role in the present paper.

Example 1.4. Let $X\subset\mathbb{P}^N$ be a nondegenerate n-dimensional variety, let $L\subset\mathbb{P}^N$, $\dim L=N-n-1$, be a general linear subspace, and let $p_L:X\to\mathbb{P}^n$ be the projection with center L. Then p_L is a finite map of degree $d=\deg X$, and we denote by $R_L\subset X$ its (apparent) ramification locus

$$R_L=\overline{\{x\in \mathrm{Sm}\ X\mid T_{X,x}\cap L\neq\varnothing\}},\tag{1.4.1}$$

where $\mathrm{Sm}\ X=X\backslash\mathrm{Sing}\ X$ is the locus of nonsingular points of X and $T_{X,x}$ denotes the tangent space to X at x. It is clear that R_L is a (Weil) divisor in X and that, as L varies, the corresponding divisors R_L are rationally equivalent to each other and there are no points in $\mathrm{Sm}\ X$ common to all R_L. If, furthermore, X is smooth, then it is easy to determine the ambient linear system $|R_L|$. To wit, $K_{\mathbb{P}^n}=\mathcal{O}(-(n+1))$ and, if ω is a rational rank-n differential form on $\mathbb{P}^n$, then $(p_L^*(\omega))\sim-(n+1)H+R_L$, where H is the divisor of a hyperplane section of X, and so

$$|R_L|=|K_X+(n+1)H|\tag{1.4.2}$$

(here $K_{\mathbb{P}^n}$ (respectively K_X) denotes the canonical class of $\mathbb{P}^n$ (respectively X); for $n\leqslant 2$ this formula was already known to Clebsch). The linear system $|R_L|$ is ample (cf. [Zak1993, ch. I, Corollary 2.14]) and even very ample on X (cf. [Ein1982]). If $X_i=X\cap\mathbb{P}^{N-n+i}$ is the i-dimensional section of X by a general linear subspace $\mathbb{P}^{N-n+i}\supset L$, then $R_L(X_i)=R_L(X)\cap\mathbb{P}^{N-n+i}$ and $(R_L(X_i)^i)=(R_L^iH^{n-i})=r_i$ (cf. Remark 1.2(i)). The number r_i will be called the ith *ramification volume* or *mixed ramification degree* of the projective variety X.

Theorem 1.1 applies to $R=R_L$ and yields

$$r_i=(R_L^iH^{n-i})\leqslant\frac{r_1^i}{d^{i-1}}.\tag{1.4.3}$$

Let $C=X_1\subset\mathbb{P}^{N-n+1}$ be a general curve section of X, so that $r_1=r_1(X)=r_1(C)$. Suppose that C is nonsingular, and let π be the genus of C. Then $\deg C=\deg X=d$ and, by the above,

$$r_1(X)=r_1(C)=\deg K_X+2H=2\pi-2+2d\tag{1.4.4}$$

(this is just the Riemann–Hurwitz formula). Furthermore, $\pi\leqslant\mathcal{C}(d,a)$, where $a=N-n=\operatorname{codim}X=\operatorname{codim}C$ and

$$\mathcal{C}(d,a)=\frac{(d-\varepsilon)(d-a+\varepsilon-2)}{2a},\quad \varepsilon\equiv d\ (\mathrm{mod}\ a), 1\leqslant\varepsilon\leqslant a\tag{1.4.5}$$

among all the n-dimensional parallelepipeds with the same perimeter (sum of lengths of edges), which again is a kind of isoperimetric inequality.

is the Castelnuovo bound (cf. for example [GH1978, p. 252] and [HE1982, Theorem 3.7]). The function

$$\phi(t) = \frac{(d-t)(d-a+t-2)}{2a}$$

attains maximal value for $t = a/2+1$ and $\phi(a/2+1) = (d-a/2-1)^2/2a$. Thus,

$$\begin{gathered} \mathcal{C}(d,a) \leqslant \frac{(d-a/2-1)^2}{2a}, \\ r_1 = 2\pi - 2 + 2d \leqslant \frac{(d-a/2-1)^2}{a} + 2d - 2 = \frac{(d+(a-2)/2)^2}{a}, \\ r_i \leqslant \frac{r_1^i}{d^{i-1}} \leqslant d\biggl(\frac{(d+(a-2)/2)^2}{ad}\biggr)^i. \end{gathered} \tag{1.4.6}$$

Corollary 1.5. *Let $X \subset \mathbb{P}^N$ be a nondegenerate nonsingular n-dimensional variety of codimension $a = N-n$ and degree d, let $R = R_L \sim K_X + (n+1)H$ (cf. Example 1.4) and let $1 \leqslant i \leqslant n$. Then $r_i = (R^i H^{n-i}) \leqslant r_1^i/d^{i-1} < d(d/a+5/4)^i$. Furthermore, if $d \geqslant (a-2)^2/8$ (which is always true provided that $a \leqslant 12$), then $r_i < d(d/a+1)^i$.*

Proof. From (1.4.3), (1.4.6) and the inequality $d \geqslant a+1$, it follows that

$$r_i \leqslant \frac{r_1^i}{d^{i-1}} \leqslant d\biggl(\frac{d}{a} + \frac{a-2}{a} + \frac{(a-2)^2}{4ad}\biggr)^i < d\biggl(\frac{d}{a} + \frac{5}{4}\biggr)^i. \tag{1.5.1}$$

If $d \geqslant (a-2)^2/8$, then $-2/a + (a-2)^2/4ad \leqslant 0$, and so

$$\frac{(d+(a-2)/2)^2}{ad} \leqslant \frac{d}{a} + 1, \tag{1.5.2}$$

which yields the second claim of Corollary 1.5. Finally, if $a \leqslant 12$, then $d \geqslant a+1 > (a-2)^2/8$. □

Corollary 1.6. *Let $X \subset \mathbb{P}^N$ be a nondegenerate nonsingular n-dimensional variety of degree d, codimension $a = N-n$ and sectional genus π, let K_X be the canonical class of X, let $R = R_L \sim K_X + (n+1)H$, and let $r_1 = \deg K_X = 2\pi + 2d - 2$.*

(i) *One has*

$$\begin{aligned} \deg(K_X^i) = (K_X^i H^{n-i}) &\leqslant d\biggl(\frac{r_1}{d} - n - 1\biggr)^i = \frac{(r_1 - d(n+1))^i}{d^{i-1}} \\ &= \frac{(2\pi - (n-1)d - 2)^i}{d^{i-1}} < d\biggl(\frac{d}{a} + \frac{1}{4} - n\biggr)^i, \quad 1 \leqslant i \leqslant n. \end{aligned}$$

If, moreover, $d \geqslant (a-2)^2/8$ (which is always true provided that $a \leqslant 12$), then $\deg(K_X^i) < d(d/a-n)^i$, $1 \leqslant i \leqslant n$.

(ii) *Suppose that $\pi \leqslant (n-1)d/2+1$. Then $K_X^i H^{n-i} \leqslant 0$ for all $1 \leqslant i \leqslant n$, $i \equiv 1 \pmod 2$.*

(iii) *If $\pi \leqslant (n-1)d/2+1$ (respectively $\pi < (n-1)d/2+1$), then no positive multiple of the canonical class K_X can be a positive divisor, so that either $K_X = 0$ or $P_m = P_m(X) = l(mK_X) = 0$ for all $m > 0$ (respectively $P_m = 0$ for all $m > 0$). If $\pi \geqslant (n-1)d/2+1$ (respectively $\pi > (n-1)d/2+1$), then no negative multiple of the canonical class can be a positive divisor, so that either $K_X = 0$ or $l(-mK_X) = 0$ for $m > 0$ (respectively $l(-mK_X) = 0$ for $m > 0$); in particular, $-K_X$ cannot be ample.*

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(iv) *Suppose that* $d \leqslant an+2$. *Then* X *satisfies the assumption of (ii), so that, in particular, no multiple of the canonical class* K_X *can be a positive divisor. Furthermore, if* $d \leqslant an+1$, *then* $P_m = P_m(X) = 0$ *for all* $m > 0$.

(v) *Suppose that* $d < an + (an)^{(n-1)/n}/(n+1) + 2$. *Then the canonical class* K_X *cannot be ample on* X.

Proof. (i) Expanding the right-hand side of the equality

$$(K_X^i H^{n-i}) = ((R_X - (n+1)H)^i H^{n-i}),$$

substituting the sequence of inequalities from Theorem 1.1 and using (1.4.6) in the same way as in the proof of Corollary 1.5, one gets

$$\begin{aligned}
\deg\,(K_X^i) &= \sum_{j=0}^{i} (-1)^j \binom{i}{j} (n+1)^j r_{i-j} \\
&= \sum_{j=0}^{[i/2]} \left[\binom{i}{2j} (n+1)^{2j} r_{i-2j} - \binom{i}{2j+1} (n+1)^{2j+1} r_{i-2j-1} \right] \\
&\leqslant \sum_{j=0}^{[i/2]} r_{i-2j-1} \left[\binom{i}{2j} (n+1)^{2j} \frac{r_{i-2j-1}}{r_{i-2j-2}} - \binom{i}{2j+1} (n+1)^{2j+1} \right] \\
&\leqslant \frac{r_1^{i-2j-1}}{d^{i-2j-2}} \left[\binom{i}{2j} (n+1)^{2j} \frac{r_1}{d} - \binom{i}{2j+1} (n+1)^{2j+1} \right] \\
&= d \sum_{j=0}^{i} (-1)^j \binom{i}{j} (n+1)^j \left(\frac{r_1}{d}\right)^{i-j} = d \left(\frac{r_1}{d} - (n+1) \right)^i \\
&= \frac{(2\pi - (n-1)d - 2)^i}{d^{i-1}} < d \left(\frac{d}{a} + \frac{1}{4} - n \right)^i
\end{aligned}$$

(here brackets denote integral part and we put $r_l = 0$ for $l < 0$). If, moreover, $d \geqslant (a-2)^2/8$, then, by (1.5.2), $(2\pi - (n-1)d - 2)/d < d(d/a - n)$, which proves the second claim of (i).

(ii) From our assumptions, it follows that $r_1 \leqslant d(n+1)$ (respectively $r_1 < d(n+1)$); thus, (ii) is an immediate consequence of (i).

(iii) In view of (1.4.4), one has

$$\deg mK_X = m \deg\,(R - (n+1)H) = m(r_1 - (n+1)d) = m(2\pi - (n-1)d - 2),$$

which yields our claim.

(iv) If $d \leqslant a(n-1) < a(n-1/4)$, then from (1.5.1) it follows that $r_1/d - n - 1 < 0$, which, in view of (ii), proves (iv) in this case.

Suppose now that $a(n-1) < d \leqslant an$. Then, by (1.4.5), $\varepsilon = d - a(n-1)$ and

$$\frac{r_1}{d} - n - 1 = \frac{(n-1)(2d - an - 2) + 2d - 2}{d} - n - 1 \leqslant -\frac{2n}{d} < 0, \tag{1.6.1}$$

which, in view of (ii), proves (iv) in this case.

Suppose finally that $an < d \leqslant a(n+1)$. Then, by (1.4.5), $\varepsilon = d - an$ and

$$\frac{r_1}{d} - n - 1 = \frac{n(d - a + \varepsilon - 2) + 2d - 2}{d} - n - 1 = \frac{d - 2 - n(a + 2 - \varepsilon)}{d}. \tag{1.6.2}$$

From (1.6.2), it follows that $r_1/d-n-1\leqslant 0$ (respectively $r_1/d-n-1<0$) if and only if

$$\varepsilon\leqslant 2\quad(\text{respectively }\varepsilon=1),\tag{1.6.3}$$

which completes the proof of (iv).

(v) Put $d=an+\varepsilon$. Using (i) and arguing as in (1.6.2), we see that K_X cannot be ample on X provided that

$$d\left(\frac{r_1}{d}-n-1\right)^n=d\left(\frac{n(d-a+\varepsilon-2)+2d-2}{d}-n-1\right)^n<1.\tag{1.6.4}$$

Expanding (1.6.4), we conclude that K_X fails to be ample for

$$\varepsilon<2+\frac{d^{(n-1)/n}}{n+1}\tag{1.6.5}$$

and *a fortiori* for

$$\varepsilon<2+\frac{(an)^{(n-1)/n}}{n+1},\tag{1.6.6}$$

which yields (v). □

Remarks 1.7. (i) Sometimes the bounds in (1.4.6) and Corollaries 1.5 and 1.6 can be improved. For example, if $a=N-n$ is odd, then $\operatorname{argmax}\phi=[a/2]+1$ and one gets slightly better upper bounds for r_1 and r_n. Thus, if $a=1$, then $r_1\leqslant(d-1)(d-2)/2+2(d-1)=d(d-1)$, $r_n\leqslant d^n(d-1)^n/d^{n-1}=d(d-1)^n$, which is the best possible bound (it is sharp if and only if the hypersurface X is smooth; cf. Theorem 1.18 below), and $(K_X^i)\leqslant d(d-n-2)^i$, which is again sharp. If $a=2$, then $r_1\leqslant d^2/2$, $r_n\leqslant d^{2n}/2^nd^{n-1}=d^{n+1}/2^n$ and $(K_X^i)\leqslant d((d-2n-2)/2)^i$, and if $a=3$, then $r_1\leqslant d(d+1)/3$, $r_n\leqslant d((d+1)/3)^n$ and $(K_X^i)\leqslant d(d-3n-2/3)^i$. We already saw that if $a\leqslant 12$ or, more generally, $d\geqslant(a-2)^2/8$, then $r_n\leqslant d(d/a+1)^n$ and $(K_X^iH^{n-i})\leqslant d(d/a-n)^i$. If, on the other hand, $d\leqslant 2a+1$, then from the Clifford theorem it follows that $r_1\leqslant 2(2d-a-2)$ (cf. [GH1978, p. 252]), Theorem 1.1 shows that $r_n\leqslant 3^nd$ and from Corollary 1.6 it follows that K_X cannot be effective for $n\geqslant 2$.

(ii) After completing this paper I learned that, under certain assumptions, Di Gennaro [DG2001] obtained Castelnuovo-type bounds for (K_X^n) in terms of degree, dimension and codimension. In particular, he had to assume that K_X is nef. Apart from being intimately connected with other important invariants, such as classes (cf. Theorem 1.12 below), R_L, unlike the canonical class, is always ample (cf. Example 1.4).

(iii) It might be interesting to extend Example 1.4 to more general finite coverings of $\mathbb{P}^n$ (not necessarily corresponding to projections); cf. [Laz2004, 6.3.D] and Remark (6) in § 4.

We apply the above results to bound the classes of projective varieties in terms of their degree and sectional genus or codimension. We recall the necessary definitions. Let $X\subset\mathbb{P}^N$ be an n-dimensional variety. The hyperplanes in $\mathbb{P}^N$ are parametrized by the dual projective space $\mathbb{P}^{N*}$. For a point $\alpha\in\mathbb{P}^{N*}$, we denote by $L_\alpha\subset\mathbb{P}^N$ the corresponding hyperplane in $\mathbb{P}^N$. We denote by $\operatorname{Sm}X$ the subset of nonsingular points of X. For a point $x\in X$, we denote by $T_{X,x}\subset\mathbb{P}^N$ the (embedded projective) tangent subspace to X at x; if $x\in\operatorname{Sm}X$, then $\dim T_{X,x}=\dim X=n$. A hyperplane $\alpha\in\mathbb{P}^{N*}$ is said to be *tangent* to X at x if $L_\alpha\supset T_{X,x}$.

Definition 1.8. Let $X\subset\mathbb{P}^N$ be an n-dimensional projective algebraic variety of degree d. The subvariety $\mathcal{P}_X\subset X\times\mathbb{P}^{N*}$, $\mathcal{P}_X=\overline{\{(x,\alpha)\mid x\in\operatorname{Sm}X,L_\alpha\supset T_{X,x}\}}$, where bar denotes projective closure, is called the *conormal variety* of X in $\mathbb{P}^N$. Let $p : \mathcal{P}_X \to X$ and $\pi : X \times \mathbb{P}^{N*} \to \mathbb{P}^{N*}$ be the canonical projections. The image $X^* = \pi(\mathcal{P}_X) \subset \mathbb{P}^{N*}$ is called the *dual variety* of X.

For any $x \in \operatorname{Sm} X$, the fiber $p^{-1}(x)$ is the $(N-n-1)$-dimensional linear subspace of $\mathbb{P}^{N*}$ dual to $T_{X,x}$, and so $\dim \mathcal{P}_X = N-1$. Thus, X^* is the locus of hyperplanes that are tangent to X at a nonsingular point, and $n^* = \dim X^* \leqslant \dim \mathcal{P}_X = N-1$. The number $\operatorname{def} X = N - n^* - 1$ is called the *defect* of X. We denote by $d^* = \deg X^*$ the degree of X^*, that is, the number of common points of X^* and a general $(\operatorname{def} X + 1)$-dimensional linear subspace in $\mathbb{P}^{N*}$. The number $d^* = \operatorname{codeg} X$ is called the *codegree* of X.

An important result about dual varieties is the following theorem.

Reflexivity Theorem 1.9. *We have $(X^*)^* = X$. More precisely, if $\alpha \in X^*$ is a nonsingular point, then the linear subspace $\mathbb{P}_\alpha = \{z \in \mathbb{P}^N \mid \Lambda_z \supset T_{X^*,\alpha}\}$ is contained in X and L_α is tangent to X at all points of $\mathbb{P}_\alpha \cap \operatorname{Sm} X$ (here Λ_z denotes the hyperplane in $\mathbb{P}^{N*}$ corresponding to a point $z \in \mathbb{P}^N$).*

Proof. See for example [Tev2005, Theorem 1.7]. $\square$

We also need the following elementary result.

Lemma 1.10. (i) *Let $L \subset \mathbb{P}^N$, $\operatorname{codim}_{\mathbb{P}^N} L \leqslant \operatorname{def} X$, be a general linear subspace, let $\Lambda = \Lambda_L \subset \mathbb{P}^{N*}$, $\dim \Lambda = N - \dim L - 1$, be the subspace corresponding to L and let $Y = X \cap L$. Then $Y^* = \pi(X^*)$, where $\pi = \pi_\Lambda : \mathbb{P}^{N*} \dashrightarrow L^*$ is the projection with center Λ.*

(ii) *Let $L \subset \mathbb{P}^N$, $\dim L < \operatorname{codim}_{\mathbb{P}^n} X - 1$, be a general linear subspace, let $\Lambda = \Lambda_L \subset \mathbb{P}^{N*}$, $\dim \Lambda = N - \dim L - 1$, be the linear subspace corresponding to L and let $Y = p(X)$, where $p = p_L : \mathbb{P}^N \dashrightarrow \Lambda^*$ is the projection with center L. Then $Y^* = X^* \cap \Lambda$.*

Proof of Lemma 1.10. (i) Clearly, it suffices to consider the case when $\operatorname{def} X > 0$ and L is a hyperplane. Since the dual of any curve is a hypersurface, $\operatorname{def} X > 0$ implies $n > 1$. Thus, from the Bertini theorem it follows that Y is irreducible. Furthermore, if $y \in Y$ is a general point, then $T_{Y,y} = T_{X,y} \cap L$ and $T_{Y,y}$ is a hyperplane in $T_{X,y}$. Let $M \subset L$, $M \supset T_{Y,y}$ be a tangent hyperplane and let $\beta = \beta_M \in Y^*$ be the corresponding point of the dual variety. Consider the hyperplane $\widetilde{M} = \langle M, T_{X,y}\rangle \subset \mathbb{P}^N$, where, for a subset $A \subset \mathbb{P}^N$, we denote by $\langle A \rangle$ its linear span, and let $\tilde{\beta} \in \mathbb{P}^{N*}$ denote the corresponding point. Then it is clear that $\pi(\tilde{\beta}) = \beta$, and so $Y^* \subset \pi(X^*)$.

Conversely, let $\widetilde{\beta} \in X^*$ be a general point, let $\widetilde{M} \subset \mathbb{P}^N$ be the corresponding hyperplane and put $\beta = \pi(\widetilde{\beta})$. Since $\operatorname{def} X > 0$, $\widetilde{M}$ is tangent to X along a positive-dimensional subvariety $Z_{\widetilde{\beta}} \subset X$ (more precisely, $\widetilde{M} \supset T_{X,x}$ for all $x \in Z_{\widetilde{\beta}} \setminus \operatorname{Sing} X$; from the Reflexivity Theorem it actually follows that $Z_{\widetilde{\beta}} = \mathbb{P}^{\operatorname{def} X} \subset \mathbb{P}^N$, cf. [Tev2005, Theorem 1.18] for more details). Clearly, $Z_{\widetilde{\beta}} \cap Y = Z_{\widetilde{\beta}} \cap L \neq \varnothing$. Since $(\alpha, \widetilde{\beta}) \in \mathbb{P}^{N*} \times X^*$ is a general point, $Z_{\widetilde{\beta}} \cap Y \not\subset \operatorname{Sing} Y$, and so there exists a point $y \in Y \setminus \operatorname{Sing} Y$ such that the hyperplane $M = L \cap \widetilde{M} \subset L$ corresponding to the point $\beta = \pi(\widetilde{\beta}) \in L^*$ is tangent to Y at y. Thus, $\beta \in Y^*$ and $\pi(X^*) \subset Y^*$. This completes the proof of (i). $\square$

(ii) can be proved in a similar way. However, one can also observe that claim (ii) is *dual* to claim (i), and so (i) $\Rightarrow$ (ii) by the Reflexivity Theorem 1.9. $\square$

The codegree is but the most important representative of a sequence of classical invariants reflecting infinitesimal properties of projective embedding called *classes*.

