A CLASS OF INTEGRAL TRANSFORMS †

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1. Introduction

In this paper we discuss a new class of integral transforms and their inversion formula. The kernel in the transform is a G-function (for a treatment of this function, see ((**1**), 5.3) and integration is performed with respect to the argument of that function. In the inversion formula, the kernel is likewise a G-function, but there integration is performed with respect to a parameter. Known special cases of our results are the Kontorovitch-Lebedev transform pair ((**2**), v. 2; (**3**))

$$g(x) = \frac{2}{\pi^2} x \sinh(\pi x) \int_0^\infty t^{-1} K_{ix}(t) f(t) dt, \quad \text{.................(1.1)}$$

$$f(x) = \int_0^\infty K_{it}(x) g(t) dt, \quad \text{.............................(1.2)}$$

and the generalised Mehler transform pair (**7**)

$$g(x) = \frac{x}{\pi} \sinh(\pi x) \Gamma(\tfrac{1}{2} - k + ix) \Gamma(\tfrac{1}{2} - k - ix) \int_1^\infty P^k_{ix-\frac{1}{2}}(t) f(t) dt, \quad \text{...(1.3)}$$

$$f(x) = \int_0^\infty P^k_{it-\frac{1}{2}}(x) g(t) dt. \quad \text{..........................(1.4)}$$

These transforms are used in solving certain boundary value problems of the wave or heat conduction equation involving wedge or conically-shaped boundaries, and are extensively tabulated in (**6**).

Section **2** contains preliminary results and definitions, and Section **3** contains the derivation of our main result. Several examples are presented in Section **4**.

2. Preliminary definitions and results

The Mellin transform (**2**)

$$g(s) = \int_0^\infty x^{s-1} f(x) dx = \mathscr{M}_s\{f(x)\}, \quad \text{....................(2.1)}$$

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and its inversion formula

$$f(x) = \mathscr{M}_x^{-1}\{g(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} g(s)ds, \qquad \text{(2.2)}$$

will be used constantly.

The G-function we define by the Mellin-Barnes contour integral

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} x^s ds, \qquad \text{(2.3)}$$

where $0 \leqq m \leqq q$, $0 \leqq n \leqq p$ and the path L runs from $-i\infty$ to $+i\infty$ so that the poles of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$, are to the right and all the poles of $\Gamma(1-a_j-s)$, $j = 1, 2, \ldots, n$ are to the left of L. Convergence is assured if $p+q<2(m+n)$ and $|\arg x| < \left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi$. All these assumptions will be retained throughout.

We will frequently use the shorthand notation for (2.3)

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) \qquad \text{(2.4)}$$

when no ambiguity can arise.

In what follows, let $\sigma_2 = \min \operatorname{Re}(b_j)$, $j = 1, 2, \ldots, m$, $\sigma_1 = \max \operatorname{Re}(a_j - 1, -\nu)$, $j = 1, 2, \ldots, n$. We have

Theorem 1. *Let* 1. $\sigma_1 < \operatorname{Re}(-s) < \sigma_2$,

2. $0 \leqq m \leqq q$, $0 \leqq n \leqq p$, $p+q<2(m+n+1)$.

Then

$$\mathscr{M}_s\left\{G_{p+2,q}^{m,n+2}\left(x \,\middle|\, \begin{matrix} 1-\nu, \frac{1}{2}-\nu, a_p \\ b_q \end{matrix}\right)\right\}$$

$$= \frac{\prod_{j=1}^{m} \Gamma(b_j+s)\Gamma(\nu-s)\Gamma(\nu-s+\frac{1}{2}) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)}. \qquad \text{(2.5)}$$

(2.5) is known ((1), (5.5.2)) and results by using the condition after (2.3) in (2.2).

Theorem 2. *Let* 1. $\sigma_1 < 0 < \sigma_2$,

2. *A real number* δ *exist*, $\sigma_1 < \delta < \sigma_2$, *such that s satisfies*

$$-\operatorname{Re}(\nu) - \delta < \operatorname{Re}(s) < \operatorname{Re}(\nu) + \delta,$$

3. $0 \leqq m \leqq q$, $0 \leqq n \leqq p$, $p+q \leqq 2(m+n+1)$,

4. $|\arg \alpha| < \left(m+n+1-\frac{p}{2}-\frac{q}{2}\right)\pi$.