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Definition 1.11. Let $X \subset \mathbb{P}^N$ be a nondegenerate n-dimensional projective variety of degree d and codegree d^* (cf. Definition 1.8). The number

$$\mu = \begin{cases} d^*, & \operatorname{def} X = 0, \\ 0, & \operatorname{def} X > 0, \end{cases}$$

is called the *class* of X. Thus, class is equal to codegree if X^* is a hypersurface and is zero otherwise.

Let $0 \leqslant i \leqslant n$, and let X_i be the section of X by a general linear subspace $L \subset \mathbb{P}^N$, $\operatorname{codim} L = n - i$, so that $X_i \subset \mathbb{P}^{N-i}$ is a nondegenerate smooth projective variety of dimension i. The class of X_i is called the *ith class* of X and is denoted by μ_i.

Thus, $\mu_n = \mu$ is the class of X, $\mu_0 = d$ and $\mu_i = 0$ if and only if $\operatorname{def} X_i > 0$, that is, if and only if $i > n - \operatorname{def} X$.

The number

$$\boldsymbol{\mu} = \sum_{i=0}^{n} \mu_i = \sum_{i=0}^{n-\operatorname{def} X} \mu_i = \sum_{i=0}^{n-\operatorname{def} X} \operatorname{codeg} X_i$$

is called the *total class* of X.

It is clear that $\mu_i(X) = \mu_i(X_{n-1}) = \cdots = \mu_i(X_i)$ and, in particular, $\boldsymbol{\mu}(X) = \mu_n(X) + \boldsymbol{\mu}(X_{n-1})$. Furthermore, μ_i can be interpreted as the degree of the i-dimensional polar locus $P_i = P_i(L) = \{x \in X \mid \dim T_{X,x} \cap L \geqslant i - 1\}$, where $L \subset \mathbb{P}^N$ is a general linear subspace of codimension $n - i + 2$, and can be computed as an intersection on the conormal variety: $\mu_i = \int_{\mathcal{P}_X} h^{n-i} {h'}^{N-n+i-1}$, where h and h' denote the liftings on $\mathcal{P}_X$ of the classes of hyperplane sections of X and X^*, respectively.

The notion of class was first introduced and studied by Poncelet and Plücker in the case of plane curves and by Salmon in the case of surfaces in $\mathbb{P}^3$. Foundations of a general theory of polar varieties and classes were laid by Severi and developed by Todd; cf. [Kle1986, Pie1978, Sev1902, Tod1937] or [Ful1998, ch. 14].

We proceed with giving a bound for classes in terms of ramification volumes.

Theorem 1.12. *Let $X \subset \mathbb{P}^N$ be an n-dimensional nondegenerate nonsingular variety, and let R_L be the ramification divisor of the projection $p_L : X \to \mathbb{P}^n$, where $L \subset \mathbb{P}^N$, $\dim L = a - 1 = N - n - 1$ is a general linear subspace. Let $0 \leqslant i \leqslant n$, and let μ_i be the ith class of X. Then $\mu_i \leqslant r_i$.*

Proof. If $\operatorname{def} X > 0$, then we replace X by its general linear section $X_{n-\operatorname{def} X}$, where $X_{n-\operatorname{def} X} = X \cap \mathbb{P}^{N-\operatorname{def} X}$. From Lemma 1.10(i), it follows that $\operatorname{def} X_{n-\operatorname{def} X} = 0$ and

$$\mu_i(X) = \begin{cases} \mu_i(X_{n-\operatorname{def} X}), & i \leqslant n - \operatorname{def} X, \\ 0, & i > n - \operatorname{def} X. \end{cases}$$

Furthermore, since $\mu_i(X) = \mu_i(X_i)$, we see that to bound μ_i we can replace X by X_i. Thus, the proof of Theorem 1.12 reduces to showing that $d^* \leqslant r_n$ provided that $\operatorname{def} X = 0$, which we will assume from now on.

Let $M \subset L$ be a general hyperplane. We denote by $p_M : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n+1}$ and $\pi_{LM} = p_{p_M(L)} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ the projections with centers M and $p_M(L)$ respectively as well as their restrictions on X and $X' = p_M(X)$; thus, $p_L = \pi_{LM} \circ p_M$.

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Consider the hypersurface $X' = p_M(X) \subset \mathbb{P}^{n+1}$ defined by vanishing of a form $F(x_0, \ldots, x_{n+1})$ of degree d. By Lemma 1.10(ii), X'^* is the section of X^* by the linear subspace of $\mathbb{P}^{n+1*}$ corresponding to M and, in particular, def $X' = 0$. On the other hand, $X'^* = \gamma(X)$, where $\gamma : X' \dashrightarrow \mathbb{P}^{n+1*}$, $\gamma = (\partial F/\partial x_0 : \cdots : \partial F/\partial x_{n+1})$ is the Gauß map defined outside of $\operatorname{Sing} X'$. From the Reflexivity Theorem 1.9, it follows that $\gamma : X' \dashrightarrow X'^*$ is birational. It is clear that, for a general linear combination $F' = \alpha_0(\partial F/\partial x_0) + \cdots + \alpha_{n+1}(\partial F/\partial x_{n+1})$, the intersection of X' with the (polar) hypersurface of degree $d-1$ defined by vanishing of F' has the form $R_{ML} + B$, where $R_{ML} = R_{p_M(L)}$ is the ramification locus of π_{LM} for a suitable linear subspace $L \supset M$ and B is a Weil divisor supported on $\operatorname{Sing} X'$. Taking a general collection of n hyperplane sections $\Xi_1, \ldots, \Xi_n$ of X'^* and applying Lemma 1.10(ii), we see that $\operatorname{card}\{\Xi_1 \cap \cdots \cap \Xi_n\} = d^*(X') = d^*(X) = d^*$ and, if $R_{ML,i}$ is the ramification locus in X' for the projection from the point $l_i \in \mathbb{P}^{n+1}$ corresponding to the hyperplane $\langle \Xi_i \rangle$, then $\{R_{ML,1} \cap \cdots \cap R_{ML,n}\}$ contains the subset $\gamma^{-1}(\Xi_1 \cap \cdots \cap \Xi_n) \backslash \operatorname{Sing} X'$ consisting of d^* distinct points. Similarly, if $L_i = \langle M, l_i \rangle$, then the intersection $R_{L_1} \cap \cdots \cap R_{L_n} \subset X$ contains the subset $p_M^{-1}(\gamma^{-1}(\Xi_1 \cap \cdots \cap \Xi_n))$ consisting of d^* distinct points. Thus, the codegree of X does not exceed the number of intersection points of n general ramification divisors in X, that is, $d^* \leqslant r_n$. □

Another approach consists in interpreting the codegree d^* of the nondefective variety X as the number of tangent hyperplanes passing through a general linear subspace of codimension two. Let $\tilde{L} \subset \mathbb{P}^N$, $\dim \tilde{L} = N-2$, be a general linear subspace, let $L_1, \ldots, L_n \subset \tilde{L}$, $\dim L_k = a-1$, be a general collection of n linear subspaces and let $R_1, \ldots, R_n$, $R_k = R_{L_k}$, $k = 1, \ldots, n$, be the corresponding ramification divisors on X. Then it is clear that $R_1 \cap \cdots \cap R_n$ contains a component $\{x \in X \mid \dim T_{X,x} \cap \tilde{L} \geqslant n-1\}$ and the set $\{x \in X \mid \dim T_{X,x} \cap \tilde{L} \geqslant n-1\}$ consists of d^* points, which yields Theorem 1.12 (for $i < n$ it suffices to pass to X_i). This approach allows us to improve the bound for μ_i for $a > 1$; cf. Remark 1.17(iv). A further elaboration of this approach allows us to give an upper bound for the difference $r_i - \mu_i$; cf. [Zak2012]. □

Theorems 1.1 and 1.12 and Corollary 1.5 yield the following corollary.

Corollary 1.13. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate nonsingular variety of degree d and codimension a, and let $1 \leqslant i \leqslant n$. Then $\mu_i \leqslant \mu_1^i/d^{i-1} < d(d/a + 5/4)^i$. If, moreover, $d \geqslant (a-2)^2/8$, then $\mu_i < d(d/a+1)^i$.*

Examples 1.14. (i) Let X be a nonsingular curve of genus g. Then $r_1 = \deg R = d^*$, which, combined with (1.4.2), yields the *Riemann–Hurwitz formula*

$$\mu = d^* = 2g + 2d - 2 \tag{1.14.1}$$

(cf. (1.4.4) and Example 2.5 below).

(ii) Let $X \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of degree d. Then, in the notation of Example 1.4, $K_X = (d-n-2)H$ and $R_L = K_X + (n+1)H = (d-1)H$. Thus, by Theorem 1.12, $d^* \leqslant r_n = r_1^n/d^{n-1} = d(d-1)^n$. It is easy to see that the above inequality is actually an equality; cf. Theorem 1.18 below.

(iii) Let X be a (nonsingular) cubic scroll in $\mathbb{P}^4$. Then, in the notation of the proof of Theorem 1.12, $L \subset \mathbb{P}^4$ is a line, $M \in L$ is a point and the double locus D_M is a conic. The cubic surface $X' = p_M(X)$ is singular along the line $D'_M = p_M(D_M)$. One has $\mu_2 = \mu_2(X) = d^* = 3$ (in fact, since X' is obtained from the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ by first taking a general hyperplane section and then a general projection, from Lemma 1.10 it follows that X'^* is obtained from the dual Segre variety $(\mathbb{P}^1 \times \mathbb{P}^2)^* \simeq \mathbb{P}^1 \times \mathbb{P}^2$ by performing the same operations in an opposite order). It is also easy to see that $\mu_1 = \mu_1(X) = 4$ (the dual of a twisted cubic is a developable quartic singular along the twisted cubic parametrizing osculating planes; the equality $\mu_1 = 4$ also follows from (i)) and $\mu_0 = \mu_0(X) = 3$. In this example, $R_L \sim H + F$, $D_M \sim H - F$ and $R_L + D_M \sim 2H$, where H is a hyperplane section and F is a fiber of X, and so $r_2 = (R^2) = 5$ and $r_1 = 4$. The reason why $\mu_2 < r_2$ is that the double locus of the projection of X from a general point $z \in \mathbb{P}^4$ is a conic $D_z \sim H - F$ containing two pinch points corresponding to the two tangent lines to D_z passing through the point z: $3 = 5 - 2$. In this example, $R_L \sim H + F$, $D_M \sim H - F$ and $R_L + D_M \sim 2H$, where H is a hyperplane section and F is a fiber of X. The reason why $\mu = \mu_2 = 3 < (R_L^2) = 5$ is that the double line D'_M on X' contains two pinch points corresponding to two tangent lines to D_M passing through the point M: $3 = 5 - 2$.

(iv) A general projection X' of the Veronese surface $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ in $\mathbb{P}^3$ is called a *Steiner* or *Roman surface*. Here, in the notation of Theorem 1.12, $L \subset \mathbb{P}^5$ is a plane, $M \subset L$ is a line and the double locus D_M consists of three conics. In fact, if $m \in M$ is a general point, $\bar{X} = p_m(X) \subset \mathbb{P}^4$ is the projected Veronese surface and $m' = p_m(M)$, then m' is a general point of $\mathbb{P}^4$ and, as noted first by Castelnuovo, m' is contained in a unique trisecant line of $\bar{X}$ and any two of the three intersection points of this line with X lie on a unique conic corresponding to a line in $\mathbb{P}^2$. Furthermore, the surface $p_{m'}(\bar{X}) = p_M(X) = X'$ is singular along $D'_M = p_M(D_M)$, where D'_M is a union of three lines meeting in a triple point (the image of the trisecant line under the projection $p_{m'}$). It is clear that $\mu_2(X) = d^*(X) = d^*(\bar{X}) = d^*(X') = d^* = 3$ (the determinant of a symmetric 3×3 matrix is a cubic form) and, by Lemma 1.10(ii), X'^* is a cubic surface with four nodes; more precisely, the Del Pezzo surface defined by the plane cubic curves passing through the six intersection points of four general lines in the plane. In this example, $R_L \sim 3\ell$, $D_M \sim 3\ell$, $H = 2\ell$, $R_L + D_M \sim 3H$ (here we use the isomorphism $X \simeq \mathbb{P}^2$ and ℓ is a line in $\mathbb{P}^2$), $\mu_1 = 6$ (the dual surface is defined by vanishing of the discriminant of a quartic binary form) and $\mu_0 = 4$. The reason why $\mu = \mu_2 = 3 < (R_L^2) = 9$ is that each of the three double lines on the Steiner surface obviously contains two pinch points (since any point in the complement of a plane conic is contained in exactly two tangent lines to the conic): $3 = 9 - 6$.

General formulas of this type for surfaces in $\mathbb{P}^3$ were first obtained by Salmon, Cayley and Zeuthen.

Corollary 1.15. *Let $X \subset \mathbb{P}^N$ be a nondegenerate nonsingular variety of codimension a, degree d and sectional genus π. Then*

$$d^* \leqslant r_{n-\operatorname{def} X} \leqslant \frac{r_1^{n-\operatorname{def} X}}{d^{n-\operatorname{def} X - 1}} < d\left(\frac{d}{a} + \frac{5}{4}\right)^{n-\operatorname{def} X}, \quad r_1 = \mu_1 = 2\pi + 2d - 2.$$

If, moreover, $d \geqslant (a-2)^2/8$, then $d^* \leqslant (d/a+1)^{n-\operatorname{def} X}$.

Proof. Corollary 1.15 follows from Theorems 1.1 and 1.12 and Corollary 1.5 by the argument in the first paragraph of the proof of Theorem 1.12 relying on Lemma 1.10(i) and showing that $d^* = \mu_{n-\operatorname{def} X}$. □

Example 1.16. Let $X^n \subset \mathbb{P}^N$ be a nonsingular rational normal scroll. Then $d = \deg X = N - n + 1$ and $\operatorname{def} X = n - 2$ (since each tangent hyperplane to X contains a linear generator $\mathbb{P}^{n-1} \subset X$ and the corresponding hyperplane section is reducible, so that the dimension of its singular locus, equal to the defect, is $n-2$). Another way to see this is to observe that X is a linear section of a Segre variety $\mathbb{P}^1 \times \mathbb{P}^a \subset \mathbb{P}^{2a+1}$, which is easily seen to be self-dual. Then, applying Lemma 1.10, one concludes that X^* is a projection of the Segre variety, and so $\dim X^* = a + 1 = N - n + 1$ and $\operatorname{def} X = N - 1 - (N - n + 1) = n - 2$. The same argument shows that $d^* = \deg X^* = a+1 = N-n+1 = d$. On the other hand, in this case $r_1 = 2d-2$ and Corollary 1.15 only gives $d^* \leqslant r_1^2/d = 4d-8-4/d$, which fails to be sharp for $d \neq 2$. It should be noted that for $d=3$ this example reduces to Example 1.14(iii) and that any smooth section of X is again a rational normal scroll.

Remarks 1.17. (i) Let $X \subset \mathbb{P}^N$ be a nondegenerate variety of dimension n and codimension a, and let $L \subset \mathbb{P}^N$, $\dim L = a+i-2$, $0 \leqslant i \leqslant n$, be a general linear subspace. The subset $P_i = P_i(L) = \overline{\{x \in \operatorname{Sm} X \mid \dim T_{X,x} \cap L \geqslant i-1\}}$ is called the *ith polar locus* of X with respect to L. If X is nonsingular, then $x \in P_i$ if and only if there exists a hyperplane passing through L and tangent to X at x (cf. [Zak2012] for an alternative definition of polar classes). It is clear that $\operatorname{codim}_X P_i = i$ and $\deg P_i = \mu_i$, which gives an alternative (classical) definition of classes (cf. for example [Pie1978]). In particular, $P_1 = R_L$ is the ramification divisor from Example 1.4 and $\mu_1 = r_1 = \deg R_L$.

(ii) The argument in the proof of Theorem 1.12 is closely related to the classical relation $R + D \sim (d-1)H$, where $R = R_L$ is the ramification divisor of a general projection $p_L : X \to \mathbb{P}^n$ defined in (1.4.1) and D is the double point divisor of a general projection $p_M : X \to \mathbb{P}^{n+1}$. This relation (known to Clebsch in the case of curves and surfaces) easily follows from the argument involving polar hypersurfaces in the proof of Theorem 1.12. Furthermore, even without referring to Theorem 1.1 and Example 1.4, from the proof of Theorem 1.12 it immediately follows that $\mu_i \leqslant d(d-1)^i$ for all $1 \leqslant i \leqslant n$ with equality holding if and only if $D = \varnothing$, that is, if and only if X is a hypersurface; cf. Theorem 1.18 for a more general statement and Remark 1.19 for its refinement.

(iii) The first inequality in Corollary 1.15 actually yields a bound for the codegree of X in terms of the degree d and the sectional genus $\pi(X)$ (defined as $\pi(X) = g(X_{n-1})$, so that one has $r_1 = 2d + 2\pi - 2$). In the case when the sectional genus of X is much less than the maximum given by Castelnuovo's theorem, this bound is much better than the general one given by the second inequality.

(iv) The bounds for codegree given in Corollary 1.15 are not optimal. The main reason for the failure of these bounds to be sharp is that the inequality $\mu_i \leqslant r_i$ proved in Theorem 1.12 is *always* strict provided that $i > 1$ and $a > 1$ (cf. Examples 1.14 and [Zak2012]). In fact, let $L_0 \subset \mathbb{P}^N$, $\dim L_0 = a-2$, be a general linear subspace, let $L_k \supset L_0$, $\dim L_k = a-1$, $k = 1, \dots, i$, $i \leqslant n$, be general linear subspaces and let $L = \langle L_1, \dots, L_i \rangle$, $\dim L = a+i-2$, be their linear span. Denote by $\mathcal{R}^m = \mathcal{R}^m(M) \subset X$, $n \leqslant m \leqslant 2n-1$, the ramification locus of the projection $p_M : X \to \mathbb{P}^m$ from a general linear subspace $M \subset \mathbb{P}^N$, $\dim M = N-m-1$. It is easy to see that $\operatorname{codim}_X \mathcal{R}^m = m-n+1$; in particular, $\mathcal{R}^n$ is the ramification divisor R from Example 1.4. It is clear that $R_{L_1} \cap \dots \cap R_{L_i} = P_i(L) \cup \mathcal{R}^{n+1}(L_0)$ for $i \geqslant 2$ (for $i = 2$ this is actually a classical formula representing the self intersection of a hyperplane section of the Grassmann variety $G(N,n)$ parametrizing n-dimensional linear subspaces of $\mathbb{P}^N$ as a sum of two Schubert subvarieties of codimension two) and $\mu_i \leqslant r_i - \deg \mathcal{R}^{n+1}$, so that $\mu_i < r_i$ provided that $\mathcal{R}^{n+1} \neq \varnothing$, which is always so if $a > 1$. If $n = 2$, then one arrives at the classical formula $r_2 = \mu_2 + \nu_2$, where $\nu_2 = \deg \mathcal{R}^3$ is the so-called *type* of the surface X (cf. [SR1949, ch. IX, Theorem 2.01]); this explains the appearance of the number of pinch points in Examples 1.14. In general, the ramification locus $\mathcal{R}^{n+1}$ can be computed via Johnson's formula (cf. for example [Ful1998, Example 9.3.13]), which, combined with the relation $R + D \sim (d-1)H$ mentioned in (ii), yields the inequality $\deg \mathcal{R}^{n+1} \leqslant d(d-1) - r_1$ (with equality holding if and only if either $n = 1$ or $a = 2$ or X is a Veronese surface); if $n = 2$, one arrives at a classical formula expressing the number of improper nodes in terms of type and classes (this formula is due to Zeuthen;

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cf. [SR1949, ch. IX, § 2.13]). One can also give lower and upper bounds for the difference $r_i - \mu_i$ in terms of other projective invariants (cf. [Zak2012]).

Also the bound $r_i \leqslant r_1^i/d^{i-1}$ obtained in Theorem 1.1 is not always sharp.

Varieties of positive defect have an extra reason for the failure of this bound to be sharp (cf. Example 1.16). Finally, the last bound in Corollary 1.15 depends on the bound $\mathcal{C}(d,a) < d(d/a + 5/4)$ (cf. (1.4.5) and (1.5.1)), which is not sharp either (cf. Remark 1.7(i)). The real import of Corollary 1.15 is that $\mu_i < d^{i+1}/a^i + \cdots$, where '$\cdots$' stand for an (easily computable) polynomial of degree i in d. Below we will give examples of series of varieties whose classes have the form $\mu_i = d^{i+1}/a^i + \cdots$ (cf. Example 3.5 and Proposition 3.11), and so in this sense our bound for classes is good. In § 3, we will obtain better bounds for classes and classify the varieties on the boundary.

One case when the bound in Corollary 1.15 *is* sharp is the case of hypersurfaces (cf. Example 1.14(ii)).

THEOREM 1.18. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate (not necessarily nonsingular) variety of degree d. Then $\mu_i \leqslant d(d-1)^i$, $0 \leqslant i \leqslant n$, with equality (for some i) holding if and only if either $i = 0$ or X is a hypersurface with $\dim(\operatorname{Sing} X) < n - i$ (that is, if and only if X_i is a nonsingular hypersurface).*

Similarly, $d^* \leqslant d(d-1)^{n-\operatorname{def} X}$ with equality holding if and only if X is a cone over a nonsingular hypersurface $Y \subset \mathbb{P}^{N-\operatorname{def} X}$ with vertex $\mathbb{P}^{\operatorname{def} X - 1}$, in which case $n = N-1$, $\dim Y = n - \operatorname{def} X$, $\operatorname{Sing} X = \mathbb{P}^{\operatorname{def} X-1}$ and $X^* = Y^* \subset \mathbb{P}^{N-\operatorname{def} X^*} \subset \mathbb{P}^{N^*}$ is the linear subspace orthogonal to the vertex of the cone X.

In particular, one always has $d^* \leqslant d(d-1)^n$ with equality holding if and only if X is a nonsingular hypersurface.

Proof. Theorem 1.18 can be proved by specializing the bound in Theorem 1.12 (cf. Remarks 1.7(i) and 1.17(i)), but to avoid making any assumptions on the singularities of X we give an independent proof.

Replacing X by its section X_i by a general linear subspace $\mathbb{P}^{N+i-n} \subset \mathbb{P}^N$, one may assume that $i = n$, that is, consider the codegree instead of class.

If $\operatorname{def} X > 0$, then one can replace X by its section $X_{n-\operatorname{def} X}$ by a general linear subspace $\mathbb{P}^{N-\operatorname{def} X} \subset \mathbb{P}^N$. Applying Lemma 1.10(i), we see that $X^*_{n-\operatorname{def} X}$ is a general projection of X^*, and so $\operatorname{def} X_{n-\operatorname{def} X} = 0$, $\deg X_{n-\operatorname{def} X} = \deg X = d$ and $\deg X^*_{n-\operatorname{def} X} = \deg X^* = d^*$. This reduces Theorem 1.18 to the case $\operatorname{def} X = 0$.

Suppose that $\operatorname{def} X = 0$, $\operatorname{codim} X = a > 1$. Then we can replace X by its projection $\bar{X} \subset \mathbb{P}^{n+1}$ from a general linear subspace $\mathbb{P}^{a-2} \subset \mathbb{P}^N$. Applying Lemma 1.10(ii), we see that $\bar{X}^*$ is the section of X^* by a general $(N+1)$-dimensional linear subspace in $\mathbb{P}^{N^*}$. Thus, $\bar{X}$ is a hypersurface of degree d in $\mathbb{P}^{n+1}$ and $\deg \bar{X}^* = \mu_n(\bar{X}) = \deg X^* = d^*$.

To prove the bound in Theorem 1.18, it now remains to verify it in the case when X is a hypersurface with $\operatorname{def} X = 0$ in $\mathbb{P}^{n+1}$ defined by an equation $F(x_0 : \cdots : x_{n+1}) = 0$, $\deg F = d$. To this end we observe that, by definition, X^* is the image of the Gauß map $\gamma : X \dashrightarrow \mathbb{P}^{n+1^*}$, $\gamma(x_0 : \cdots : x_{n+1}) = (\partial F/\partial x_0 : \cdots : \partial F/\partial x_{n+1})$ defined outside the singular locus $\operatorname{Sing} X \subset X$. Since $\partial F/\partial x_0, \ldots, \partial F/\partial x_{n+1}$ are forms of degree $d-1$, this means that $X^* = \varpi(v_{d-1}(X))$, where $v_{d-1}(X) \subset \mathbb{P}^{\binom{n+d}{n+1}-1}$ is the $(d-1)$st Veronese embedding of X and $\varpi : \mathbb{P}^{\binom{n+d}{n+1}-1} \dashrightarrow \mathbb{P}^{n+1\,*}$

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is a linear projection. Thus,

$$\deg X^* = \deg \gamma(X) \leqslant \deg v_{d-1}(X) = d(d-1)^n. \tag{1.18.1}$$

To classify the varieties on the boundary, we need first to classify the hypersurfaces for which the inequality in (1.18.1) turns into equality. Clearly, this happens if and only if the projection ϖ is a regular map, that is, if and only if the Gauß map $\gamma : X \dashrightarrow X^*$ does not have fundamental points or, equivalently, if $\operatorname{Sing} X = \varnothing$. Furthermore, being linearly normal, a nonsingular hypersurface cannot be a birational regular projection of a nondegenerate variety. In view of our reductions, this completes classification of varieties for which $d^* = d(d-1)^n$.

Arguing in a similar way, we see that to complete the proof of Theorem 1.18 it remains to classify the hypersurfaces $X^n \subset \mathbb{P}^{n+1}$ whose intersection with a general linear subspace of codimension $\operatorname{def} X$ is nonsingular. But, for such a hypersurface, $\dim \operatorname{Sing} X < \operatorname{def} X$ and a general tangent hyperplane $\alpha \in X^*$ is tangent to X at all the points of $\mathbb{P}_\alpha \subset X$, where $\mathbb{P}_\alpha$ is a linear subspace of dimension $\operatorname{def} X$ (cf. Theorem 1.9 or [Tev2005, Theorem 1.18]). Clearly, this is only possible if $\dim \operatorname{Sing} X = \operatorname{def} X - 1$ and all the $\mathbb{P}_\alpha$ pass through a component of $\operatorname{Sing} X$. But then, for general $\alpha, \alpha' \in X^*$, $\mathbb{P}_\alpha \cap \mathbb{P}_{\alpha'} = \mathbb{P}^{\operatorname{def} X - 1} \subset \operatorname{Sing} X$ and each point $x \in X$ is contained in a linear subspace $\mathbb{P}_x^{\operatorname{def} X}$ passing through this $\mathbb{P}^{\operatorname{def} X-1}$. This means that $\operatorname{Sing} X = \mathbb{P}^{\operatorname{def} X-1}$ and X is a cone with vertex $\mathbb{P}^{\operatorname{def} X - 1}$. □

Remark 1.19. For X nonsingular, Theorem 1.12 combined with Remark 1.7(i) yields a much better bound for classes than Theorem 1.18, *viz.* $\mu_i < d^{i+1}/2^i$ for $a > 1$, $i \geqslant 1$.

Theorem 1.18 has two generalizations to varieties of given codimension $a \geqslant 1$. In Theorem 3.16(i), we show that $\mu_i < d^{i+1}/a^i$, at least for $d \geqslant (a+1)^2$. We end this section with another generalization in which the property of having degree d is substituted by the property of being defined by equations of degree at most d.

DEFINITION 1.20. Let $X \subset \mathbb{P}^N$ be a projective variety and let $\mathcal{I}_X$ be the sheaf of ideals defining X. We say that X is defined by equations of degree (not exceeding) d if the sheaf $\mathcal{I}(d)$ is generated by its global sections.

It is clear that if $d' \geqslant d$ and X is defined by equations of degree d, then X is also defined by equations of degree d'.

For example, a complete intersection of a hypersurfaces of degrees $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_a = d$ is defined by equations of degree d (of course, if $d_1 < d$ and we wish to represent X (scheme theoretically) as an intersection of hypersurfaces of degree d, then we need more than a hypersurfaces).

THEOREM 1.21. *Let $X^n \subset \mathbb{P}^N$, $N = n + a$, be a (not necessarily nondegenerate or nonsingular) variety defined by equations of degree d, and let i, $0 \leqslant i \leqslant n$ be an integer. Then $\mu_i \leqslant \binom{a+i-1}{i} d^a (d-1)^i = \binom{a+i-1}{i} d^{a+i} + \cdots$ (where '$\cdots$' stand for a polynomial in d of degree at most $a+i-1$) with equality holding if and only if X is a complete intersection of a hypersurfaces of degree d with $\dim(\operatorname{Sing} X) < n - i$ (so that X_i is a nonsingular complete intersection). In particular, $d^* \leqslant \binom{N-1}{n} d^a (d-1)^n = \binom{N-1}{n} d^N + \cdots$ with equality holding if and only if X is a nonsingular complete intersection of a hypersurfaces of degree d.*

Sketch of proof. Replacing X by its linear section, we see that it suffices to consider the case when $i = n$ and $\operatorname{def} X = 0$, that is, to bound the class. We pick a general equations $F_1, \ldots, F_a$

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of the variety X, so that $F_i = F_i(x_0 : \cdots : x_N)$, $i = 1, \ldots, a$, is a form of degree d and the differentials $d_xF_1, \ldots, d_xF_a$ are linearly independent for $x \in U$, where $U \subset X$ is an open subset. Let $\gamma_i : U \to \mathbb{P}^{N\,*}$, $\gamma_i(x) = ((\partial F_i/\partial x_0)(x) : \cdots : (\partial F_i/\partial x_N)(x))$, $i = 1, \ldots, a$, and let $\Xi_i = \overline{\gamma_i(X)} = \overline{\gamma_i(U)} \subset \mathbb{P}^{N\,*}$, so that $\gamma_i : X \dashrightarrow \Xi_i$ is a rational map. By definition, $X^* = \overline{\bigcup_{x\in U}\langle\gamma_1(x), \ldots, \gamma_a(x)\rangle}$, where $\langle\gamma_1(x), \ldots, \gamma_a(x)\rangle = \mathcal{P}_x$ is the conormal space to X at the point x (cf. Definition 1.8). From Theorem 1.9, it follows that the maps $\gamma_i : X \dashrightarrow \Xi_i$, $i = 1, \ldots, a$, are actually birational. Let $\mathbf{P}$ be a projective space of dimension $a(N+1) - 1$, and let $\varkappa_i : \mathbb{P}^{N\,*} \hookrightarrow \mathbf{P}$, $i = 1, \ldots, a$, be linear embeddings such that $\varkappa_i(\mathbb{P}^{N\,*}) \cap \langle\varkappa_1(\mathbb{P}^{N\,*}), \ldots, \widehat{\varkappa_i(\mathbb{P}^{N\,*})}, \ldots, \varkappa_n(\mathbb{P}^{N\,*})\rangle = \varnothing$, $1 \leqslant i \leqslant a$. Put $\tilde{\gamma}_i = \varkappa_i \circ \gamma_i : X \dashrightarrow \mathbf{P}$, $\tilde{\Xi}_i = \tilde{\gamma}_i(X)$, $i = 1, \ldots, a$, $\widetilde{X^*} = \overline{\bigcup_{x\in U}\langle\tilde{\gamma}_1(x), \ldots, \tilde{\gamma}_a(x)\rangle}$. It is clear that X^* is a birational projection of $\widetilde{X^*}$, and so $\deg X^* \leqslant \deg \widetilde{X^*}$ with equality if and only if the center of projection does not meet $\widetilde{X^*}$ (the points of the form $(\tilde{\gamma}_1(x), \ldots, \tilde{\gamma}_N(x))$, $x \in \operatorname{Sm} X$, such that the differentials $d_xF_1, \ldots, d_xF_a$ are linearly dependent clearly lie in the center of projection). Arguing as in the proof of Theorem 1.18, we see that $\deg \widetilde{\Xi}_i = \deg \Xi_i \leqslant (d-1)^n \deg X$, $i = 1, \ldots, a$, with equality holding if and only if the hypersurface in $\mathbb{P}^N$ defined by vanishing of F_i is nonsingular along X. Furthermore, $\deg X \leqslant d^a$ with equality holding if and only if X is a complete intersection of hypersurfaces of degree d. To prove Theorem 1.21, it now suffices to reproduce the argument in [Har1992, 19.5–19.7]. □

2. Bounds for Betti numbers

Let $X^n \subset \mathbb{P}^N$ be an n-dimensional complex projective algebraic variety of degree $d = \deg X > 1$. The variety X has the structure of a finite simplicial complex of dimension $2n$ and, if X is nonsingular, then X can be viewed as a compact topological manifold of dimension $2n$. We denote by $b_i(X) = \dim H_i(X, \mathbb{Q})$ the ith Betti number of X and by $b(X) = \sum_{i=0}^{2n} b_i(X)$ the *total Betti number* of X.

If X is nonsingular, then Poincaré duality yields

$$b_i(X) = b_{2n-i}(X), \quad i = 0, \ldots, 2n. \tag{2.0.2}$$

Denote by $e(X) = \sum_{i=0}^{2n}(-1)^i b_i(X)$ the *Euler–Poincaré characteristic* of X. From (2.0.2), it follows that

$$b(X) = b_n + 2\sum_{i=0}^{n-1} b_i, \tag{2.0.3}$$

$$e(X) = (-1)^n b_n + 2\sum_{i=0}^{n-1}(-1)^i b_i. \tag{2.0.4}$$

To compare the Betti numbers of X with those of its general hyperplane section, we need some facts from Lefschetz theory.

Definition 2.1. Let $X^n \subset \mathbb{P}^N$ be a nonsingular complex projective variety. A line $\ell \subset \mathbb{P}^{N*}$ is called a *Lefschetz pencil* if and only if ℓ meets X^* in exactly μ points, where $\mu = \mu_n$ is the class of X (cf. Definition 1.11). This means that either $\operatorname{def} X > 0$ and $\ell \cap X^* = \varnothing$ or $\operatorname{def} X = 0$ and ℓ transversely intersects X^* meeting it in $d^* = \deg X^* = \mu$ nonsingular points.

The linear subspace $L_\ell = \bigcap_{\alpha\in\ell} L_\alpha$ of codimension two in $\mathbb{P}^N$ is called the *axis* of the pencil ℓ.

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To each Lefschetz pencil ℓ, we associate a variety

$$\widetilde{X} = \widetilde{X}_\ell \subset X \times \ell, \quad \widetilde{X} = \{(x, \alpha) \mid x \in L_\alpha\}.$$

The projection of $X \times \ell$ onto the second factor yields a morphism $p : \widetilde{X} \to \ell$ whose fibers are sections by hyperplanes from the pencil ℓ. The morphism $\sigma : \widetilde{X} \to X$ induced by the projection of $X \times \ell$ onto the first factor is the inverse of the blowing up of the subvariety $X'' = X_{n-2} = L_\ell \cap X \subset X$; it is clear that X'' is nonsingular and that the exceptional divisor $E = \sigma^{-1}(X'')$ is a line bundle over X''.

From the above description, it is clear that the sections of X by hyperplanes from ℓ fall into two types: the nonsingular sections that are all diffeomorphic between themselves and the singular ones, which exist only if $\operatorname{def} X = 0$. We denote by $X' = X_{n-1}$ a smooth hyperplane section of X (or a general fiber of $p : \widetilde{X} \to \ell$) and by Y_k, $k = 1, \dots, \mu$, $Y_k = X \cap L_{\alpha_k}$, $\ell \cap X^* = \alpha_1 \cup \dots \cup \alpha_\mu$, the singular hyperplane sections (or the singular fibers of $p : \widetilde{X} \to \ell$). There are a closed embedding $j : X' \hookrightarrow X$ and a topological retraction $\rho_k : X' \to Y_k$ defined by the trajectories orthogonal to a nonsingular fiber in a small neighborhood of the fiber Y_k of the map p.

We summarize what we need from Lefschetz theory in the following proposition.

PROPOSITION 2.2. *Let $X^n \subset \mathbb{P}^N$ be a nonsingular variety, let X^* be its dual variety and let $\ell \subset \mathbb{P}^{N*}$ be a Lefschetz pencil. Then, in the above notation:*

(i) *the following conditions are equivalent:*
   - (a) $\operatorname{def} X = 0$ *and* $\alpha \in \operatorname{Sm} X^*$;
   - (b) *the hyperplane section $Y_\alpha = L_\alpha \cap X$ corresponding to α has a unique singular point x_α, and x_α is a nondegenerate quadratic singularity of Y_α (which means that, in a suitable coordinate system in a small complex neighborhood U_α of the point x_α in X, Y_α is defined by the equation $x_1^2 + \dots + x_n^2 = 0$);*

(ii) *each of the singular fibers Y_k, $k = 1, \dots, \mu$, has a unique nondegenerate quadratic singular point x_k with $x_k \notin X''$;*

(iii) *the homology map $j_i : H_i(X', \mathbb{Q}) \to H_i(X, \mathbb{Q})$ is an isomorphism for $i < n - 1$ and is surjective for $i = n - 1$. In particular, $b_i' \geqslant b_i$ for $0 \leqslant i \leqslant n - 1$ and $b_i = b_i'$ for $i < n - 1$, where $b_i = b_i(X)$ and $b_i' = b_i(X')$ are the Betti numbers of X and X', respectively;*

(iv) $b_i \geqslant b_{i-2}$ *for* $2 \leqslant i \leqslant n$;

(v) $\rho_{k,i} : H_i(X', \mathbb{Q}) \to H_i(Y_k, \mathbb{Q})$ *is an isomorphism for $i < n - 1$, $k = 1, \dots, \mu$. Furthermore, one of the following conditions holds:*
   - (a) $\ker \rho_{k,n-1} = \mathbb{Q}\delta_k$ *is generated by a single element δ_k called the* Lefschetz vanishing cycle *associated to the point $\alpha_k \in \ell$ and $\rho_{k,n}$ is an isomorphism;*
   - (b) $\rho_{k,n-1}$ *is an isomorphism and* $\operatorname{coker} \rho_{k,n} = \mathbb{Q}\Delta_k$ *is generated by a single element* Δ_k.

   Moreover, if one of the above conditions holds for some k, $1 \leqslant k \leqslant \mu$, then it holds for all k and

$$\langle \delta_k^2 \rangle = \begin{cases} 0, & n \equiv 0 \pmod 2, \\ 2(-1)^{(n-1)/2}, & n \equiv 1 \pmod 2, \end{cases} \qquad k = 1, \dots, \mu, \tag{2.2.1}$$

   where $\langle\ \rangle$ denotes the intersection pairing on $H_{n-1}(X', \mathbb{Q})$. In particular, (b) is possible only if $n \equiv 0 \pmod 2$;

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(vi) $\ker j_{n-1} = \mathrm{Van}$, *where* $\mathrm{Van} \subset H_{n-1}(X', \mathbb{Q})$ *is the subspace of vanishing cycles spanned by* $\delta_1, \dots, \delta_\mu$;

(vii) *for* $k = 1, \dots, \mu$, *one has* $e(Y_k) = e(X') + (-1)^n$.

Proof. See [DK1973, Lam1981, Lef1924, Moi1967]. Geometrically, if Y_k is defined in a neighborhood of x_k by an equation $x_1^2 + \cdots + x_n^2 = 0$, then δ_k can be thought of as a sphere defined in real coordinates by the equation $x_1^2 + \cdots + x_n^2 = \varepsilon$. If the cycle δ_k happens to be homologous to zero in X', then the chain bounding it tends to a limit cycle Δ_k in Y_k. □

Definition 2.3. In the notation and assumptions of Proposition 2.2, the vector space Van is called the *vanishing homology subspace*. Its dimension is called the *vanishing number* of the pair (X, X') and is denoted by $\lambda = \lambda_{n-1}(X') = b'_{n-1}(X') - b_{n-1}(X)$.

The following well-known proposition expressing the classes μ_i in purely topological terms is an immediate generalization of the Riemann–Hurwitz formula for curves (cf. Example 2.5 below).