Then

$$\frac{2^{2\nu-1}}{\sqrt{\pi}}\mathcal{M}_s\left\{\left[\frac{x}{(1+x)^2}\right]^{\nu} G^{m,\,n+2}_{p+2,\,q}\left(\frac{4\alpha x}{(1+x)^2}\,\middle|\,\begin{matrix}1-\nu,\ \tfrac{1}{2}-\nu,\ a_p\\ b_q\end{matrix}\right)\right\}$$
$$= G^{m,\,n+2}_{p+2,\,q}\left(\alpha\,\middle|\,\begin{matrix}1-\nu+s,\ 1-\nu-s,\ a_p\\ b_q\end{matrix}\right). \quad \text{.........(2.6)}$$

Proof. We use the representation (2.3), with appropriate change of variable, in (2.1). Condition (1) assures the separation of poles while condition (2) assures the absolute convergence of the double integral. Thus the order of integration may be interchanged and the result is (2.6).

3. The main result

We wish to solve the integral equation

$$g(x) = \int_0^\infty G^{m,\,n+2}_{p+2,\,q}\left(t\,\middle|\,\begin{matrix}1-\nu+ix,\ 1-\nu-ix,\ a_p\\ b_q\end{matrix}\right)f(t)dt, \quad \text{.........(3.1)}$$

where the conditions on the parameters are those of Theorem 2. Replace x by $-iu$. Multiply both sides of (3.1) by $x^{-u}du$ and integrate from $c-i\infty$ to $c+i\infty$. If we invoke (2.6) and let

$$\frac{4x}{(1+x)^2} = \bar{x}, \quad x = \frac{(2-\bar{x})-2\sqrt{1-\bar{x}}}{\bar{x}}, \quad \text{.....................(3.2)}$$

on the right-hand side of the equation, we have

$$2\sqrt{\pi}\,\bar{x}^{-\nu}\bar{g}(x) = \int_0^\infty G^{m,\,n+2}_{p+2,\,q}\left(\bar{x}t\,\middle|\,\begin{matrix}1-\nu,\ \tfrac{1}{2}-\nu,\ a_p\\ b_q\end{matrix}\right)f(t)dt, \quad \text{.........(3.3)}$$

where

$$\bar{g}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-u}g(-iu)du. \quad \text{.......................(3.4)}$$

Using the notation of ((**10**), p. 315), we have

$$\mathfrak{G}(s) = 2\sqrt{\pi}\int_0^\infty \bar{x}^{(s-\nu-1)}\bar{g}(x)d\bar{x},$$
$$= 2\sqrt{\pi}\,4^{s-\nu}\int_{e^{\pi i}}^0 (x-1)x^{s-\nu-1}(1+x)^{2\nu-2s-1}\bar{g}(x)dx, \quad \text{.........(3.5)}$$

or

$$\mathfrak{G}(1-s) = \frac{4^{1-s-\nu}}{\sqrt{\pi}\,i}\int_{c-i\infty}^{c+i\infty} g(-iu)du\int_{e^{\pi i}}^0 (x-1)x^{-u-s-\nu}(1+x)^{2s+2\nu-3}dx,$$
$$= \frac{e^{\pi i(1-s-\nu)}}{2\pi i}\Gamma(\nu+s-1)\Gamma(\nu+s-\tfrac{1}{2})\int_{c-i\infty}^{c+i\infty} e^{-\pi i u}g(-iu)$$
$$\cdot\left\{\frac{\Gamma(2-u-s-\nu)}{\Gamma(s+\nu-u)}+\frac{\Gamma(1-u-s-\nu)}{\Gamma(s+\nu-u-1)}\right\},$$
$$= \frac{e^{-\pi i(s+\nu)}}{\pi i}\Gamma(\nu+s-1)\Gamma(\nu+s-\tfrac{1}{2})\int_{c-i\infty}^{c+i\infty} e^{-\pi i u}ug(-iu)\frac{\Gamma(1-u-s-\nu)}{\Gamma(s+\nu-u)}du. \quad \text{(3.6)}$$