Proposition 2.4. *Let* $X \subset \mathbb{P}^N$ *be a nonsingular nondegenerate variety. Then*

$$\mu_i = (-1)^i [e(X_i) - 2e(X_{i-1}) + e(X_{i-2})], \quad i = 0, \dots, n \tag{2.4.1}$$

(here, as usual, X_k *denotes the intersection of* X *with a general linear subspace of codimension* $n - k$ *in* $\mathbb{P}^N$*).*

Proof. Clearly, it suffices to prove the proposition in the case $i = n$. In the notation after Definition 2.1, we compute $e(\widetilde{X})$ in two different ways, using the two projections $\sigma : \widetilde{X} \to X$ and $p : \widetilde{X} \to \ell$ and the excision (or additivity) property of the Euler–Poincaré characteristic. Since σ is an isomorphism on the complement of $E = \sigma^{-1}(X'')$ and E is a line bundle over X'', one has

$$\begin{aligned} e(\widetilde{X}) &= e(\widetilde{X} \backslash E) + e(E) \\ &= e(X \backslash X'') + e(\ell) \cdot e(X'') = e(X \backslash X'') + 2e(X'') = e(X) + e(X''). \end{aligned} \tag{2.4.2}$$

On the other hand, similar arguments applied to the projection p combined with Proposition 2.2(vii) yield

$$\begin{aligned} e(\widetilde{X}) &= e\left(\widetilde{X} \Big\backslash \bigcup_{k=1}^{\mu} Y_k\right) + \sum_{k=1}^{\mu} e(Y_k) \\ &= e\left(\ell \Big\backslash \bigcup_{k=1}^{\mu} \alpha_k\right) \cdot e(X') + \mu[e(X') + (-1)^n] = 2e(X') + (-1)^n \mu. \end{aligned} \tag{2.4.3}$$

Combining (2.4.2) and (2.4.3), we get

$$e(X) + e(X'') = 2e(X') + (-1)^n \mu,$$

which yields (2.4.1) in the case $i = n$. Replacing $X = X_n$ by X_i, we complete the proof of Proposition 2.4. □

Example 2.5. Suppose that $n = 1$ and $X = C$ is a nonsingular projective curve of degree d and genus g. Then a general projection of C onto a projective line has precisely d^* simple ramification points. In this case $e(X) = 2 - 2g$, $e(X') = d$, $e(X'') = 0$ and Proposition 2.4 yields

$$\mu = d^* = 2d + 2g - 2, \tag{2.5.1}$$

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which is just the Riemann–Hurwitz formula expressing genus in terms of ramification (cf. Example 1.14(i)).

Example 2.6. Let $X^n \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of degree d. Since X is isomorphic to a smooth hyperplane section of the variety $v_d(\mathbb{P}^{n+1})$ (where, as usual, $v_d : \mathbb{P}^{n+1} \to \mathbb{P}^{\binom{n+d+1}{d}-1}$ is the dth Veronese embedding of $\mathbb{P}^{n+1}$), all nonsingular hypersurfaces of dimension n and degree d are diffeomorphic and, from Proposition 2.2(iii) and the Poincaré duality, it follows that

$$b_i(X) = \begin{cases} 0, & i \neq n, i \equiv 1 \pmod 2; \\ 1, & i \neq n, i \equiv 0 \pmod 2. \end{cases}$$

Thus,

$$b(X) = \begin{cases} n + b_n(X) + 1, & n \equiv 1 \pmod 2; \\ n + b_n(X), & n \equiv 0 \pmod 2, \end{cases} \tag{2.6.1}$$

$$e(X) = \begin{cases} n - b_n(X) + 1, & n \equiv 1 \pmod 2; \\ n + b_n(X), & n \equiv 0 \pmod 2. \end{cases} \tag{2.6.2}$$

Combining (2.6.1) and (2.6.2) with Theorem 1.18 and Proposition 2.4, we get the following recurrent formula for $b(X)$:

$$b(X) = d(d-1)^n - 2b(X') - b(X'') + 4n, \tag{2.6.3}$$

where X' (respectively X'') is a nonsingular hypersurface of degree d and dimension $n-1$ (respectively $n-2$). Using (2.6.3), we get the following general formula for $b(X)$ (cf. also [KK1989, ch. IV, § 5.10]):

$$\begin{aligned} b(X) &= \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + n + 1 + (-1)^n \\ &= d\left(\sum_{i=0}^{n} (-1)^i \binom{n+2}{i} d^{n-i}\right) + (1 + (-1)^{n+1})(n+1). \end{aligned} \tag{2.6.4}$$

In particular, if X^n is a nonsingular hypersurface of degree d, $b(X^n)$ has the following values for $1 \leqslant n \leqslant 4$:

$$\begin{aligned} b(X^1) &= d^2 - 3d + 4; \\ b(X^2) &= d^3 - 4d^2 + 6d; \\ b(X^3) &= d^4 - 5d^3 + 10d^2 - 10d + 8; \\ b(X^4) &= d^5 - 6d^4 + 15d^3 - 20d^2 + 15d \end{aligned} \tag{2.6.5}$$

(the first of these formulas is equivalent to the well-known computation of the genus of nonsingular plane curves).

We will also need the formulas for the Betti numbers of nonsingular quadrics (the case $d = 2$ in the above computations):

$$b_i(X) = \begin{cases} 0, & i \equiv 1 \pmod 2, \\ 1, & i \equiv 0 \pmod 2, i \neq n, \\ 2, & i = n \equiv 0 \pmod 2, \end{cases} \qquad b(X) = \begin{cases} n+1, & n \equiv 1 \pmod 2, \\ n+2, & n \equiv 0 \pmod 2. \end{cases} \tag{2.6.6}$$

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Corollary 2.7. *In the assumptions of Proposition 2.4, one has*

(i) $\mu_i = (b_i(X_i) - b_{i-2}(X_i)) + 2\lambda_{i-1}(X_{i-1}) + \lambda_{i-2}(X_{i-2})$;

(ii) $\operatorname{def} X_i > 0$ *if and only if* $\lambda_{i-2}(X_{i-2}) = \lambda_{i-1}(X_{i-1}) = b_i(X_i) - b_{i-2}(X_i) = 0$.

Proof. To get (i), it suffices to substitute in (2.4.1) the expressions for $e(X)$, $e(X_{n-1})$ and $e(X_{n-2})$ obtained from (2.0.4). To get (ii), we observe that the three summands on the right are nonnegative by Proposition 2.2(iii) and (iv). □

Corollary 2.8. *Let X be a nonsingular nondegenerate complex projective variety of dimension n and codimension a. Put $b_k = b_k(X)$, $0 \leqslant k \leqslant 2n$, $\lambda_{n-1} = \lambda_{n-1}(X')$. Then:*

(i) *if* $\operatorname{def} X > 0$, *then* $b_n = b_{n-2}$, *and if* $\operatorname{def} X = 0$, *then*

$$b_{n-2} \leqslant b_n < b_{n-2} + \mu_n \leqslant b_{n-2} + \frac{\mu_1^n}{d^{n-1}} < b_{n-2} + d\left(\frac{d}{a} + \frac{5}{4}\right)^n; \tag{2.8.1}$$

(ii) *if* $\operatorname{def} X > 0$, *then* $\lambda_{n-1} = 0$, *and if* $\operatorname{def} X = 0$, *then*

$$\lambda_{n-1} \leqslant \frac{1}{2}\mu_n \leqslant \frac{1}{2}r_n \leqslant \frac{\mu_1^n}{2d^{n-1}} < \frac{d}{2}\left(\frac{d}{a} + \frac{5}{4}\right)^n. \tag{2.8.2}$$

Proof. The first inequality in (2.8.1) follows from Proposition 2.2(iv), and the second one from Proposition 2.2(iii) and Corollary 2.7(i). If $\operatorname{def} X > 0$, then, from Corollary 2.7(ii) it follows that $b_n = b_{n-2}$, and if $\operatorname{def} X = 0$, then, by Corollary 2.7(i), $b_n \leqslant b_{n-2} + \mu_n$. To prove (i), it now remains to apply Theorem 1.12 and Corollaries 1.5 and 1.15.

Similarly, (2.8.2) follows from Corollary 2.7(i), Proposition 2.2(iii) and (iv), Theorem 1.12 and Corollaries 1.5 and 1.15. □

Theorem 2.9. *In the above notation, one has*

(i) $e(X) = (-1)^n[\mu_n - 2\mu_{n-1} + \cdots + (-1)^n(n+1)\mu_0]$;

(ii) $b_i(X) = b_{2n-i}(X) \leqslant \mu_i + \mu_{i-2} + \cdots + \mu_{i-2[i/2]}$, $i = 1, \ldots, n$;

(iii) $b(X) \leqslant \mu_n + 2\mu_{n-1} + 3\mu_{n-2} + \cdots + (n+1)\mu_0$.

Proof. To prove (i), it suffices to substitute in the right-hand part the equalities (2.4.1) for $i = 0, \ldots, n$.

To prove (ii), we sum up the equalities in Corollary 2.7(i) for $k = i, i-2, \ldots, i-2[i/2]$, where brackets denote integral part, and use the nonnegativity of λ_k (cf. Definition 2.3).

(iii) is obtained by summing up the inequalities in (ii). □

Remarks 2.10. (i) In view of Proposition 2.2 and the Poincaré duality, all homologies of X except for the middle one are bounded by homologies of a general hyperplane section X'. The inequality (2.8.1) bounds the middle homology and thus allows us to bound the total Betti number $b(X)$; alternatively, this can be done using Theorem 2.9(iii) (cf. Corollary 2.14 below).

(ii) The first inequality in (2.8.2) (the author recently learned that this inequality was earlier proved as [KP2004, Theorem 4]) shows that, whatever the embedding of X, the μ Lefschetz vanishing cycles $\delta_k \in \operatorname{Van} \subset H_{n-1}(X', \mathbb{Q})$, $k = 1, \ldots, \mu$, satisfy numerous linear relations: at most half of them are linearly independent. Similarly to Proposition 2.2(v)(b), one sees that these relations yield generators of the cokernel of j_n.

(iii) Asymptotically (for d large), the bound (2.8.2) is far from being optimal. Below (cf. Corollary 2.14) we show that not only λ_{n-1} but even $b_{n-1}(X') > \lambda_{n-1}$ is bounded by a polynomial of degree n in d.

(iv) The formula in Theorem 2.9(i) looks simple, is easily proved and involves classes which are classical invariants, but I have not encountered it in this form in the literature. However, careful examination revealed that similar formulas were being repeatedly discovered under different guises.

For curves, the Riemann–Hurwitz formula shows that $b_1 = \mu_1 - 2\mu_0 + 2 \leqslant \mu_1$ and $e = 2 - b_1 = 2\mu_0 - \mu_1$ in accordance with Theorem 2.9. The case of surfaces is more interesting. The role of the Euler–Poincaré characteristic e was classically played by the *Zeuthen–Segre invariant* I defined by the formula $I = \delta - \sigma - 4p$, where, for a (general) pencil of curves, δ is the number of nodal curves, σ is the number of base points and p is the genus of a generic curve (cf. [SR1949, ch. IX, § 7.1]). This number was shown to be independent of the choice of pencil of curves. By methods anticipating those of modern Lefschetz theory, Alexander proved (in two different ways) the equality $I = e - 4$ (cf. [Ale1914, §§ 2 and 3]). In our setup, that is, for a Lefschetz pencil, using the Riemann–Hurwitz formula $2p = \mu_1 - 2\mu_0 + 2$, one gets, in our notation, that $I = \mu_2 - \mu_0 - 4p = \mu_2 - \mu_0 - 2(\mu_1 - 2\mu_0 + 2) = \mu_2 - 2\mu_1 + 3\mu_0 - 4$. By Alexander's interpretation of I, this classical formula (cf. for example [SR1949, ch. IX, § 7, Formula (5)]) can be rewritten as $e = 3\mu_0 - 2\mu_1 + \mu_2$, which is a special case of Theorem 2.9(i) for $n = 2$. Furthermore, by Lefschetz's theorem (cf. Proposition 2.2(iii)) and the formula for curves in the preceding paragraph, $b_3 = b_1 \leqslant 2p = \mu_1 - 2\mu_0 + 2 \leqslant \mu_1$ and $b_2 = e + 2b_1 - 2 \leqslant \mu_2 - 2\mu_1 + 3\mu_0 + 2(\mu_1 - 2\mu_0 + 2) - 2 = \mu_2 - \mu_0 + 2 \leqslant \mu_2$, which is a special case of Theorem 2.9(ii). It is amusing to observe that the now obvious formula $b_2 = I + 4(p_g - p_a) + 2$ (where p_g and p_a are respectively the geometric and arithmetic genus of our surface) was obtained only by Alexander [Ale1914] who corrected an erroneous relation $b_2 = I + 2(p_g - p_a) + 2$ published by Poincaré in 1906.

In [Seg1895/96, § 11], Segre introduced a generalization II of the invariant I to varieties of arbitrary dimension by using (in our notation) a recurrent formula $II_n = \mu_n - 2II_{n-1} - II_{n-2}$, $II_0 = \mu_0 - 1$, $II_1 = \mu_1 - 2\mu_0 + 2$ (thus $II_2 = I + 1$; the normalization chosen by Segre is such that the invariants II_n vanish for all linear spaces). Computing by induction, one can show that

$$\begin{aligned} II_n &= (-1)^n\biggl(\sum_{i=0}^{n}(n-i+1)\mu_i - n - 1\biggr) \\ &= (-1)^n(\mu_n - 2\mu_{n-1} + \cdots + (-1)^{n-1}n\mu_1 + (-1)^n(n+1)(\mu_0+1)). \end{aligned}$$

On the other hand, using Alexander's method, one can verify that $II_n = (-1)^n(e - n - 1)$ (Alexander himself used a different generalization I_n of I defined by the same recurrent formula, but based on a different value $I_0 = II_0 + 1$, which does not make much difference). Putting together the above two formulas, one arrives at Theorem 2.9(i).

As was kindly pointed out to me by Lê Dũng Tráng, Theorem 2.9(i) can also be deduced from the formulas for the local Euler obstructions for singularities obtained in [LT1981]. Different proofs of Theorem 2.9(i) can be found in [KhZ, Theorem 2.13] and [Zak2012, Corollary 1.8].

We do not know of any classical bounds for Betti numbers in terms of classes similar to those in Theorem 2.9(ii) and (iii).

(v) One can view Theorem 2.9(i) as a far-reaching generalization of Hopf's celebrated formula for the index of a vector field to the setup of projective algebraic geometry.

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(vi) The inequalities in Theorem 2.9(ii) and (iii) are not sharp. In fact, by Corollary 2.7(i) they could only be sharp if the vanishing homology subspaces were trivial for all linear sections of X, but this is never the case since for example $\lambda_0(X_0) = d-1$. Already this observation shows that the bound in Theorem 2.9(ii) can be improved by $2(d-1)$, *viz.*

$$b_i(X) = b_{2n-i}(X) \leqslant \sum_{k=0}^{[i/2]} \mu_{i-2k} - 2(d-1), \quad i = 1, \dots, n, \tag{2.10.1}$$

and the bound in Theorem 2.9(iii) can be improved by $(3n-1)(d-1) - n - 1$, *viz.*

$$b(X) \leqslant \sum_{k=0}^{n} (k+1)\mu_{n-k} - (3n-1)(d-1) + n + 1. \tag{2.10.2}$$

More generally, from (2.2.1) it follows that $\lambda_k(X_k) > 0$ for $k \equiv 0 \pmod 2$, which allows us to further improve the bounds in Theorem 2.9(ii) and (iii).

PROPOSITION 2.11. *Let $X \subset \mathbb{P}^N$ be a nonsingular nondegenerate variety of dimension $n \geqslant 2$ and let X' be its general hyperplane section. Then $b(X) = b(X') + b_{n-1} + b_n - \lambda_{n-1}(X') \leqslant b(X') + b_{n-1} + b_n$.*

Proof. This follows immediately from (2.0.3) and Proposition 2.2(iii). □

COROLLARY 2.12. *In the assumptions of Proposition 2.11, $b(X) \leqslant b(X') + \boldsymbol{\mu}(X)$.*

Proof. This is an immediate consequence of Proposition 2.11 and Theorem 2.9(ii). □

Combining Theorem 2.9 with Theorems 1.12 and 1.1, one can get various bounds for the Betti numbers in terms of dimension, codimension, degree and sectional genus, trade-off being between compactness and sharpness. We proceed with giving some simple formulas of this type.

COROLLARY 2.13. *Let X be a nondegenerate nonsingular variety of dimension n, codimension a and degree $d > 2$, let $r_1 = \mu_1 = 2\pi + 2d - 2$, where $\pi = g(X_1)$ is the sectional genus (cf. (1.4.4)), and let $t = r_1/d$. Then*

$$b_i(X) = b_{2n-i}(X) < d\frac{t^{i+2}}{t^2-1}, \quad i = 1, \dots, n;$$
$$b(X) = \sum_{i=0}^{2n} b_i(X) < d\frac{t^{n+2}}{(t-1)^2},$$
$$b(X) = \sum_{i=0}^{2n} b_i(X) < d(t+1)^{n-1}(t+2)$$

(in the case $d = 2$, the Betti numbers were computed in (2.6.6)).

Proof. Since $d > 2$, one has $t = 2\pi/d - 2/d + 2 > 1$.

From Theorems 2.9(ii), 1.12 and 1.1, it follows that

$$\begin{aligned} b_i(X) = b_{2n-i}(X) &\leqslant \mu_i + \mu_{i-2} + \dots + \mu_{i-2[i/2]} \\ &\leqslant r_i + r_{i-2} + \dots + r_{i-2[i/2]} \leqslant d(t^i + t^{i-2} + \dots + t^{i-2[i/2]}) < d\frac{t^{i+2}}{t^2-1}. \end{aligned} \tag{2.13.1}$$

From Theorems 2.9(iii), 1.12 and 1.1, it follows that

$$\begin{aligned} b(X) &\leqslant \mu_n + 2\mu_{n-1} + \cdots + n\mu_1 + (n+1)\mu_0 \\ &\leqslant r_n + 2r_{n-1} + \cdots + nr_1 + (n+1)r_0 \\ &\leqslant d(1+t+\cdots+t^n) + d(1+t+\cdots+t^{n-1}) + \cdots + d(1+t) + d \\ &< \frac{dt}{t-1}(t^n + t^{n-1} + \cdots + t + 1) < d\frac{t^{n+2}}{(t-1)^2}. \end{aligned} \tag{2.13.2}$$

The first three lines of (2.13.2) also yield

$$\begin{aligned} b(X) &\leqslant d(1+t+\cdots+t^n) + d(1+t+\cdots+t^{n-1}) + \cdots + d(1+t) + d \\ &= \frac{d}{t-1}(t+\cdots+t^n - n - 1) = \frac{d}{t-1}\left(t\frac{t^{n+1}-1}{t-1} - (n+1)\right) \\ &= \frac{d(t^{n+2} - (n+2)t + n + 1)}{(t-1)^2}. \end{aligned} \tag{2.13.3}$$

From (2.13.3), it follows that to prove the last inequality in Corollary 2.13 it suffices to verify that

$$\frac{t^{n+2} - (n+2)t + n + 1}{(t-1)^2} \leqslant (t+1)^{n-1}(t+2), \quad n \geqslant 1, t > 1. \tag{2.13.4}$$

We prove (2.13.4) by induction on n. For $n = 1$, (2.13.4) obviously turns into equality. Assuming that (2.13.4) holds for $n-1$, to check that it holds for n it clearly suffices to show that

$$\frac{t^{n+2} - (n+2)t + n + 1}{t^{n+1} - (n+1)t + n} \leqslant t + 1, \quad n \geqslant 2, t > 1$$

or, equivalently, that

$$\lambda_n(t) = t^{n+1} - (n+1)t^2 + (n+1)t - 1 \geqslant 0, \quad n \geqslant 2, t > 1. \tag{2.13.5}$$

Since $\lambda_n''(t) = n(n+1)t^{n-1} - 2(n+1)$, $\lambda_n'(t) = (n+1)t^n - 2(n+1)t + n + 1$ is a strictly increasing function for $n \geqslant 2$, $t \geqslant 1$, while $\lambda_n'(1) = \lambda_n(1) = 0$. This completes the proof of (2.13.5) and thus of Corollary 2.13. □

Corollary 2.14. *In the setup of Corollary 2.13, one has*

(i)
$$b_i(X) = b_{2n-i}(X) < \begin{cases} a\left(\dfrac{d}{a} + \dfrac{5}{4}\right)^{i+1}, \\ a\left(\dfrac{d}{a} + 1\right)^{i+1} & \text{if } d \geqslant \dfrac{(a-2)^2}{8}, \end{cases} \qquad i = 1, \ldots, n.$$

(ii)
$$b(X) < \begin{cases} \dfrac{a^2}{d}\left(\dfrac{d}{a} + \dfrac{5}{4}\right)^{n+2}, \\ \dfrac{a^2}{d}\left(\dfrac{d}{a} + 1\right)^{n+2} & \text{if } d \geqslant \dfrac{(a-2)^2}{8}. \end{cases}$$

Proof. By (2.6.6), both (i) and (ii) are obvious in the case $d = 2$, so we assume that $d > 2$ and so $t = (2\pi + 2d - 2)/d > 1$.

(i) By Corollary 2.13,

$$b_i(X) = b_{2n-i}(X) < d\phi(t), \quad \phi(t) = \frac{t^{i+2}}{t^2 - 1}. \tag{2.14.1}$$

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One has

$$\phi'(t) = \frac{(i+2)t^{i+1}(t^2-1) - 2t^{i+3}}{(t^2-1)^2} = \frac{it^{i+3} - (i+2)t^{i+1}}{(t^2-1)^2}, \tag{2.14.2}$$

and so the function $\phi(t)$ is monotonically increasing on the interval $t \geqslant \sqrt{(i+2)/i}$. Thus, for

$$t \geqslant \sqrt{\frac{i+2}{i}}, \tag{2.14.3}$$

our inequalities for $b_i(X)$ follow from (2.14.1) and Corollary 1.5. Since $t = (2\pi + 2d - 2)/d$, the condition (2.14.3) is always satisfied if the sectional genus π is positive. Consider now the case when $\pi = 0$, $t = (2d-2)/d$. By (2.14.1),

$$b_i(X) < d\phi(t) = \frac{(2d-2)^{i+2}}{d^{i-1}(3d^2 - 8d + 4)}. \tag{2.14.4}$$

Put $\psi(a) = a(d/a + 1)^{i+1}$. Then

$$\psi'(a) = \left(1 - \frac{id}{a}\right)\left(\frac{d}{a} + 1\right)^i \tag{2.14.5}$$

and, since $id/a \geqslant d/a > 1$, $\psi(a)$ is a monotonically decreasing function for $1 \leqslant a \leqslant d-1$. Thus, to prove (i), it suffices to verify that $d\phi(t) \leqslant \psi(d-1)$, which is immediate from (2.14.4). □

(ii) If $n = 1$, then, by (1.4.6),

$$b(X) = 2 + b_1(X) \leqslant \frac{d^2}{a} - d - \frac{2d}{a} + \frac{a}{4} + \frac{1}{a} + 3 < \frac{a^2}{d}\left(\frac{d}{a} + 1\right)^3 \tag{2.14.6}$$

and we are done. For $n \geqslant 2$, the proof is similar to that of (i). By Corollary 2.13,

$$b(X) = \sum_{i=0}^{2n} b_i(X) < d\phi(t), \quad \phi(t) = \frac{t^{n+2}}{(t-1)^2}. \tag{2.14.7}$$

One has

$$\phi'(t) = \frac{t^{n+1}}{(t-1)^3}(nt - (n+2)), \tag{2.14.8}$$

and so the function $\phi(t)$ is monotonically increasing on the interval $t \geqslant (n+2)/n$. Thus, for

$$t \geqslant \frac{n+2}{n}, \tag{2.14.9}$$

our inequalities for $b(X)$ follow from (2.14.7) and Corollary 1.5. Since $t = (2\pi + 2d - 2)/d$, the condition (2.14.9) is always satisfied if the sectional genus π is positive. Consider now the case when $\pi = 0$, $t = (2d-2)/d$. By (2.14.7),

$$b(X) < d\phi(t) = \frac{(2d-2)^{n+2}}{d^{n-1}(d-2)^2}. \tag{2.14.10}$$

Put $\psi(a) = (a^2/d)(d/a + 1)^{n+2}$. Then

$$\psi'(a) = \left(\frac{2a}{d} - n\right)\left(\frac{d}{a} + 1\right)^{n+1}, \tag{2.14.11}$$

and so $\psi(a)$ is a monotonically decreasing function for $1 \leqslant a \leqslant d-2$. Thus, to prove (ii), for $a \leqslant d-2$ it suffices to verify that $d\phi(t) \leqslant \psi(d-2)$, which is immediate from (2.14.10). Since $a \leqslant d-1$, it remains to deal with varieties of minimal degree for which both inequalities in (ii) are trivially satisfied. □

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Remarks 2.15. (i) The bounds in Corollary 2.14 can be proved directly, without deducing them from Corollary 2.13. However, Corollary 2.13 gives bounds for the Betti numbers of X in terms of its degree, dimension and sectional genus π. In the case when the sectional genus of X is much less than the maximum given by Castelnuovo's theorem, this bound is much better than the general one given in Corollary 2.14 (cf. Remark 1.17(iii)).

(ii) The bounds for the total Betti number given in Corollaries 2.13 and 2.14 are not optimal for several reasons. Firstly, if the codimension a of X is larger than one, then the bounds $\mu_i \leqslant r_i$ proved in Theorem 1.12 are not sharp for $i > 1$ (cf. Remark 1.17(iv)). Also, the bounds $r_i \leqslant r_1^i/d^{i-1}$ in Theorem 1.1 are not always sharp. Secondly, the basic inequalities in Theorem 2.9 are never sharp because of the existence of nontrivial vanishing cycles (cf. Remark 2.10(vi)). Thirdly, the bound for r_1 from Corollary 1.5 that we used in the proof of Corollary 2.14 can be improved for certain values of a (cf. Remark 1.7(i)). For example, combining Remark 1.7(i) with Corollary 2.13, we see that $b(X) < d^{n+3}/2^{n+2}(d-2)^2$ for $a = 2$ and $b(X) < d(d+1)^{n+2}/3^n(d-2)^2$ for $a = 3$.

(iii) The true import of Corollary 2.14 is that $b(X) < d^{n+1}/a^n + O(d^n)$ with a rather precise bound on $O(d^n)$. In the next section, we will give examples of (series of) varieties whose middle Betti number has the form $b_n(X) = d^{n+1}/a^n + O(d^n)$, and so the above bound is asymptotically sharp. Moreover, we will see that, using known bounds for curves, one can improve the above bound, deduce from it an optimal one and classify the varieties on the boundary. In particular, for sufficiently large d, one has $b(X) < d^{n+1}/a^n$ (cf. Remark 2.19 and Theorem 3.16(ii) below).

(iv) Let $b_-(X) = \sum_{i \neq n} b_i(X) = b(X) - b_n(X)$ be the sum of all Betti numbers of X except the middle one. Then, from Proposition 2.2(iii) and Corollary 2.14, it follows that $b_-(X) \leqslant b(X') + b_{n-1}(X) \leqslant b(X') + b_{n-1}(X') < 2b(X')$ is bounded by a polynomial of the form $dP_{n-1}(d/a)$, where P_{n-1} is a polynomial of degree $n-1$ with leading coefficient 2 (for d large, one can take $P_{n-1}(t) = 2t^n$; cf. (iii), Remark 2.19 and Theorem 3.16 below). On the other hand, Examples 2.6 and 3.5 show that it is impossible to give a bound of this form for the middle Betti number. In other words, if the degree and the total Betti number of a variety are large, then the contribution of all homologies except the middle one to the total Betti number is much less than that of the middle homology. More generally, if $b_{-l} = \sum_{|i-n| \geqslant l} b_i$, $1 \leqslant l \leqslant n-1$, then, for d large with respect to a, $b_{-l}(X) < 2d(d/a)^{n-l}$.

Theorem 2.16. *Let X be a nonsingular projective variety of dimension n and degree d. Then $b(X) \leqslant b(\boldsymbol{X})$, where $\boldsymbol{X} \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of the same dimension n and degree d. Furthermore, $b(X) = b(\boldsymbol{X})$ if and only if X is itself a hypersurface (that is, $a = \operatorname{codim} X = 1$).*

Proof. If $n = 1$, then, by (1.4.5), $b(X) \leqslant (d-2)^2/2 + 2 < (d-1)(d-2) + 2 = b(\boldsymbol{X})$. Thus, in what follows, we may assume that $n \geqslant 2$.

Suppose first that $a \geqslant 6$. By (2.6.4) and the first bound in Corollary 2.14(ii), one has

$$b(\boldsymbol{X}) > \frac{(d-1)^{n+2}}{d}; \qquad b(X) < \frac{a^2}{d}\left(\frac{d}{a} + \frac{5}{4}\right)^{n+2}. \tag{2.16.1}$$

Considering the quotient $\phi(a, d) = a^n((d-1)/(d + (5/4)a))^{n+2}$ of the right-hand sides of (2.16.1), we see that to prove the theorem it suffices to show that

$$\phi(a, d) \geqslant 1, \quad a \geqslant 6. \tag{2.16.2}$$

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It is immediate that, for given a, ϕ is a monotonically increasing function of d, and so it suffices to check (2.16.2) for $d = a + 1$, that is, to show that

$$\phi(a, a+1) = \frac{a^{2n+2}}{((9/4)a+1)^{n+2}} = a^n \frac{1}{(9/4+1/a)^{n+2}} \geqslant 1. \tag{2.16.3}$$

Since $\phi(a, a+1)$ is a monotonically increasing function of a, it suffices to verify (2.16.3) for $a = 6$, in which case it is immediate provided that $n \geqslant 2$.

If $a < 6$, then $d > (a-2)^2/8$ (this inequality actually holds if $a \leqslant 12$; cf. Corollary 1.5), and in (2.16.1) one can replace the first bound from Corollary 2.14(ii) by the second one:

$$\begin{aligned} b(\boldsymbol{X}) &> \frac{(d-1)^{n+2}}{d}; \\ b(X) &< \frac{a^2}{d}\left(\frac{d}{a}+1\right)^{n+2}. \end{aligned} \tag{2.16.4}$$

In this case, we can consider the quotient $\psi(a, d) = a^n((d-1)/(d+a))^{n+2}$ of the right-hand sides of (2.16.4), and to prove the theorem it suffices to show that

$$\psi(a, d) \geqslant 1. \tag{2.16.5}$$

As above, it is obvious that, for given a, ψ is a monotonically increasing function of d. Hence, to prove (2.16.5) for an arbitrary $d \geqslant a+2$, it suffices to check it for $d = a+2$, that is, to show that

$$\psi(a, a+2) = \frac{1}{4}\left(\frac{a}{2}\right)^n \geqslant 1. \tag{2.16.6}$$

It is clear that (2.16.6) is satisfied for all $a \geqslant 4$ provided that $n \geqslant 2$ (and $a = 3$ provided that $n \geqslant 4$).

If $d = a+1$, then X is a variety of minimal degree and, from (2.6.4), it follows that $b(X) \leqslant 2n < b(\boldsymbol{X})$ for $d \geqslant 3$.

In the case $a = 3$, Remark 2.15(ii) yields a better bound for $b(X)$, so that in place of (2.16.4) we get the following inequalities:

$$\begin{aligned} b(\boldsymbol{X}) &> \frac{(d-1)^{n+2}}{d}; \\ b(X) &< \frac{d(d+1)^{n+2}}{3^n(d-2)^2}. \end{aligned} \tag{2.16.7}$$

Denote by $\eta(d)$ the quotient of the right-hand sides of (2.16.7). Then $\eta(d) = 3^n((d-2)/d)^2 ((d-1)/(d+1))^{n+2}$, and one sees immediately that $\eta(d) > 1$ for $d \geqslant 5 = a+2$, $n \geqslant 3$ or $d \geqslant 6$, $n \geqslant 2$. In the remaining case when $d = 5$ and $n = 2$, (1.4.5) yields $\pi \leqslant 1$ and $r_1 \leqslant 10$; hence, by Corollary 2.7(i), $b(X) \leqslant b_2(X) + 6 \leqslant \mu_2 + 3 < r_2 + 3 \leqslant 100/5 + 3 = 23$ while $b(\boldsymbol{X}) > 4^4/5 > 50$, which completes the proof in the case $a = 3$.

Suppose finally that $a = 2$. Then, by Remark 2.15(ii), in place of (2.16.4) we get the following inequalities:

$$\begin{aligned} b(\boldsymbol{X}) &> \frac{(d-1)^{n+2}}{d}; \\ b(X) &< \frac{d^{n+3}}{2^n(d-2)^2}. \end{aligned} \tag{2.16.8}$$

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Denote by $\xi(d)$ the quotient of the right-hand sides of (2.16.8). Then $\xi(d) = 2^n((d-2)/d)^2((d-1)/d)^{n+2}$, and one sees immediately that $\xi(d) > 1$ for $d \geqslant 4$, $n \geqslant 5$ or $d \geqslant 5$, $n \geqslant 4$ or $d \geqslant 6$, $n \geqslant 3$ or $d \geqslant 7$, $n \geqslant 2$, which proves the theorem for those values of d and n. If $d = 6$, $n = 2$, then, by (1.4.5), $\pi \leqslant 4$ and $r_1 \leqslant 18$; hence, by Corollary 2.7(i), $b(X) \leqslant b_2(X) + 18 < \mu_2 + 14 < r_2 + 14 \leqslant 18^2/6 + 14 = 68$ while $b(\boldsymbol{X}) > 625/6$ and we are done. Finally, if $d = 5$, then, by (1.4.5), $\pi \leqslant 2$, $r_1 \leqslant 12$ and $t \leqslant 12/5$. If $n = 3$, then, by Corollary 2.13, $b(X) < dt^5/(t-1)^2 = 12^5/(25 \cdot 49) < 204$ while $b(\boldsymbol{X}) > 4^5/5 > 204$. It remains to consider the case $n = 2$. By Corollary 2.7(i), in this case $b(X) \leqslant b_2(X) + 10 < \mu_2 + 7 < r_2 + 7 \leqslant 12^2/5 + 7 < 36$ while $b(\boldsymbol{X}) > 4^4/5 > 51$. □

Corollary 2.17. *Let X be a nonsingular projective variety of dimension n and degree d. Then $b_n(X) \leqslant b_n(\boldsymbol{X})$, where $\boldsymbol{X} \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of the same dimension n and degree d. Furthermore, $b_n(X) = b_n(\boldsymbol{X})$ if and only if X is itself a hypersurface.*

Proof. By Theorem 2.16,

$$b(X) \leqslant b(\boldsymbol{X}) \tag{2.17.1}$$

with equality holding if and only if $a = 1$. Corollary 2.17 follows from (2.17.1), Example 2.6 and the following inequality:

$$b_i(X) \geqslant 1, \quad i \equiv 0 \pmod 2. \tag{2.17.2}$$

□

Theorem 2.18. *Let X be a nonsingular projective variety of dimension n and degree d. Then $b(X) < d^{n+1}$.*

Proof. From Theorem 2.16, it follows that it suffices to prove Theorem 2.18 in the case when X is a hypersurface.

If X is a quadric, then, by Example 2.6, $b(X) \leqslant n + 2 < 2^{n+1}$. If $d > 2$, then, by (2.6.4),

$$b(X) = \frac{(d-1)^{n+2} + (-1)^{n+1}}{d} + n + 1 + (-1)^n < \frac{(d-1)^{n+2}}{d} + (n+2). \tag{2.18.1}$$

In view of (2.18.1), to prove Theorem 2.18 it now suffices to show that $(d-1)^{n+2}/d + (n+2) \leqslant d^{n+1}$ or, more generally, that

$$\phi_d(x) = d^x - (d-1)^x - dx \geqslant 0, \quad d \geqslant 3, x \geqslant 3. \tag{2.18.2}$$

Differentiating the left-hand side of (2.18.2), we see that

$$\begin{aligned} \phi'_d(x) &= \ln d \cdot d^x - \ln (d-1) \cdot (d-1)^x - d \\ &> \ln d[d^x - (d-1)^x] - d > d^x - (d-1)^x - d \\ &\geqslant d^2 - (d-1)^2 - d = d - 1 > 0, \quad d \geqslant 3, x \geqslant 2, \end{aligned} \tag{2.18.3}$$

so that the function $\phi_d(x)$ is increasing for $d \geqslant 3$, $x \geqslant 2$ and it suffices to check (2.18.2) for $x = 3$, in which case it is clear that $d^3 - (d-1)^3 - 3d = d^2 + d(d-1) + (d-1)^2 - 3d > d^2 - 3d \geqslant 0$. □

Remark 2.19. In the next section, we show (cf. Theorem 3.16) that if the degree d is sufficiently large (for example $d \geqslant 2(a+1)^2$), then $b(X) < d^{n+1}/a^n$, which gives a nice generalization of Theorem 2.18 for varieties of arbitrary codimension.

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The following result extends Theorem 2.18 to varieties of arbitrary codimension defined by equations of degree d.

Theorem 2.20. *Let $X^n \subset \mathbb{P}^N$ be a (not necessarily nondegenerate) variety defined by equations of degree d. Then $b(X) < \binom{N-1}{n} d^N$.*

Proof. From Theorem 2.9(iii), Remark 2.10(vi) and Theorem 1.21, it follows that

$$b(X) < d^a \sum_{i=0}^{n} (n+1-i) \binom{a+i-1}{i} (d-1)^i. \tag{2.20.1}$$

On the other hand,

$$\binom{N-1}{n} d^N = d^a \binom{N-1}{n} ((d-1)+1)^n = d^a \binom{N-1}{n} \sum_{i=0}^{n} \binom{n}{i} (d-1)^i. \tag{2.20.2}$$

Using the obvious relations

$$\begin{aligned} \binom{r}{k} &= \binom{r-1}{k} + \binom{r-1}{k-1}, \\ \binom{n}{k} &\geqslant k+1, \quad k \leqslant n-1, \end{aligned} \tag{2.20.3}$$

one can check that, for $a > 1$,

$$\binom{n}{i} \binom{N-1}{n} \geqslant (n+1-i) \binom{a+i-1}{i}, \quad i = 0, \ldots, n, \tag{2.20.4}$$

which yields our claim. □

Example 2.21. Let $X^n \subset \mathbb{P}^N$ be a nonsingular complete intersection of hypersurfaces of degree d. From Proposition 2.2(iii), it follows that

$$b_i(X) = \begin{cases} 0, & i \neq n, i \equiv 1 \pmod 2, \\ 1, & i \neq n, i \equiv 0 \pmod 2, \end{cases}$$

$$b(X) = \begin{cases} e(X), & n \equiv 0 \pmod 2, \\ 2(n+1) - e(X), & n \equiv 1 \pmod 2. \end{cases}$$

A generating function for the Euler characteristic of complete intersections was found by Hirzebruch (cf. [Hir1966, § 22] or [DK1973, exp. XI]), but the corresponding formulas are rather involved. An easier way to compute the Euler characteristic is to use the formulas for classes obtained in Theorem 1.21 and to apply Proposition 2.4 to get a recurrent formula. This seems to be a general rule: classes appear to be more basic objects than Betti numbers, and formulas for classes look nicer than those for Betti numbers (and even than those for the Euler characteristic). We will see another manifestation of this rule in Proposition 3.11.

In any case, the computation sketched above shows that the bound in Theorem 2.20 is asymptotically sharp. However, like its counterparts in Theorems 2.18 and 3.16 and many other results in this paper, this bound fails to be the best possible; we sacrificed sharpness for beauty. One expects that complete intersections have maximal Betti number among all varieties defined by equations of degree d; cf. Remark (8) in § 4.

From Theorem 2.20 and Stirling's formula, we derive the following bound valid for arbitrary smooth subvarieties of $\mathbb{P}^N$.

Corollary 2.22. *Let $X \subset \mathbb{P}^N$ be a nonsingular variety defined by equations of degree $\leqslant d$. Then $b(X) < \binom{N-1}{[(N-1)/2]} d^N \sim \sqrt{2/\pi N} \cdot (2d)^N$.*

Remark 2.23. In view of Example 2.21, the bounds in Theorem 2.20 and Corollary 2.22 are asymptotically sharp. In particular, they are better than the bounds obtained in [Mil1964, Tho1965] (cf. for example [Mil1964, Theorem 2 and Corollaries 1–3]); furthermore, the bound in Theorem 2.20 takes into account the codimension of X. On the other hand, while Theorem 1.21 is true for arbitrary varieties, to apply the theory developed in the present section we, unlike Milnor and Thom, need to assume that X is nonsingular.

3. Optimal bounds and varieties on the boundary

As pointed out in Remarks 1.17(iv) and 2.15(ii), the bounds for the classes and Betti numbers obtained in Corollaries 1.13, 2.13 and 2.14 are not optimal. In this section, we improve these bounds and describe the varieties on the boundary.

Theorem 3.1. *Let $X \subset \mathbb{P}^N$ be a nonsingular nondegenerate variety of dimension n, codimension $a = N - n$ and degree $d > 2a + 2$, let b_i, $i = 0, \ldots, 2n$, and $b = \sum_{i=0}^{2n} b_i$ be its Betti numbers, and let μ_i (respectively r_i), $i = 0, \ldots, n$, denote its classes (respectively ramification volumes).*

(i) *Suppose that, for some $i > 0$, $\mu_i > d(d/(a+1) + (d-2)/d + (a+1)/4d)^i$ (since $d > 2a + 2$, this condition is satisfied if $\mu_i > d(d/(a+1) + 9/8)^i$). Then $X \subset V$, where $V \subset \mathbb{P}^N$, $\dim V = n + 1$, $\deg V = a$, is a variety of minimal degree.*

(ii) *Suppose that, for some $i > 0$, $r_i > d(d/(a+1) + (d-2)/d + (a+1)/4d)^i$ (since $d > 2a + 2$, this condition is satisfied if $r_i > d(d/(a+1) + 9/8)^i$). Then $X \subset V$, where $V \subset \mathbb{P}^N$, $\dim V = n + 1$, $\deg V = a$, is a variety of minimal degree.*

(iii) *Suppose that $b > d\phi(d/(a+1) + (d-2)/d + (a+1)/4d)$, where, as in the proof of Corollary 2.14, $\phi(t) = t^{n+2}/(t-1)^2$ (this condition is satisfied provided that $b > ((a+1)^2/d)(d/(a+1) + 9/8)^{n+2}$). Then $X \subset V$, where $V \subset \mathbb{P}^N$, $\dim V = n + 1$, $\deg V = a$, is a variety of minimal degree.*

Proof. By Theorem 1.12, (ii) follows from (i).