Also, from (2.5)

$$\mathfrak{K}(1-s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+1-s)\Gamma(\nu+s-1)\Gamma(\nu+s-\tfrac{1}{2})\prod_{j=1}^{n}\Gamma(-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(-b_j+s)\prod_{j=n+1}^{p}\Gamma(1+a_j-s)},\quad\ldots(3.7)$$

and so, by ((**10**), p. 316),

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathfrak{G}(1-s)}{\mathfrak{K}(1-s)}x^{-s}ds=\frac{1}{2(\pi i)^2}\int_{c-i\infty}^{c+i\infty}x^{-s}ds\int_{c-i\infty}^{c+i\infty}e^{-\pi i(s+u+\nu)}ug(-iu)$$

$$\cdot\frac{\prod_{j=m+1}^{q}\Gamma(-b_j+s)\prod_{j=n+1}^{p}\Gamma(1-a_j+s)\Gamma(1-u-s-\nu)}{\prod_{j=1}^{m}\Gamma(b_j+1-s)\prod_{j=1}^{n}\Gamma(-a_j+s)\Gamma(s+\nu-u)}du,$$

$$=\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}e^{-\pi i(u+\nu)}ug(-iu)$$

$$\cdot\mathscr{M}^{-1}_{xe^{\pi i}}\left\{\frac{\prod_{j=m+1}^{q}\Gamma(-b_j+s)\prod_{j=n+1}^{p}\Gamma(1+a_j-s)\Gamma(1-u-s-\nu)}{\prod_{j=1}^{m}\Gamma(b_j+1-s)\prod_{j=1}^{n}\Gamma(-a_j+s)\Gamma(s+\nu-u)}\right\}du,$$

$$=\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}e^{-\pi i(\nu+u)}ug(-iu)G^{q-m,p-n+1}_{p+2,q}\left(xe^{\pi i}\,\middle|\,\begin{matrix}-a_{n+1},\ -a_{n+2},\ \ldots,\\ -b_{m+1},\ -b_{m+2},\ \ldots,\end{matrix}\right.$$

$$\left.\begin{matrix}-a_p,\ u+\nu,\ -a_1,\ -a_2,\ \ldots,\ -a_n,\ \nu-u\\ -b_q,\ -b_1,\ -b_2,\ \ldots,\ \quad -b_m\end{matrix}\right)du.\quad\ldots\ldots(3.8)$$

With $c=0$, we have the transform pair

$$g(x)=\int_0^\infty\Omega(t\,|\,1-\nu+ix,\ 1-\nu-ix)f(t)dt,\quad\ldots\ldots\ldots\ldots\ldots(3.9)$$

$$f(x)=\frac{i}{\pi}\int_0^\infty te^{-\nu\pi i}\{e^{\pi t}\Lambda(xe^{\pi i}\,|\,\nu+it,\ \nu-it)-e^{-\pi t}\Lambda(xe^{\pi i}\,|\,\nu-it,\ \nu+it)\}g(t)dt,\quad(3.10)$$

where

$$\Lambda(z\,|\,\alpha,\beta)=G^{q-m,p-n+1}_{p+2,q}\left(z\,\middle|\,\begin{matrix}-a_{n+1},\ -a_{n+2},\ \ldots,\\ -b_{m+1},\ -b_{m+2},\ \ldots,\end{matrix}\right.$$

$$\left.\begin{matrix}-a_p,\ \alpha,\ \quad -a_1,\ -a_2,\ \ldots,\ -a_n,\ \beta\\ -b_q,\ -b_1,\ -b_2,\ \ldots,\ -b_m\end{matrix}\right),\quad\ldots(3.11)$$

$$\Omega(z\,|\,\alpha,\beta)=G^{m,n+2}_{p+2,q}\left(z\,\middle|\,\begin{matrix}\alpha,\ \beta,\ a_p\\ b_q\end{matrix}\right).\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.12)$$

4. Applications

Example 1. $n=p=0$, $m=q=1$.