If $N = n + 1$, then all the claims are obviously true (with $V = \mathbb{P}^{n+1}$), even without any assumptions on μ_i, r_i and b.

Let $N \geqslant n + 2$. Put $C = X_1$, and let $\pi = g(C)$ denote the sectional genus of X. We claim that under our assumptions, one has

$$\pi > \frac{d^2}{2(a+1)} - \frac{d}{2} + \frac{a+1}{8}. \tag{3.1.1}$$

In fact, since, by (1.4.4), $\pi = r_1/2 - d + 1$, to show (3.1.1) it suffices to check that

$$r_1 > \frac{d^2}{a+1} + d + \frac{a-7}{4}. \tag{3.1.2}$$

In case (i), Corollary 1.13 and Remark 1.17(i) yield

$$d\left(\frac{r_1}{d}\right)^i \geqslant \mu_i > d\left(\frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d}\right)^i, \tag{3.1.3}$$

which implies (3.1.2).

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Consider now case (iii). Since, obviously, $t^3/(t-1)^2 > t$, for $n=1$ one has

$$\begin{aligned} r_1 \geqslant r_1 - 2d + 4 = b > d\phi\left(\frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d}\right) \\ > \frac{d^2}{a+1} + \frac{a+1}{4} + d - 2. \end{aligned} \tag{3.1.4}$$

For arbitrary n, from Corollary 2.13 it follows that

$$d\phi\left(\frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d}\right) < b < d\phi\left(\frac{r_1}{d}\right). \tag{3.1.5}$$

The function $\phi(t)$ is monotonically increasing for $t \geqslant (n+2)/n$ (cf. (2.14.8) and (2.14.9)). Since $d > 2a+2$ and $r_1 = 2\pi + 2d - 2$, one has

$$\begin{aligned} \frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d} \geqslant 3 + \left(\frac{a-7}{4d} + \frac{1}{a+1}\right) > 3, \\ t = \frac{r_1}{d} \geqslant 2 + 2\frac{\pi - 1}{d} \geqslant \begin{cases} 2, & \pi > 0, \\ 1 + \dfrac{a}{a+1}, & \pi = 0. \end{cases} \end{aligned} \tag{3.1.6}$$

From (3.1.6), it follows that (3.1.2) holds for $n \geqslant 3$ and also for $n=2$ provided that $\pi > 0$. If $n=2$ and $\pi = 0$, then, by (1.4.4), $\mu_1 = r_1 = 2d-2$ and, by Proposition 2.2(iii), $b_1(X) = 0$. Thus, from Theorems 2.9(iii), 1.12 and 1.1, it follows that

$$b(X) = e(X) = \mu_2 - d + 4 \leqslant 4\frac{(d-1)^2}{d} - d + 4 = 3d - 4\left(1 - \frac{1}{d}\right) < 3d. \tag{3.1.7}$$

On the other hand, since $\phi(t) = t^4/(t-1)^2 > t^2$ for all $t > 1$ and $d > 2a+2$, the first inequality in (3.1.5) yields

$$\begin{aligned} b(X) > d\phi\left(\frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d}\right) > d\left(\frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d}\right)^2 \\ > d\left(3 + \frac{a-7}{4d}\right)^2 > 3d. \end{aligned} \tag{3.1.8}$$

Thus, (3.1.8) contradicts (3.1.7), and the case $n=2$, $\pi=0$ in (iii) does not occur (by the way, it is well known that the only smooth surfaces with sectional genus zero are scrolls and the Veronese surface (cf. for example [Zak1973]), so that for such surfaces $b \leqslant 4$). This completes the proof of (3.1.1).

Suppose now that

$$\begin{aligned} b > \frac{(a+1)^2}{d}\left(\frac{d}{a+1} + \frac{9}{8}\right)^{n+2} > \frac{(a+1)^2}{d}\left(\frac{d}{d + (a+1)/8}\right)^2\left(\frac{d}{a+1} + \frac{9}{8}\right)^{n+2} \\ = d\frac{(d/(a+1) + 9/8)^{n+2}}{(d/(a+1) + 1/8)^2} = d\phi\left(\frac{d}{a+1} + \frac{9}{8}\right). \end{aligned} \tag{3.1.9}$$

One has

$$\begin{aligned} \frac{d}{a+1} + \frac{d-2}{d} + \frac{a+1}{4d} = \frac{d}{a+1} + 1 + \frac{a-7}{4d} \\ < \frac{d}{a+1} + 1 + \frac{a-7}{8a+8} < \frac{d}{a+1} + \frac{9}{8}. \end{aligned} \tag{3.1.10}$$

From (3.1.9), (3.1.10), (3.1.6) and the monotonicity of $\phi(t)$ for $t \geqslant (n+2)/n$, it follows that the inequality $b > ((a+1)^2/d)(d/(a+1)+9/8)^{n+2}$ implies the inequality $b > d\phi(d/(a+1)+(d-2)/d+(a+1)/4d)$.

We now recall the following theorem.

Theorem (Halphen–Harris). *Let $C \subset \mathbb{P}^r$ be a nondegenerate curve of genus g and degree $d > 2r$ such that*

$$g > \frac{(d-\varepsilon)(d+\varepsilon-r)}{2r} + \left[\frac{\varepsilon}{r}\right], \quad \varepsilon \equiv d \ (\mathrm{mod}\ r), 1 \leqslant \varepsilon \leqslant r.$$

Then C is contained in a surface S of minimal degree in $\mathbb{P}^r$: $C \subset S \subset \mathbb{P}^r$, $\deg S = r-1$.

Proof. See [HE1982, Theorem 3.15] or [Cil1987, Theorem 3.4]. □

We observe that

$$\frac{(d-\varepsilon)(d+\varepsilon-r)}{2r} + \left[\frac{\varepsilon}{r}\right] = \frac{d^2}{2r} - \frac{d}{2} + \frac{\varepsilon(r-\varepsilon)}{2r} + \left[\frac{\varepsilon}{r}\right] \leqslant \frac{d^2}{2r} - \frac{d}{2} + \frac{r}{8}, \tag{3.1.11}$$

and so for the validity of the Halphen–Harris theorem it suffices to assume that $g > d^2/2r - d/2 + r/8$. Thus, (3.1.1) shows that the curve section $C = X_1$ satisfies the hypothesis of the Halphen–Harris theorem for $r = a+1$.

Another property of the curve section $C = X_1$ is that it is linearly normal. In fact, if C were an isomorphic projection of a curve $\widetilde{C} \subset \mathbb{P}^{a+2}$, then, by (3.1.1) and (1.4.6),

$$\begin{aligned}\frac{d^2}{2(a+1)} - \frac{d}{2} + \frac{a+1}{8} < \pi = g(C) = g(\widetilde{C}) &\leqslant \frac{(d-(a+1)/2-1)^2}{2(a+1)} \\ &< \frac{(d-(a+1)/2)^2}{2(a+1)} = \frac{d^2}{2(a+1)} - \frac{d}{2} + \frac{a+1}{8},\end{aligned} \tag{3.1.12}$$

a contradiction.

We now need the following easy lemma.

Lemma 3.2. *Let $X \subset \mathbb{P}^N$ be a nondegenerate variety, and let $X' \subset \mathbb{P}^{N-1}$ be its general hyperplane section. Denote by $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^N}$ (respectively $\mathcal{I}_{X'} \subset \mathcal{O}_{\mathbb{P}^{N-1}}$) the ideal sheaf of X (respectively that of X'). Then:*

(i) *if X' is linearly normal, then X is also linearly normal;*

(ii) *if X is linearly normal, then the restriction $H^0(\mathbb{P}^N, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^{N-1}, \mathcal{I}_{X'}(2))$ is an isomorphism. In particular, the complete linear system of quadrics in $\mathbb{P}^{N-1}$ containing X' is cut by the complete linear system of quadrics in $\mathbb{P}^N$ containing X.*

Proof. (i) If X' is linearly normal, then

$$H^1(\mathbb{P}^{N-1}, \mathcal{I}_{X'}(1)) = 0. \tag{3.2.1}$$

On the other hand, in the exact cohomology sequence

$$H^0(\mathbb{P}^N, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X) \to H^1(\mathbb{P}^N, \mathcal{I}_X) \to H^1(\mathbb{P}^N, \mathcal{O}_X) \tag{3.2.2}$$

the first arrow is an isomorphism and the last term is zero; hence,

$$H^1(\mathbb{P}^N, \mathcal{I}_X) = 0. \tag{3.2.3}$$

Substituting (3.2.1) and (3.2.3) in the exact cohomology sequence

$$H^1(\mathbb{P}^N, \mathcal{I}_X) \to H^1(\mathbb{P}^N, \mathcal{I}_X(1)) \to H^1(\mathbb{P}^{N-1}, \mathcal{I}_{X'}(1)), \tag{3.2.4}$$

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we conclude that

$$H^1(\mathbb{P}^N, \mathcal{I}_X(1)) = 0, \tag{3.2.5}$$

that is, X is linearly normal. □

(ii) follows from the exact cohomology sequence

$$H^0(\mathbb{P}^N, \mathcal{I}_X(1)) \to H^0(\mathbb{P}^N, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^{N-1}, \mathcal{I}_{X'}(2)) \to H^1(\mathbb{P}^N, \mathcal{I}_X(1)) \tag{3.2.6}$$

whose first and last terms are trivial (the first one by the nondegeneracy of X and the last one by the linear normality of X). □

We return to the proof of Theorem 3.1. Since $C = X_1$ is linearly normal, a repeated application of Lemma 3.2(i) shows that all linear sections $X = X_n$, $X' = X_{n-1}, \ldots, C = X_1$ of the variety X are linearly normal and the linear system of quadrics containing X_i is obtained by restricting the linear system of quadrics containing X on the linear subspace $\langle X_i \rangle \subset \mathbb{P}^N$. From the Halphen–Harris theorem it follows that $C \subset S$, where $S \subset \mathbb{P}^{a+1}$, $\deg S = a$, is a surface of minimal degree. By [Zak1999, Corollary 5.8(a)] and the assumption $d > 2a + 2$, C is defined by

$$e_2(C) = h^0(\mathbb{P}^{a+1}, \mathcal{I}_C(2)) = h^0(\mathbb{P}^{a+1}, \mathcal{I}_S(2)) = e_2(S) = \frac{a(a-1)}{2} \tag{3.1.13}$$

quadratic equations. Using Lemma 3.2(ii), we can lift these quadratic equations to the quadratic equations of X in $\mathbb{P}^N$. Denoting by $V \subset \mathbb{P}^N$ the intersection of the corresponding quadrics, we immediately see that $V \supset X$, $\dim V = n + 1$, $\deg V = a$ and $V \cap \mathbb{P}^{a+1} = S$, which completes the proof of Theorem 3.1. □

Remark 3.3. Since $b > b_n$, the statement of Theorem 3.1(iii) remains true if b is replaced by b_n. On the other hand, for $i < n$ there are no conditions similar to those in Theorem 3.1(iii) ensuring that if $b_i(X)$ is large enough, then X is a codimension-one subvariety in a variety V of minimal degree in $\mathbb{P}^N$ (in fact, from Proposition 2.2 it follows that, except for the middle one, the homologies of X are the same as those of V; cf. (3.5.12)). However, such conditions can be given if $b_i(X)$ is replaced by $b_i(X_i)$. This point is further discussed in Remark (9) in § 4.

Theorem 3.1 shows the importance of studying codimension-one subvarieties in varieties of minimal degree for the problem of describing the varieties with big numerical invariants. We start with recalling classification of varieties of minimal degree (due essentially to P. Del Pezzo and E. Bertini).

Theorem 3.4. (i) *Let $V^{n+1} \subset \mathbb{P}^N$, $n \geqslant 1$, be a variety of minimal degree $a = N - n$. Then V is a cone with vertex $L = \mathbb{P}^l$ over a nonsingular variety of minimal degree $V_0 \subset \mathbb{P}^{N-l-1}$, $\dim V_0 = n - l$, $\deg V_0 = a$, where V_0 is a point, a quadric, a Veronese surface of degree four in $\mathbb{P}^5$ or a rational normal scroll.*

(ii) *Any nonsingular rational normal scroll $V_0 \subset \mathbb{P}^{N-l-1}$ has the form $V_0 = S_{a_1,\ldots,a_{n-l}}$, where $a_i > 0$, $i = 1, \ldots, n-l$, $a_1 + \cdots + a_{n-l} = a$ and $S_{a_1,\ldots,a_{n-l}}$ is constructed as follows. Take $n - l$ subspaces $\mathbb{P}^{a_i}$, $i = 1, \ldots, n-l$, in general position in $\mathbb{P}^{N-l-1}$, let $C_i \subset \mathbb{P}^{a_i}$ be a rational normal curve of degree a_i and let $\phi_i : \mathbb{P}^1 \to C_i$ be an isomorphism. We define $S_{a_1,\ldots,a_{n-l}}$ to be the locus of $(n-l-1)$-dimensional linear subspaces F_t spanned by the corresponding points of C_i, $i = 1, \ldots, n-l$, viz. $S_{a_1,\ldots,a_{n-l}} = \bigcup_{t \in \mathbb{P}^1} F_t$, $F_t = \langle \phi_1(t), \ldots, \phi_{n-l}(t) \rangle$, $t \in \mathbb{P}^1$.*

More generally, the cone with vertex $\mathbb{P}^l$ over $S_{a_1,\ldots,a_{n-l}}$ is denoted by $S_{\underbrace{0,\ldots,0}_{l+1},a_1,\ldots,a_{n-l}}$.

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(iii) *The rational normal scroll* $V = S_{a_1,\ldots,a_{n+1}} \subset \mathbb{P}^N$ *with* $a_1, \ldots, a_{n+1} > 0$ *is a linear section of the Segre variety* $\mathbb{P}^1 \times \mathbb{P}^{a-1} \subset \mathbb{P}^{2a-1}$, $a = a_1 + \cdots + a_{n+1}$.

(iv) *Let* $V = S_{a_1,\ldots,a_{n+1}}$, *let* $H \subset V$ *and* $F \subset V$ *be the divisors of the hyperplane section and the* n*-space* F_t, *respectively, and let* K_V *be the canonical class of* V. *Then* $K_V \sim -(n+1)H + (a-2)F$, *where* $a = \deg V = a_1 + \cdots + a_{n+1}$.

(v) *The dual variety* $S^*_{0,\ldots,0,a_1,\ldots,a_{n-l}} = S^*_{a_1,\ldots,a_{n-l}}$ *is contained in the linear subspace of codimension* $l+1$ *in* $\mathbb{P}^{N*}$ *formed by the hyperplanes passing through the vertex* $\mathbb{P}^l$, *and one has* $V^* = \bigcup_{t\in\mathbb{P}^1} \mathbb{P}^{a-1}_t$, *where* $a = a_1 + \cdots + a_{n+1}$ *and* $\mathbb{P}^{a-1}_t \subset \mathbb{P}^{N*}$ *is the subspace formed by the hyperplanes passing through the* n*-space* F_t, $t \in \mathbb{P}^1$ *(cf. (ii)).*

Furthermore, $\dim V^* = \deg V^* = a$, *and so* $\operatorname{def} V = n-1$ *(cf. Definition 1.8). In particular, all scrolls of dimension larger than two are defective.*

Proof. For the most part, this is well known, cf. for example [EH1987] or [Har1981, § 3]; (v) can be easily verified by hand. □

According to Theorem 3.1, varieties with sufficiently big classes, ramification volumes or Betti numbers are codimension-one subvarieties in varieties of minimal degree. However, we do not yet know whether varieties of minimal degree actually contain subvarieties with big invariants and whether the upper bounds for ramification volumes, classes and Betti numbers obtained in Corollaries 1.5, 1.13 and 2.14 are asymptotically sharp. The following example gives an affirmative answer to these questions.

Example 3.5. Let $V^{n+1} = S_{a_1,\ldots,a_{n+1}} \subset \mathbb{P}^N$, $a_1, \ldots, a_{n+1} > 0$, $\deg V = N - n = a = a_1 + \cdots + a_{n+1}$, be a nonsingular rational normal scroll (cf. Theorem 3.4(ii)), let $W \subset \mathbb{P}^N$ be a general hypersurface of degree m and let $X = W \cap V$ be the corresponding codimension-one subvariety of degree $d = ma$ in V. By Bertini's theorem, X is smooth and irreducible. Let $F = \mathbb{P}^n \subset V$ be a fiber, and let $H \subset V$ denote a hyperplane section, so that $X \sim mH$ as divisors on V.

From Theorem 3.4(iv) and the adjunction formula, it easily follows that

$$\begin{aligned} r_i &= (mH \cdot (mH + (a-2)F)^i \cdot H^{n-i})_V \\ &= m^i(am + i(a-2)) = \left(\frac{d}{a}\right)^i (d + i(a-2)), \end{aligned} \tag{3.5.1}$$

which is a better bound than the general one obtained in Corollary 1.5.

It is clear that

$$X^* = \bigcup_{x\in X} \langle \mathbb{P}^{a-2}_{V,x}, \gamma_W(x)\rangle, \tag{3.5.2}$$

where $\mathbb{P}^{a-2}_{V,x} \subset \mathbb{P}^{N*}$ is the linear subspace of hyperplanes passing through the (embedded) tangent space $T^{n+1}_{V,x}$ and $\gamma_W : W \to \mathbb{P}^{N*}$ is the Gauß map which is birational and defined by forms of degree $m-1$. An easy check shows that, for any $m > 1$, X^* is a hypersurface (that is, $\operatorname{def} X = 0$).

In the notation of Definition 1.8, for any subvariety $Y \subseteq X$ not contained in a fiber F_t, we put

$$\mathcal{P}_V|_Y = p^{-1}(Y), \quad \mathcal{P}_V(Y) = \pi(\mathcal{P}_V|_Y) = \bigcup_{y\in Y} \mathbb{P}^{a-2}_{V,y}. \tag{3.5.3}$$

From Definition 1.8 and Theorem 3.4(v), it easily follows that

$$\dim \mathcal{P}_V|_Y = \dim Y + a - 2,$$
$$\dim \mathcal{P}_V(Y) = \begin{cases} a, & \dim Y \geqslant 2, \\ a-1, & \dim Y = 1, \\ a-2, & \dim Y = 0, \end{cases} \tag{3.5.4}$$
$$\mathcal{P}_V(Y) = V^*, \quad \dim Y \geqslant 2.$$

For $i = 1, \dots, m$, let $X_i^\gamma \subset X$ be the i-dimensional subvariety cut on X by $n-i$ general hypersurfaces of degree $m-1$; in particular, we put $C = X_1^\gamma$. Then

$$\deg X_i^\gamma = d(m-1)^{n-i} \tag{3.5.5}$$

and, arguing as in [Har1992, 19.5] (cf. also the proof of Theorem 1.21) and using Theorem 3.4(v), [Har1992, 19.4] and (3.5.4), we see that

$$\begin{aligned} \mu_n &= \deg X^* = \sum_{i=0}^{n} d_0(\pi|_{p^{-1}(X_i^\gamma)}) \\ &= \deg \gamma_W(X) + \deg \mathcal{P}_V(X_1^\gamma) + m(m-1)^{n-2} \deg V^* \\ &= am(m-1)^n + \deg \mathcal{P}_V(C) + am(m-1)^{n-2}, \quad n \geqslant 2, \\ \mu_1 &= am(m-1) + \deg \mathcal{P}_V(C). \end{aligned} \tag{3.5.6}$$

Thus, it remains to compute $\deg \mathcal{P}_V(C)$. An easy way to do this almost without computations is to consider a general projection $\varpi : V \to \mathbb{P}^{n+2}$. Put $V' = \varpi(V)$, $X' = \varpi(X)$, $C' = \varpi(C)$. By Lemma 1.10(ii), V'^* is a linear section of V^*, $\dim V'^* = 2$. The double locus $D = D_\pi \subset V$ of the projection ϖ, that is, the locus on V, where ϖ fails to be an isomorphism, has the form

$$D \sim (a-n-3)H - K_V \sim (a-2)(H-F) \tag{3.5.7}$$

(cf. [Ili1998] and Theorem 3.4(iv)). Let $\gamma_{V'} : V' \dashrightarrow V'^*$ be the Gauß map defined out of $D' = \varpi(D) = \operatorname{Sing} V'$, and put $\mathcal{P}_{V'}(C') = \gamma_{V'}(C')$. Then

$$\begin{aligned} \deg \mathcal{P}_V(C) &= \deg \mathcal{P}_{V'}(C') = \deg \gamma_{V'}(C') \\ &= (\deg V' - 1) \deg C' - \operatorname{card}(C' \cap D') = (\deg V - 1) \deg C - (C \cdot D)_V \\ &= (a-1)(m-1)^{n-1} d - (a-2) m (m-1)^{n-1} (H^n \cdot H - F)_V \\ &= 2m(a-1)(m-1)^{n-1}. \end{aligned} \tag{3.5.8}$$

Substituting (3.5.8) in (3.5.6), we get

$$\begin{aligned} \mu_n &= \deg X^* \\ &= am(m-1)^n + 2m(a-1)(m-1)^{n-1} + am(m-1)^{n-2} \\ &= m(m-1)^{n-2}(am^2 - 2m + 2) \\ &= \frac{d}{a}\left(\frac{d}{a} - 1\right)^{n-2}\left(\frac{d^2}{a} - 2\frac{d}{a} + 2\right), \quad n \geqslant 2. \end{aligned} \tag{3.5.9}$$

Moreover, since, for each $i \geqslant 1$, X_i is a variety of the same type as X, we conclude that

$$\mu_i = \mu_i(X) = \begin{cases} \dfrac{d}{a}\left(\dfrac{d}{a} - 1\right)^{i-2}\left(\dfrac{d^2}{a} - 2\dfrac{d}{a} + 2\right), & 2 \leqslant i \leqslant n, \\ \dfrac{d^2}{a} + (a-2)\dfrac{d}{a}, & i = 1. \end{cases} \tag{3.5.10}$$

Thus,

$$\mu_i = \frac{d^{i+1}}{a^i} - \left(i - \frac{2(a-1)}{a}\right)\frac{d^i}{a^{i-1}} + P_{i-1}(d), \quad 1 \leqslant i \leqslant n, \tag{3.5.11}$$

where P_{i-1} is a polynomial of degree $i-1$ in d (with coefficients depending on a), and so the bound in Corollary 1.13 is asymptotically sharp for all $1 \leqslant i \leqslant n$.

Comparison of (3.5.10) with (3.5.1) illustrates Theorem 1.12 in this special case. Of course, both (3.5.1) and (3.5.11) combined with Theorem 1.12 show that the bound for ramification volumes r_i given in Corollary 1.5 is asymptotically sharp for all $1 \leqslant i \leqslant n$.

By Proposition 2.2(iii),

$$b_i(X) = b_i(V) = \begin{cases} 2, & i \equiv 0 \pmod 2, i \neq 0, n, 2n \\ 1, & i = 0, 2n \\ 0, & i \equiv 1 \pmod 2, i \neq n \end{cases} \leqslant 2 \tag{3.5.12}$$

and, by Theorem 2.9(ii),

$$b_n(X) \leqslant \mu_n + \mu_{n-2} + \cdots. \tag{3.5.13}$$

We observe that in our setup (3.5.13) can be substantially strengthened. In fact, in view of (3.5.12) and the obvious lower bounds for the vanishing homology, Proposition 2.2 and Corollary 2.7(i) yield

$$b_n(X) \leqslant \mu_n(X). \tag{3.5.14}$$

From (3.5.14) and (3.5.10) or (3.5.11), it is clear that the bound for $b_n(X)$ in Corollary 2.14(i) is asymptotically sharp. Similarly, from (3.5.10) or (3.5.11) and Theorem 2.9(iii), it follows that the bound for $b(X)$ in Corollary 2.14(ii) is asymptotically sharp (this is also evident from the sharpness of the bound for $b_n(X)$). On the other hand, from (3.5.12) and Remark 3.3, it is clear that the bounds for $b_i(X)$, $i < n$, given in Corollary 2.14(i) fail to be asymptotically sharp (cf. also Remark (9) in § 4).

Theorem 3.1 shows that nonsingular varieties with big classes, ramification volumes or Betti number are codimension-one subvarieties in varieties of minimal degree. On the other hand, according to Example 3.5, (at least some of the) varieties of minimal degree contain smooth codimension-one subvarieties of arbitrarily high degree with big invariants. Although we do not believe that the assumption of nonsingularity is crucial, it is essential for some of our arguments (cf. also Remark (4) in § 4). Thus, one needs to know which of the varieties of minimal degree contain nonsingular subvarieties of codimension one. Now we turn to settling this question.

Proposition 3.6. *Let $X \subset V^{n+1}$ be a nonsingular subvariety of a variety of minimal degree $V^{n+1} \subset \mathbb{P}^N$ with $\dim \operatorname{Sing} V = l$. Then either $X \subset L$ (and, consequently, $\dim X \leqslant l$) or $l \leqslant 2 \operatorname{codim}_V X - 1$.*

Proof. In the notation of Theorem 3.4, V is a cone with vertex $L = \mathbb{P}^l$ over a nonsingular variety V_0 of minimal degree in $\mathbb{P}^{N-l-1}$. Suppose that $X \not\subset L$. Pick a point $x \in X \backslash L$, and let $v \in V_0$ be the point for which $x \in L_v$, where $L_v = \langle L, v\rangle$, $\dim L_v = l+1$, is the linear subspace in V corresponding to v. Put $T_v = \langle T_{V_0,v}, L\rangle$. Then $T_{V,w} = T_v$ for each $w \in L_v \backslash L$, and therefore the $(n+1)$-dimensional linear subspace T_v is tangent to X along the subvariety $X_v = \overline{X \cap (L_v \backslash L)}$. Since $x \in \operatorname{Sm} V$, one has

$$\dim X_v \geqslant \dim L_v - \operatorname{codim}_V X = l - \operatorname{codim}_V X + 1. \tag{3.6.1}$$

On the other hand, by the Theorem on tangencies [Zak1993, ch. I, Corollary 1.8],

$$\dim X_v \leqslant \dim T_v - \dim X = \operatorname{codim}_V X. \tag{3.6.2}$$

Combining (3.6.1) and (3.6.2), one gets

$$l \leqslant 2 \operatorname{codim}_V X - 1. \tag{3.6.3}$$

□

COROLLARY 3.7. *In the notation of Proposition 3.6, suppose that V contains a nonsingular subvariety X^n of codimension one. Then $l \leqslant 1$.*

Proof. Since $l = \dim L \leqslant \dim V - 2 = n - 1$, $X \not\subset L$. Thus, our claim follows from (3.6.3). □

PROPOSITION 3.8. *In the notation of Proposition 3.6, suppose that $l = \dim L = 1$ and V contains a nonsingular subvariety X^n, $n \geqslant 2$, of codimension one. Then $L \subset X$ and $V = S_{0,0,a_1,\dots,a_{n-1}}$ is a cone with vertex L over a rational normal scroll $V_0 = S_{a_1,\dots,a_{n-1}} \subset \mathbb{P}^{N-2}$, $a_1 + \cdots + a_{n-1} = a$. Thus, X is a Roth variety (cf. [Ili1998, Definition 3.1]) and so $d \equiv 1 \pmod{a}$ and X has the form described in [Ili1998, Theorem 3.8].*

Proof. Suppose that $L \not\subset X$. Then, in the notation of the proof of Proposition 3.6, there exists a point $x \in L$ such that

$$x \in \bigcap_{v \in V_0} X_v, \tag{3.8.1}$$

and so

$$T^n_{X,x} \subset \bigcap_{v \in V_0} T_v = \left\langle L, \bigcap_{v \in V_0} T_{V_0,v} \right\rangle = L \tag{3.8.2}$$

(the intersection of the tangent spaces of V_0 is empty because V_0 is not a cone). Since $l = 1$ and $n \geqslant 2$, (3.8.2) leads to a contradiction.

Thus, $L \subset X$, and therefore $S(x, X) = S(y, X) = S(L, X) = V$ for arbitrary points $x, y \in L$ (here, for a subvariety $Y \subset X$, $S(Y, X)$ denotes the relative secant variety, that is, the join of Y with X). Proposition 3.8 now follows from [Ili1998, Proposition 4.1] (cf. also [Ili1998, Theorem 3.8]). □

Remarks 3.9. (i) Recall that, by [Ili1998, Theorem 3.8], the Roth varieties $X \subset V$, $V = S_{0,0,a_1,\dots,a_{n-l}}$, can be described as follows. Consider the variety $\widetilde{V} \subset \mathbb{P}^1 \times \mathbb{P}^N$, $\widetilde{V} = \bigcup_{t \in \mathbb{P}^1} (t, \langle L, \phi_1(t), \dots, \phi_{n-1}(t) \rangle)$ and its projections $\pi_2 : \widetilde{V} \to V \subset \mathbb{P}^N$ and $\pi_1 : \widetilde{V} \to \mathbb{P}^1$. Clearly, $\widetilde{V}$ is a (nonsingular) scroll over $\mathbb{P}^1$, and we denote by F its fiber and by H the pullback under π_2 of a hyperplane section of V. Let $b \geqslant 1$, and let $\tilde{X} \in |bH + F|$ be a smooth irreducible variety meeting the quadric $\mathbb{P}^1 \times L$ along an irreducible curve (a general $\tilde{X} \in |bH + F|$ has this property by Bertini's theorem). Put $X = \pi_2(\tilde{X})$. Then $X \simeq \tilde{X}$ is a Roth variety of degree $d = ab + 1$ and, conversely, any Roth variety $L \subset X \subset V$ has this form.

(ii) Any variety $V^{n+1} \subset \mathbb{P}^N$ of minimal degree with $l = \dim \operatorname{Sing} V = 0$ contains nonsingular subvarieties of codimension one. Indeed, it suffices to take the intersection of V with a general hypersurface in $\mathbb{P}^N$ avoiding the singularity of V. There also exist nonsingular subvarieties $X^n \subset V$ passing through the vertex of the cone V; in view of [Ili1998, Remark 5.14], these subvarieties admit the same description as in (i).

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COROLLARY 3.10. *A nondegenerate variety $V^{n+1} \subset \mathbb{P}^N$ of (minimal) degree $a = N - n$ and a nonsingular subvariety $X^n \subset V$ of degree d exist if and only if one of the following conditions holds:*

(a) $a = 1$, $V = \mathbb{P}^{n+1}$, *X is a hypersurface;*
(b) $a = 2$, $V = Q$ *is a quadric with at most one singular point* ($l \leqslant 0$, *X is a complete intersection if $n - l > 2$; the case $l = 0$, $n = 2$ fits into the case (e) while the case $n = 1$ is classical and easy);*
(c) $a = n - 1$, $l = 1$, $V_0 = \mathbb{P}^1 \times \mathbb{P}^{n-2} \subset \mathbb{P}^{2n-3}$, $X \supset L = \operatorname{Sing} V$, $d \equiv 1 \pmod{n-1}$ *and X is a Roth variety described in [Ili1998, Theorem 3.8]; cf. also Remark 3.9(i);*
(d) $a = n$, $l = 1$, $V = S_{0,0,\underbrace{1,\ldots,1}_{n-2},2}$, $X \supset L = \operatorname{Sing} V$, $d \equiv 1 \pmod{n}$ *and X is a Roth variety;*
(e) $a = n$, $l = 0$ *and either* $X \not\ni L$ *or* $X \ni L$ *and X is a variety described in Remark 3.9(ii);*
(f) $a \geqslant n + 1$.

Proof. This follows from Proposition 3.8, Remark 3.9 and the fact that $a = \operatorname{codim} X = \operatorname{codim} V + 1 = \operatorname{codim} V' + 1 = \deg V' = a_1 + \cdots + a_{n-l}$ (cf. Theorem 3.4). □

The varieties considered in Example 3.5 satisfy the condition $d \equiv 0 \pmod{a}$. However, similar examples exist for arbitrary degrees. In what follows, we estimate the invariants of codimension-one subvarieties of varieties of minimal degree; the bounds obtained are asymptotically equivalent to, but better than, the general bounds for arbitrary varieties obtained in the preceding sections.

We start with subvarieties of rational normal scrolls. Let $V = S_{a_1,\ldots,a_{n+1}} \subset \mathbb{P}^N$, $\deg V = N - n = a = a_1 + \cdots + a_{n+1} = a$ be a rational normal scroll (cf. Theorem 3.4(ii)), and let $X^n \subset V^{n+1}$ be an arbitrary smooth subvariety (cf. Corollary 3.7, Proposition 3.8 and Remark 3.9). If $F = \mathbb{P}^n \subset V$, $F = \langle L, F_0\rangle$, where $L = \operatorname{Sing} V$, $F_0 \subset V_0$ is a fiber, and H is a hyperplane section of V, then $X \cap F$ is a hypersurface whose degree we denote by m (clearly, $m \leqslant d = (H^n \cdot X)_V$). One may assume that $X \sim mH - \epsilon F$. If $\epsilon \geqslant 0$, then X is the residual intersection of V with a hypersurface W of degree m passing through ϵ n-dimensional linear subspaces $F_1, \ldots, F_\epsilon \subset V$, and if $\epsilon < 0$, then X is the residual intersection of V with a hypersurface W of a higher degree $m + m'$ passing through a subvariety $Y \subset V$, $Y \sim m'H - \epsilon$. We will not pursue here the problem of giving a detailed description of the pairs (m, ϵ) for which there exists an irreducible smooth subvariety $X \subset V$, $X \sim mH - \epsilon F$ (cf. Remark 3.13 below). Rather, we will give better bounds for the invariants of such subvarieties X in terms of m and ϵ (and eventually in terms of d and a).

PROPOSITION 3.11. (i) *$r_i \leqslant a(d/a + \rho_i)^{i+1}$, where $\rho_i = (i/(i+1)) \cdot ((a-2)/a) < 1$, $1 \leqslant i \leqslant n$. Furthermore, this bound is asymptotically sharp (and even sharp if $i + 1$ divides $a - 2$).*

(ii)
$$\mu_i(X) = \mu_i(m, d) = \begin{cases} -am^2 + (2d + a - 2)m, & i = 1, \\ m(m-1)^{i-2}(-iam^2 + ((i+1)d + 2(a-1))m - 2(d-1)), & i \geqslant 2. \end{cases}$$

For given d, one has

$$\mu_1 < d\left(\frac{d}{a} + 1\right), \quad d \geqslant \frac{(a-2)^2}{8};$$
$$\mu_2 < \frac{d^2}{a^2}(d-1), \quad d \geqslant \frac{a(a+6)}{3};$$
$$\mu_i < d\left(\frac{d-1}{a}\right)^i, \quad i \geqslant 3,$$

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and the function $\mu_i = \mu_i(m)$ *attains maximum at a point* $m_i^0 = m_i^0(d)$ *with*

$$\frac{d}{a} + \frac{a-2}{a(i+1)} \leqslant m_i^0 \leqslant \frac{d}{a} + \frac{2(a-1)}{a(i+1)}, \quad \lim_{d\to\infty} m_i^0 - \frac{d}{a} = \frac{a-2}{a(i+1)}.$$

Furthermore,

$$\mu_i(X) < \frac{d^{i+1}}{a^i} - \left(i - 2\frac{a-1}{a}\right)\frac{d^i}{a^{i-1}} + P_{i-1}(d),$$

where P_{i-1} *is a polynomial of degree* $i-1$ *in* d *whose coefficients depend only on* a*, and this bound is asymptotically sharp, that is, there exist varieties* X *as above with* $(1/d^i)(\mu_i(X) - d^{i+1}/a^i + (i-2(a-1)/a)(d^i/a^{i-1}))$ *arbitrarily small.*

In our setup $m = m(X)$ *is an integer, and the maximal value of* $\mu_i(X) = \mu_i(m(X), d)$ *is attained either for* $m = [d/a]$ *or for* $m = [d/a]+1$.

(iii)
$$\begin{aligned} b(X) &= (-1)^n e(X) + 2n(1 + (-1)^{n+1}) \\ &= \mu_n - 2\mu_{n-1} + 3\mu_{n-2} - \cdots + (-1)^n(n+1)\mu_0 + 2n(1+(-1)^{n+1}) \end{aligned}$$

and

$$b(X) < \frac{d^{n+1}}{a^n}.$$

Furthermore,

$$b(X) < \frac{d^{n+1}}{a^n} - \left(n + \frac{2}{a}\right)\frac{d^n}{a^{n-1}} + Q_{n-1}(d),$$

where $Q_{n-1}(d)$ *is a polynomial of degree* $n-1$ *in* d *whose coefficients depend only on* a*, and this bound is asymptotically sharp, that is, there exist varieties* X *as above with* $(1/d^n)(b(X) - d^{n+1}/a^n + (n+2/a)(d^n/a^{n-1}))$ *arbitrarily small.*

In our setup $m = m(X)$ *is an integer, and the maximal value of* $b(X) = b(m(X), d)$ *is attained either for* $m = [d/a]$ *or for* $m = [d/a]+1$.

Proof. The arguments in the proof of Proposition 3.11 are similar to those used in Example 3.5, and we will not go into petty computational details.

(i) From Theorem 3.4(iv) and the adjunction formula, it follows that

$$\begin{aligned} r_i &= (mH - \epsilon F \cdot (mH + (a-\epsilon-2)F)^i \cdot H^{n-i})_V \\ &= m^i(am + (a-2)i - (i+1)\epsilon) = \left(\frac{d+\epsilon}{a}\right)^i (d + i(a-\epsilon-2)). \end{aligned} \tag{3.11.1}$$

It is easy to see that, for fixed a, d and i, r_i attains maximal value for $\epsilon = \epsilon_0 = i(a-2)/(i+1)$ (here we assume that $r_i = r_i(\epsilon)$ is defined by (3.5.1) for arbitrary (real) values of ϵ, although only integral values make sense geometrically). Substituting ϵ_0 in (3.5.1), we get

$$r_i \leqslant a\left(\frac{d}{a} + \rho_i\right)^{i+1} \quad \text{where } \rho_i = \frac{i}{i+1}\cdot\frac{a-2}{a} < 1. \tag{3.11.2}$$

(ii) Adding to the equation of W a suitable form from the homogeneous ideal of V, we may assume that W is smooth. Clearly,

$$d = \deg X = am - \epsilon. \tag{3.11.3}$$

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An argument similar to that in Example 3.5 shows that in our setting (3.5.6) is replaced by the formula

$$\begin{aligned}\mu_n = \deg X^* &= (mH - F)((m-1)H - \epsilon F)^n \\ &\quad + \deg \mathcal{P}_V(C) + m(m-1)^{n-2} \deg V^*, \quad n \geqslant 2,\end{aligned} \tag{3.11.4}$$

where

$$\begin{aligned}C &\sim (mH - \epsilon F) \cdot ((m-1)H - \epsilon F)^{n-1} \\ &\sim m(m-1)^{n-1}H^n - \epsilon(m-1)^{n-2}(nm-1)H^{n-1}F.\end{aligned} \tag{3.11.5}$$

From (3.5.7) and (3.11.5), it follows that

$$\begin{aligned}(C \cdot D)_V &= ((a-2)(H-F) \cdot (m(m-1)^{n-1}H^n - \epsilon(m-1)^{n-2}(nm-1)H^{n-1}F)) \\ &= (a-2)(m-1)^{n-2}(m(m-1)(a-1) - \epsilon(nm-1))\end{aligned} \tag{3.11.6}$$

and

$$\deg C = (m-1)^{n-2}(ma(m-1) - \epsilon(nm-1)). \tag{3.11.7}$$

Arguing as in the proof of (3.5.8), we get

$$\begin{aligned}\deg \mathcal{P}_V(C) &= (a-1)\deg C - (C \cdot D)_V \\ &= (m-1)^{n-2}(2m(m-1)(a-1) - \epsilon(nm-1)),\end{aligned} \tag{3.11.8}$$

and so, by (3.11.4),

$$\begin{aligned}\mu_n(X) &= (m-1)^n(ma-\epsilon) + (m-1)^{n-2}(2m(m-1)(a-1) \\ &\quad - \epsilon(nm-1) + ma) - \epsilon mn(m-1)^{n-1} \\ &= m(m-1)^{n-2}((am^2 - 2m + 2) - \epsilon((n+1)m - 2)), \quad n \geqslant 2.\end{aligned} \tag{3.11.9}$$

Moreover, since, for each $i \geqslant 1$, X_i is a variety of the same type as X, arguing as above, we conclude that

$$\begin{aligned}\mu_i &= \mu_i(X) \\ &= \begin{cases} m(m-1)^{i-2}((am^2 - 2m + 2) - \epsilon((i+1)m - 2)), & i \geqslant 2, \\ m(a(m+1) - 2(\epsilon + 1)), & i = 1. \end{cases}\end{aligned} \tag{3.11.10}$$

Substituting $\epsilon = am - d$ from (3.11.3) in (3.11.10), we get the first formula in (ii):

$$\mu_i = \begin{cases} m(m-1)^{i-2}(-iam^2 + ((i+1)d + 2(a-1))m - 2(d-1)), & i \geqslant 2, \\ -am^2 + (2d + a - 2)m, & i = 1. \end{cases} \tag{3.11.11}$$

From (3.11.11), it is easy to deduce that, for fixed d, the function $\mu_i = \mu_i(m)$ attains maximum at a single point $m_i^0 = m_i^0(d)$ with

$$\frac{d}{a} + \frac{a-2}{a(i+1)} < m_i^0 < \frac{d}{a} + \frac{2(a-1)}{a(i+1)}, \tag{3.11.12}$$

and

$$\lim_{d \to \infty} m_i^0(d) = \frac{d}{a} + \frac{a-2}{a(i+1)} \tag{3.11.13}$$

($m_1^0 = d/a + (a-2)/2a$). Of course, only integral values of m make sense in our geometric setup, and analyzing the behavior of the function μ_i and its derivatives, it is easy to see that the maximal value of μ_i at an integral point is attained either for $m = [d/a]$ or for $m = [d/a] + 1$.