We need the formulæ

$$G_{12}^{10}\left(\omega \,\middle|\, \begin{matrix}\alpha\\ \beta,\gamma\end{matrix}\right) = \frac{e^{\omega/2}\omega^{(\gamma+\beta-1)/2}}{\Gamma(1+\beta-\gamma)\Gamma(\alpha-\beta)} M_{\alpha-\frac{\beta+\gamma+1}{2},\frac{\beta-\gamma}{2}}(\omega), \quad\text{.........(4.1)}$$

$$G_{12}^{21}\left(\omega \,\middle|\, \begin{matrix}\alpha\\ \beta,\gamma\end{matrix}\right) = \Gamma(\beta-\alpha+1)\Gamma(\gamma-\alpha+1)e^{\omega/2}\omega^{(\gamma+\beta-1)/2} W_{\alpha-\frac{\beta+\gamma+1}{2},\frac{\beta-\gamma}{2}}(\omega). \quad\text{...(4.2)}$$

Let $b_1 = \frac{1}{2}-\nu-\kappa$. Since

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix}a_p\\ b_q\end{matrix}\right) = G_{q,p}^{n,m}\left(\frac{1}{x} \,\middle|\, \begin{matrix}1-b_q\\ 1-a_p\end{matrix}\right), \quad\text{.....................(4.3)}$$

we have

$$\Lambda(xe^{\pi i}\mid \nu\pm it, \nu\mp it) = G_{21}^{01}\left(xe^{\pi i} \,\middle|\, \begin{matrix}\nu\pm it,\ \nu\mp it\\ \kappa+\nu-\frac{1}{2}\end{matrix}\right) = G_{12}^{10}\left(\frac{e^{-\pi i}}{x} \,\middle|\, \begin{matrix}\frac{3}{2}-\kappa-\nu\\ 1-\nu\mp it,\ 1-\nu\pm it\end{matrix}\right),$$

$$= \frac{(xe^{\pi i})^{\nu-\frac{1}{2}}e^{-1/2x}}{\Gamma(1\mp 2it)\Gamma(\frac{1}{2}-\kappa\pm it)} M_{-\kappa,\mp it}\left(\frac{e^{-\pi i}}{x} \,\middle|\, x\right). \quad\text{...........(4.4)}$$

Using first the fact that

$$e^{(m-\frac{1}{2})\pi i}M_{-\kappa,-m}(x) = M_{\kappa,-m}(xe^{-\pi i}), \quad\text{.....................(4.5)}$$

and then

$$W_{\kappa,m}(x) = \frac{\pi}{\sin(2m\pi)}\left\{\frac{-M_{\kappa,m}(x)}{\Gamma(\frac{1}{2}-m-\kappa)\Gamma(1+2m)} + \frac{M_{\kappa,-m}(x)}{\Gamma(\frac{1}{2}+m-\kappa)\Gamma(1-2m)}\right\},$$

$$= W_{\kappa,-m}(x), \quad\text{...(4.6)}$$

(see [(8), (1.7)]), we arrive at the transform pair

$$g(x) = \Gamma(\tfrac{1}{2}-\kappa-ix)\Gamma(\tfrac{1}{2}-\kappa+ix)\int_0^\infty e^{1/2t}t^{\frac{1}{2}-\nu}W_{\kappa,it}\left(\frac{1}{t}\right)f(t)dt, \quad\text{......(4.7)}$$

$$f(x) = \frac{e^{-1/2x}x^{\nu-\frac{1}{2}}}{\pi^2}\int_0^\infty t\sinh(2\pi t)W_{\kappa,it}\left(\frac{1}{x}\right)g(t)dt. \quad\text{..................(4.8)}$$

In (4.7) replace t by $1/t$; in (4.8), x by $1/x$, and then replace $e^{t/2}t^{\nu-\frac{3}{2}}f(t^{-1})$ by $f(t)$. The transform pair is

$$g(x) = \Gamma(\tfrac{1}{2}-\kappa-ix)\Gamma(\tfrac{1}{2}-\kappa+ix)\int_0^\infty W_{\kappa,ix}(t)f(t)dt, \quad\text{...........(4.9)}$$

$$f(x) = \frac{1}{(x\pi)^2}\int_0^\infty t\sinh(2\pi t)W_{\kappa,it}(x)g(t)dt. \quad\text{.....................(4.10)}$$

Since

$$W_{0,m}(x) = \sqrt{\frac{x}{\pi}}\,K_m\left(\frac{x}{2}\right), \quad\text{...........................(4.11)}$$

putting $\kappa = 0$ in (4.9) and (4.10) we obtain the Kontorovich-Lebedev transforms (1.1), (1.2). For a theoretical study of how such transforms arise from second order differential equations, see (5), (9).

Another transform pair involving Whittaker functions has been given by Erdélyi (**4**).

Example 2. $n = p = 0, m = 1, q = 2.$
Here we use

$$G_{22}^{1n}\left(x\,\middle|\,\begin{matrix}a_1, a_2\\ b_1, b_2\end{matrix}\right) = \frac{\prod_{j=1}^{n}\Gamma(1+b_1-a_j)x^{b_1}}{\Gamma(1+b_1-b_2)\prod_{j=n+1}^{2}\Gamma(a_j-b_1)}$$

$$\cdot\, {}_2F_1\left(\begin{matrix}1+b_1-a_1,\ 1+b_1-a_2\\ 1+b_1-b_2\end{matrix}\,\middle|\,(-)^{n+1}x\right). \quad\text{.........(4.12)}$$

Let $b_1 = \xi - \nu$ and $b_2 = \sigma - \nu$. A straightforward application of (3.9)-(3.12) gives

$$g(x) = \frac{\Gamma(\xi-ix)\Gamma(\xi+ix)}{\Gamma(1+\xi-\sigma)}\int_0^\infty t^{\xi-\nu}{}_2F_1\left(\begin{matrix}\xi-ix,\ \xi+ix\\ 1+\xi-\sigma\end{matrix}\,\middle|\,-t\right)f(t)dt, \quad\text{...(4.13)}$$

$$f(x) = \frac{x^{\nu-\sigma}}{\Gamma(1+\xi-\sigma)}\int_0^\infty \frac{t\sinh(2\pi t){}_2F_1\left(\begin{matrix}1-\sigma+it,\ 1-\sigma-it\\ 1+\xi-\sigma\end{matrix}\,\middle|\,-x\right)}{\Gamma(\sigma-it)\Gamma(\sigma+it)(\cosh^2\pi t-\cos^2\pi\sigma)}g(t)dt. \quad\text{...(4.14)}$$

If we let $\sigma = \frac{1}{2}$ and use the relationships

$${}_2F_1\left(\begin{matrix}1-\beta+\alpha,\ -\alpha-\beta\\ 1-\beta\end{matrix}\,\middle|\,-y\right) = \Gamma(1-\beta)(y+y^2)^{\beta/2}P_\alpha^\beta(2y+1), \quad\text{...(4.15)}$$

$${}_2F_1\left(\begin{matrix}-\alpha,\ \alpha+1\\ \beta\end{matrix}\,\middle|\,-y\right) = \Gamma(\beta)\left(\frac{y}{1+y}\right)^{\frac{1}{2}-\beta/2}P_\alpha^{1-\beta}(2y+1), \quad\text{.........(4.16)}$$

in (4.13) and (4.14) respectively we obtain the generalised Mehler transform pair (1.3), (1.4).

Example 3. $m = n = p = q = 1, a_1 = b_1 = \kappa.$
Here, the kernels of the transform pairs are Lommel's functions ((**1**), (7.5.5.)). We have

$$\Omega(t\mid 1-\nu+ix,\ 1-\nu-ix)$$
$$= t^{-\nu}2^{2(\kappa+\nu)}\Gamma(\kappa+\nu+ix)\Gamma(\kappa+\nu-ix)S_{1-2\kappa-2\nu,\,2ix}(2t^{-\frac{1}{2}}), \quad\text{...(4.17)}$$

$$\Lambda(z^{-1}\mid \nu\pm it,\ \nu\mp it) = \frac{z^{1\mp it-\nu}{}_0F_1(1\mp 2it\mid z)}{\Gamma(1-\nu-\kappa\mp it)\Gamma(\kappa+\nu\pm it)\Gamma(1\mp 2it)}$$
$$= \frac{z^{1-\nu}}{\pi}\sin\pi(\kappa+\nu\pm it)I_{\mp 2it}(2z^{\frac{1}{2}}), \quad\text{...............(4.18)}$$

by ((**1**), (5.3(5))) and ((**1**), (7.2.2(12))). If we use the well-known properties of Lommel's and Bessel functions, the analysis is straightforward and we omit