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The remaining claims in (ii) can now be obtained by estimating the value $\mu_i(m_i^0)$. For example, for $i \geqslant 3$ our claims follow from a stronger bound

$$\mu_i \leqslant \mu_i(m_i^0) < \frac{d^2}{a}\left(\frac{d}{a} - \frac{i-1}{i+1}\right)\left(\frac{d}{a} - \frac{ai+2}{a(i+1)}\right)^{i-2}, \tag{3.11.14}$$

which can be obtained from (3.11.11), (3.11.12) and a study of the derivative μ_i'; we will incorporate (3.11.14) in Corollary 3.12. We observe that one gets progressively better bounds for μ_i for higher values of i and a.

(iii) The equality

$$\begin{aligned} b(X) &= (-1)^n e(X) + 2n(1 + (-1)^{n+1}) \\ &= \mu_n - 2\mu_{n-1} + 3\mu_{n-2} - \cdots + (-1)^n (n+1)\mu_0 + 2n(1 + (-1)^{n+1}) \end{aligned} \tag{3.11.15}$$

is an immediate consequence of (3.5.12) and Theorem 2.9(i). The claims concerning the Betti number can be obtained from (3.11.15) and the already known properties of classes. For example, using (3.11.11), one can show that

$$\begin{aligned} &((n-i+1)\mu_i - (n-i+2)\mu_{i-1})'_m > 0, \quad m \leqslant \frac{d}{a}, \\ &((n-i+1)\mu_i - (n-i+2)\mu_{i-1})'_m < 0, \quad m \geqslant \frac{d}{a} + 1, \end{aligned} \tag{3.11.16}$$

after which the proof of (iii) is reduced to a computational check.

The cases $n=1$ and $n=2$ are somewhat special. For $n=1$,

$$b(X) = \mu_1(X) - 2d + 4 \leqslant \frac{d^2}{a} - \frac{a+2}{a}d + \frac{a}{4} + 3 + \frac{1}{a} < \frac{d^2}{a}, \quad d \geqslant a+1. \tag{3.11.17}$$

For $n=2$, a combination of (3.11.11) and (3.11.15) yields

$$b(X) < \frac{d^2(d-a)}{a^2} < \frac{d(d-1)(d-2)}{a^2} < d\left(\frac{d-1}{a}\right)^2 < \frac{d^3}{a^2}. \tag{3.11.18}$$

□

Proposition 3.11 and its proof yield other useful bounds for classes. For example, one has the following corollary.

Corollary 3.12.

(i) $\mu_i < \dfrac{d^2}{a}\left(\dfrac{d}{a} - \dfrac{i-1}{i+1}\right)\left(\dfrac{d}{a} - \dfrac{ai+2}{a(i+1)}\right)^{i-2}$ *for* $i \geqslant 3$.

(ii) $\mu_i < d\dfrac{d-1}{a} \cdot \dfrac{d-2}{a} \cdots \dfrac{d-i}{a}$ *for* $i > 3$ *or* $i = 3, a > 4$.

Proof. (i) was already stated in (3.11.14), and (ii) follows from (i). □

Remark 3.13. It turns out that for $m = [d/a]$ and $m = [d/a] + 1$ there indeed exist smooth irreducible subvarieties $X^n \subset V^{n+1}$ such that $X \sim mH - \epsilon F$, $\epsilon = am - d$; cf. [Har1981, § 4].

As we recalled in Theorem 3.4(i), there are other nonsingular varieties of minimal degree besides rational normal scrolls, *viz.* quadrics and the Veronese surface. Leaving aside the classical case $n=1$ already considered in Example 1.4, and the case when the quadric is a scroll, we get the following proposition.

Proposition 3.14. *Let $V = Q^{n+1}$, $n > 1$, be a quadric such that* $\dim V^0 > 2$, *and let $X^n \subset Q$ be a nonsingular subvariety. Then $d = \deg X \equiv 0 \pmod 2$, $a = 2$ and X is a complete intersection of Q and a hypersurface $W \subset \mathbb{P}^{n+2}$ of degree $m = d/2$. Furthermore,*

$$r_i(X) = \frac{d^{i+1}}{2^i}, \quad 1 \leqslant i \leqslant n,$$

$$\mu_i(X) = 2m(1 + (m-1) + \cdots + (m-1)^i) < \begin{cases} d\left(\dfrac{d}{2}+1\right), & i = 1, \\ \dfrac{d^2}{4}(d-1), & i = 2, \\ d\left(\dfrac{d-1}{2}\right)^i, & i \geqslant 3, \end{cases}$$

$$\begin{aligned} b(X) &= (-1)^n e(X) \\ &= \mu_n - 2\mu_{n-1} + \cdots + (-1)^n (n+1)\mu_0 < \frac{d^{n+1}}{a^n}. \end{aligned}$$

Proof. Under our assumptions, $\operatorname{Pic} Q = \mathbb{Z}$ and X is a complete intersection (Klein's theorem); furthermore, one may assume W to be smooth.

The first claim follows from the adjunction formula.

Denoting by $\gamma_Q : Q \to \mathbb{P}^{n+2*}$ and $\gamma_W : W \to \mathbb{P}^{n+2*}$ the Gauß maps defined respectively by linear forms and forms of degree $m-1$, one sees that

$$X^* = \bigcup_{x \in X} \langle \gamma_Q(x), \gamma_W(x) \rangle \tag{3.14.1}$$

and, from [Har1992, 19.5–19.7], it follows that

$$\begin{aligned} \mu_n &= \deg \gamma_W(X) + \mu_{n-1} \\ &= d(1 + (m-1) + \cdots + (m-1)^n) = 2m\frac{(m-1)^{n+1} - 1}{m-2}. \end{aligned} \tag{3.14.2}$$

Replacing X by its intersection with a general linear subspace of codimension $n - i$, that is, replacing n by i, and making obvious verifications, we get the second claim.

From Proposition 2.2 and Corollary 2.7(i), it follows that

$$b(X) \leqslant b_n(X) + n + 1 \leqslant \mu_n(X) + n + 1 \tag{3.14.3}$$

(cf. also (3.5.14)). For $n \geqslant 2$, the bound for $b(X)$ follows from (3.14.3) and the bound for $\mu_n(X)$. For $n = 1$,

$$b(X) = \mu_1 - 2\mu_0 + 4 < d\left(\frac{d}{2}+1\right) - 2d + 4 = \frac{d^2}{2} - (d-4) \leqslant \frac{d^2}{2}. \tag{3.14.4}$$

□

The remaining case of (cones over) the Veronese surface is the simplest one, and we will not discuss it in detail (cf. also [Har1981, pp. 63–64] and [DG2001, p. 150]; we remark that, by Corollary 3.7 and Proposition 3.8, a nonsingular subvariety X^n of codimension one in such a variety of minimal degree exists if and only if $n \leqslant 2$). We sum up some of the above results in the following theorem.

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Theorem 3.15. *Let $X^n \subset V$ be a nonsingular subvariety of codimension one and degree d in a variety $V^{n+1} \subset \mathbb{P}^N$, $\deg V = a = N - n$. Then:*

(i) $r_i(X) \leqslant a\left(\dfrac{d}{a} + \rho_i\right)^{i+1}$, *where* $\rho_i = \dfrac{i}{i+1} \cdot \dfrac{a-2}{a} < 1, 1 \leqslant i \leqslant n$;

(ii) $\mu_1(X) < d\left(\dfrac{d}{a} + 1\right), \quad d \geqslant \dfrac{(a-2)^2}{8}$;

$$\mu_2(X) < \frac{d^2}{a^2}(d-1), \quad d \geqslant \frac{a(a+6)}{3};$$

$$\mu_i(X) < d\left(\frac{d-1}{a}\right)^i, \quad i \geqslant 3;$$

(iii) $b(X) < \dfrac{d^{n+1}}{a^n}$.

Furthermore, the above bounds are asymptotically sharp.

The following result strengthens Theorems 1.18 and 2.18 in the case of varieties of fixed codimension a.

Theorem 3.16. *Let $X^n \subset \mathbb{P}^N$ be a nondegenerate nonsingular variety of degree d and codimension $a = N - n$. Then:*

(i) $\mu_i(X) < \dfrac{d^{i+1}}{a^i}, \quad d \geqslant (a+1)^2, \quad 2 \leqslant i \leqslant n$;

$$\mu_1(X) \leqslant \frac{1}{a}\left(d + \frac{a-2}{2}\right)^2;$$

(ii) $b(X) < \dfrac{d^{n+1}}{a^n}, \quad d \geqslant 2(a+1)^2, \quad n \geqslant 2$;

$$b(X) < \frac{d^2}{a}, \qquad n = 1.$$

Furthermore, the above bounds are asymptotically sharp.

Proof. (i) In view of Theorem 3.1(i), the claim follows from Theorem 3.15(ii).

(ii) For $n = 1$, this claim was proved in (3.11.17).

Suppose that $n \geqslant 2$. By (2.14.8) and (2.14.9), the function $\phi(t) = t^{n+2}/(t-1)^2$ is monotonically increasing for $t \geqslant (n+2)/n$. By Theorems 3.1(iii) and 3.15(iii), to prove our claim it suffices to show that there exists a number α such that, for $d > 2(a+1)^2$ and $t_\alpha = (d-\alpha)/a$, the following conditions hold:

(a) $t_\alpha = \dfrac{d-\alpha}{a} \geqslant \dfrac{n+2}{n}$;

(b) $t_\alpha = \dfrac{d-\alpha}{a} > \dfrac{d}{a+1} + \dfrac{a-7}{4d} + 1$;

(c) $d\phi(t_\alpha) < \dfrac{d^{n+1}}{a^n}$.

Condition (a) is clearly satisfied for $d \geqslant a + \alpha + 2a/n$.

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Since $(a+1)(a+\alpha+1) > (a+1)(a+\alpha) + (a(a-7)(a+1))/4d$ for $d > a(a-7)/4$, condition (b) is satisfied for $d \geqslant (a+1)(a+\alpha+1)$.

Condition (c) is equivalent to the inequality

$$(d-\alpha)^{n+2} < d^n(d-(a+\alpha))^2. \tag{3.16.1}$$

If $\alpha > 0$, then it is clear that if (3.16.1) holds for some $n = n_0$, then it also holds for all $n > n_0$. Putting $n_0 = 2$, one sees that (c) is satisfied for example for $\alpha = a+1$ provided that $d \geqslant (a+2)^2$.

Putting all this together, one concludes that conditions (a), (b) and (c) are satisfied for

$$\alpha = a+1, \quad d \geqslant 2(a+1)^2 \tag{3.16.2}$$

(for larger n, one can take smaller α, which leads to an (insignificant) improvement of the lower bound for d; cf. also Remark 3.17(ii)).

Asymptotic sharpness follows from a similar claim in Theorem 3.15. □

Remarks 3.17. (i) Arguing as in the proof of Theorem 3.16, one can use Theorems 3.1(ii) and 3.15(i) to obtain a bound for the ramification volumes $r_i(X)$ of arbitrary smooth varieties of sufficiently large degree d. However, this bound is only slightly better than the bound $r_i < d(d/a+1)^i$ for $d \geqslant (a-2)^2/8$ from Corollary 1.5, and so we do not give it here.

(ii) Theorem 3.16 is stated in the above way for the sake of uniformity and brevity. The bounds can be sharpened for specific values of i and n. For example, from Theorems 3.1(i) and 3.15(ii), it follows that $\mu_i(X) < d((d-1)/a)^i$ for $i \geqslant 3$ and $d \geqslant (a+1)(a+2)$. More generally, (3.11.14) and other similar inequalities yield progressively better bounds for the codegree $d^* = \mu_n$ and the Betti numbers b and b_n as n grows, but these sharper bounds are valid for somewhat larger d.

In a different direction, it is evident that, for large d, (3.16.1) holds provided that $2(a+\alpha) < (n+2)\alpha$, that is, $\alpha > 2a/n$. Taking smaller α, one can slightly improve the lower bound for d in Theorem 3.16, but it still remains quadratic in a.

4. Further remarks and open problems

Here we give a list of possible extensions of the above results and some open problems.

(1) Our bounds for ramification volumes, classes and Betti numbers hold, under mild assumptions, for projective varieties defined over an algebraically closed field of arbitrary characteristic. In fact, the proof of Theorem 1.1 is based on the Hodge index theorem, which holds in arbitrary characteristic (cf. for example [Gr1958]), and the bounds for Betti numbers can be derived from Theorems 1.1 and 1.12 by using Lefschetz theory for ℓ-adic cohomologies (cf. [DK1973]).

(2) Similar bounds hold for the Betti numbers of real algebraic varieties. There are two ways to obtain them. First, one can apply Smith theory to the bounds for complex varieties given in the present paper. Unfortunately, this method applies only to homologies with coefficients $\mathbb{Z}/2\mathbb{Z}$. Second, one can develop a relative version of Morse theory for morphisms to a projective line valid for both real and complex varieties; this is done in a forthcoming joint paper with Kharlamov [KhZ].

It should be mentioned that, unlike in the complex setup, in the real case our methods yield bounds only for the total Betti number and not for the individual ones. Also, while it is rather easy to prove asymptotic sharpness, getting sharp bounds for the total Betti number of real varieties is much harder. In particular, it is not easy to show the existence of M-varieties among hypersurfaces in varieties of minimal degree (cf. [KhZ]).

(3) In this paper we give bounds for the Betti numbers of nonsingular projective varieties. However, there are similar (and even better) bounds for nonsingular affine varieties. To wit, if $V^n \subset \mathbb{A}^N$ is a nonsingular affine variety, X is its closure in $\mathbb{P}^N$ and μ_i, $i = 0, \ldots, n$, is the ith class of X, then in [KhZ] we show that V is homotopy equivalent to a CW-complex with at most μ_i cells of dimension i, so that, in particular, $b_i(V) \leqslant \mu_i$. This gives a far-reaching generalization of Lefschetz's weak theorem, and it is desirable to give an 'abstract' proof of this result by methods similar to those used in the present paper. However, we do not know how to do that, particularly in the case when X has singularities at infinity.

(4) The nonsingularity hypothesis is essential for our methods. It seems less important for the validity of bounds for ramification volumes and classes (although the definition and ampleness of the ramification divisor present certain problems), but it is crucial for the application of Lefschetz theory and its consequences. Nevertheless, we feel that 'morally' our bounds should be true for varieties with arbitrary singularities. It is also possible that in the singular case it is worthwhile to study other invariants whose relation to Betti numbers can be compared to that of arithmetic genus to geometric genus in Castelnuovo's theory.

(5) Our proof of Theorem 3.16 uses the Halphen–Harris theorem, and so it does not work for small d. The bound $b(X) < d^{n+1}$ proved in Theorem 2.18 is true without any assumptions on d. It would be nice to find out whether the bound $b(X) < d^{n+1}/a^n$ proved in Theorem 3.16(ii) for $d \geqslant 2(a+1)^2$ holds for smaller d when one cannot guarantee that X is contained in a variety of minimal degree.

(6) Varieties of dimension n and degree d are a special case of d-fold coverings of the projective space $\mathbb{P}^n$ (cf. for example [Laz2004, 6.3 D]). It would be nice to study the appropriate notions and extend our bounds to this wider class of varieties.

(7) It would be nice to extend the results of this paper to subvarieties of abelian and toric varieties.

(8) We already mentioned that while we considered the problem of bounding numerical invariants, such as classes or Betti numbers, on the set of varieties $X^n \subset \mathbb{P}^N$ of a given degree d, Milnor [Mil1964], Oleinik [Ole1951], Thom [Tho1965] and others studied the problem of bounding the (total) Betti number of a variety $X^n \subset \mathbb{P}^N$ defined by equations whose degree does not exceed d (cf. Definition 1.20). In Theorems 1.21 and 2.20, we found asymptotically sharp bounds for classes and Betti numbers in the setting of Milnor and Thom. However, while in the case of classes our bounds are sharp and attained only by nonsingular complete intersections of hypersurfaces of degree d (cf. Theorem 1.21), we did not give classification of varieties with maximal Betti numbers. It would be nice to classify such varieties, with respect to both b_n and b (presumably, they also are complete intersections). More generally, one can study bounds and extremal varieties for classes and Betti numbers of varieties whose defining equations include (at least) e_i (linearly independent) equations of degree d_i (with possible variations).

(9) We already gave examples showing that our bounds for $b(X)$ and $b_n(X)$ as well as those for $r_i(X)$ and $\mu_i(X)$ are asymptotically sharp (cf. e.g. Example 3.5). However, this is not the case for the bounds for the other Betti numbers $b_i(X)$ in Corollary 2.14(i). As an example, consider the case $i = 1$. If the irregularity of X were close to the upper bound given by

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Corollary 2.14(i), then, by Proposition 2.2(iii), the sectional genus of X would be still closer to this bound, so that, by the arguments in the proof of Theorem 3.1, X is a divisor in a variety of minimal degree, whence X is regular provided that $n > 1$ (cf. (3.5.12) and Proposition 2.2(iii)). Thus, the sharp upper bound for irregularity is smaller than the one given in Corollary 2.14(i), and it is desirable to find this bound and to classify maximally irregular varieties of given dimension, codimension and degree. By Lefschetz theory, this problem reduces to the case of surfaces. There are reasons to believe that the bound in question is roughly twice better than Castelnuovo's bound for curves. However, this bound should be much better for small a. In fact, complete intersections are regular and, by a conjecture of Peskine and Van de Ven, irregularity of all nonsingular surfaces in $\mathbb{P}^4$ is uniformly bounded.

Similar questions arise with respect to all other Betti numbers $b_i(X)$ with $1 \leqslant i \leqslant n-1$. Thus, it is desirable to obtain sharp (or asymptotically sharp) inequalities for individual Betti numbers other than the middle one and classify the varieties on the boundary.

(10) It is not accidental that the principal term in a sharp bound for $b(X)$ (or $e(X)$; cf. Theorem 2.9 and Corollary 2.14) is the same as the one in a sharp bound for $r_n(X)$ (or $\mu_n(X)$ or (K_X^n); cf. Corollaries 1.5 and 1.6), *viz.* d^{n+1}/a^n. In **[Zak2012]** we showed, among other things, that for any two multi-indices I, I' with $|I| = |I'| = n$, the difference $c_I - c_{I'}$ between the corresponding Chern numbers is bounded by a polynomial of degree n in d. Applying this to c_n and c_1^n or, with some deliberation, to ramification and polar classes, one gets an 'explanation' of the above observation.

(11) Another natural generalization of Castelnuovo's work for curves is to bound, for fixed p and q, the Hodge number $h^{p,q}$ of nondegenerate nonsingular projective varieties $X^n \subset \mathbb{P}^N$ of a given degree d and to classify the varieties on the boundary. For $p = n$, $q = 0$ (the case of geometric genus) this was done in [Har1981]. *A priori*, even in the most important case when $p + q = n$, one might think that one thus gets $[(n+2)/2]$ independent maximization problems (clearly, $h^{p,q} = h^{q,p}$). However, it turns out that asymptotically (for large d), all these problems are equivalent, *viz.* $h^{p,q}$ attains maximal value simultaneously with b_n (hence with all the $h^{p',q'}$, $p' + q' = n$) and the extremal varieties are again codimension-one subvarieties in varieties of minimal degree (cf. [Zak2012]).

More precisely, let $\alpha^p = h^{p,n-p}/b_n$, $\sum_{p=0}^n \alpha^p = 1$, denote the 'weight' of the corresponding Hodge number. Then, in [Zak2012], we showed that the weights $\alpha^p(X)$ are close to $\alpha^p(\boldsymbol{X})$ provided that $b_n(X)$ is large enough (here, as in Theorem 2.16, $\boldsymbol{X}$ denotes a nonsingular hypersurface of dimension n and degree d). Thus, the limit weights α_∞^p are given by the volumes of slices of the unit cube $\mathbf{K} \subset \mathbb{R}^{n+1}$, *viz.*

$$\alpha_\infty^p = \operatorname{vol}\{(x_0, \ldots, x_n) \in \mathbf{K} \mid p \leqslant x_0 + \cdots + x_n \leqslant p+1\}$$

(cf. for example [KK1989, ch. 4, § 5.10]).

In general, for given n, a, d and $0 \leqslant p, q \leqslant n$, it is an interesting problem to find a sharp (or asymptotically sharp) bound for the Hodge number $h^{p,q}$ and to classify the varieties on or near the boundary. Since $\sum_{p+q=i} h^{p,q} = b_i$, this problem is closely related to the one considered in (9).

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