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details. After appropriate changes of variables, we get

$$g(x) = \int_0^\infty S_{\mu,ix}(t)f(t)dt, \quad \text{.........................(4.19)}$$

$$f(x) = \frac{2^{-2\mu-1}}{\pi^2 x}\int_0^\infty t\sinh \pi t\,\frac{\Gamma\left(\frac{1-\mu+it}{2}\right)\Gamma\left(\frac{1-\mu-it}{2}\right)}{\Gamma\left(\frac{1+\mu+it}{2}\right)\Gamma\left(\frac{1+\mu-it}{2}\right)}[S_{\mu,it}(x)-s_{\mu,it}(x)]g(t)dt. \quad \text{...(4.20)}$$

Example 4. $m = n = 0, p = q = 1$.
Here $a_1 = 1-\nu$, $b_1 = \frac{1}{2}-\nu$,

$$\Omega(t \mid 1-\nu+ix, 1-\nu-ix) = \frac{t^{-\nu}\sqrt{\pi}}{2i\sinh(\pi x)}[J^2_{-ix}(t^{-\frac{1}{2}})-J^2_{ix}(t^{-\frac{1}{2}})], \quad \text{...(4.21)}$$

and

$$G^{12}_{31}\left(z \,\middle|\, \begin{matrix}\nu-1, u+\nu, \nu-u \\ \nu-\frac{1}{2}\end{matrix}\right) = [\delta-(\nu-1)]G^{12}_{31}\left(z \,\middle|\, \begin{matrix}\nu, u+\nu, \nu-u \\ \nu-\frac{1}{2}\end{matrix}\right)$$

$$= 2\sqrt{\pi}[\delta-(\nu-1)]z^{\nu-1}I_{-u}(z^{-\frac{1}{2}})K_{-u}(z^{-\frac{1}{2}}), \quad \text{...(4.22)}$$

where $\delta = zd/dz$ [see [(1), (5.3(13)), (5.4(25), (26))]]. If $c = 0$ in (3.8) and we make the necessary changes of variable, the transform pair is

$$g(x) = \frac{i\sqrt{\pi}}{\sinh(\pi x)}\int_0^\infty [J^2_{ix}(t)-J^2_{-ix}(t)]f(t)dt, \quad \text{..............(4.23)}$$

$$f(x) = \frac{\sqrt{\pi}}{2}\int_{-\infty}^\infty te^{-\pi t}\frac{d}{dx}\{J_{it}(x)H^{(1)}_{it}(x)\}g(t)dt. \quad \text{...........(4.24)}$$

Example 5. $m = 0, p = q = n = 1, a_1 = 1-\nu, b_1 = \frac{1}{2}-\nu$.
Since

$$\Omega(t \mid 1-\nu+ix, 1-\nu-ix) = \frac{2}{\sqrt{\pi}}t^{-\nu}K^2_{ix}(t^{-\frac{1}{2}}), \quad \text{..............(4.25)}$$

and

$$G^{11}_{31}\left(z \,\middle|\, \begin{matrix}u+\nu, \nu-1, \nu-u \\ \nu-\frac{1}{2}\end{matrix}\right) = -\sqrt{\pi}[\delta-(\nu-1)]z^{\nu-1}J^2_{-u}(z^{-\frac{1}{2}}),\ \delta = \frac{zd}{dz}, \quad \text{...(4.26)}$$

elementary manipulations with Bessel functions give

$$g(x) = \frac{2}{\sqrt{\pi}}\int_0^\infty K^2_{ix}(t)f(t)dt, \quad \text{.............................(4.27)}$$

$$f(x) = \frac{i}{\sqrt{\pi}}\int_{-\infty}^\infty t\frac{d}{dx}\{I^2_{it}(x)\}g(t)dt. \quad \text{....................(4.28)}$$

In closing, we note that the development in Sections 3 and 4 has been formal only, and conditions for the validity of each transform pair must be decided individually. We defer these considerations to a future paper.

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