ON A q-ANALOGUE OF THE LOG-Γ-FUNCTION

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0. Introduction

For complex numbers q and u, Carlitz defined the q-Bernoulli numbers $\{\beta_k(q)\}$ and the q-Euler numbers $\{H_k(u, q)\}$ associated to u by

$$\beta_0(q) = 1, \ \sum_{j=0}^{k} \binom{k}{j} q^{j+1}\beta_j(q) - \beta_k(q) = \begin{cases} 1 & (k = 1) \\ 0 & (k \geq 2) \end{cases}$$

and

$$H_0(u, q) = 1, \ \sum_{j=0}^{k} \binom{k}{j} q^j H_j(u, q) - u H_k(u, q) = 0 \quad (k \geq 1).$$

(See [2]). Note that if $q \to 1$, then $\beta_k(q) \to B_k$ and $H_k(u, q) \to H_k(u)$ where $\{B_k\}$ and $\{H_k(u)\}$ are the ordinary Bernoulli and Euler numbers defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!},$$

and

$$\frac{1 - u}{e^t - u} = \sum_{k=0}^{\infty} H_k(u) \frac{t^k}{k!}.$$

In [4], Kobiltz constructed a q-analogue of the p-adic L-function $L_{p,q}(s, \chi)$ which interpolated the q-Bernoulli numbers at non positive integers, and suggested the following two problems.

(1) Are there complex analytic q-L-series which $L_{p,q}(s, \chi)$ can be viewed as interpolating, in the same way that $L_p(s, \chi)$ interpolates $L(s, \chi)$?

(2) Do Carlitz's $\beta_k(q)$ occur in the coefficients of some Stirling type series for p-adic or complex analytic q-log-Γ-functions?

In [5], Satoh gave an answer to the problem (1). Namely he constructed a q-L-series $L_q(s, \chi)$ which interpolated the generalized q-Bernoulli numbers defined in [4]. An answer to the problem (2) in the p-adic case was given by the author in [7]. But the problem (2) in the complex case is unsolved.

In the present paper, we give an answer to the problem (2) in the complex case. In §1, we construct a locally analytic function $g(x, u, q)$ in which the q-Euler numbers occur as the coefficients of the Stirling expansion. In §2, we construct a locally analytic function $G(x, q)$ in which the q-Bernoulli numbers occur as the coefficients of the Stirling expansion. In §3, we calculate the values of $L_q(s, \chi)$ at positive integers by using $G(x, q)$. The result is a q-analogue of the classical relation between the Dirichlet L-series and the $\log$-Γ-function. So we can regard $G(x, q)$ as the q-$\log$-Γ-function which the above problem (2) in the complex case requires.

1.   The function $g(x, u, q)$

Let $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ be the sets of rational integers, real numbers and complex numbers. Let q be a complex number with $|q| < 1$, and u be a complex number with $|u| > 1$. For a complex number z, we use the notation $[z] = [z ; q] = (1 - q^z)/(1 - q)$. Note that

$$(1.1) \qquad \qquad \lim_{n \to \infty} [n] = \frac{1}{1 - q}.$$

Let

$$(1.2) \qquad \qquad l(s, u, q) = \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^s},$$

for $s \in \mathbf{C}$.

LEMMA 1 (Satoh).   *$l(s, u, q)$ is analytic in the whole complex plane. For $k \in \mathbf{Z}$ with $k \geq 0$,*

$$l(- k, u, q) = \begin{cases} \dfrac{1}{u - 1} & (k = 0) \\[2ex] \dfrac{u}{u - 1} H_k(u, q) & (k \geq 1) \end{cases}$$

Proof.   See [5].

Now we define the function

(1.3)
$$g(x, u, q) = \sum_{n=0}^{\infty} u^{-n}(x + [n])\log(x + [n]).$$

By the condition $|u| > 1$, we can see that $g(x, u, q)$ is a locally analytic function.

PROPOSITION 1. *For $x \in \mathbf{C}$ with $|x| > 1/(1 - |q|)^2$,*

$$g(x, u, q) = \frac{u}{u - 1}\left\{x\log x + \frac{1}{u - q}(\log x + 1)\right\}$$

$$+ \frac{u}{u - 1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k + 1)}H_{k+1}(u, q)\frac{1}{x^k}.$$

Proof. By (1.3),

(1.4)
$$g(x, u, q) = \sum_{n=0}^{\infty} u^{-n}(x + [n])\log x$$

$$+ \sum_{n=0}^{\infty} u^{-n}(x + [n])\log\left(1 + \frac{[n]}{x}\right).$$

By Lemma 1, we have

$$\sum_{n=0}^{\infty} u^{-n}[n] = \frac{u}{u - 1}H_1(u, q)$$

$$= \frac{u}{u - 1}\frac{1}{u - q}.$$

So the first term of (1.4) can be calculated and equals

$$\frac{u}{u - 1}\left\{x\log x + \frac{1}{u - q}\log x\right\}.$$

On the other hand, by using the series expansion formula

$$\log(1 + z) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}z^k}{k},$$

the second term of (1.4) equals

(1.5)
$$\sum_{n=0}^{\infty} u^{-n}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\frac{[n]^k}{x^{k-1}} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\frac{[n]^{k+1}}{x^k}\right\}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \left\{ \sum_{n=0}^{\infty} u^{-n} [n]^{k+1} \right\} \frac{1}{x^k} + \frac{u}{u-1} \frac{1}{u-q} .$$

Note that $|[n]| < 1/(1 - |q|)$ for $n = 1, 2, 3, \ldots$. So the equation (1.5) holds for x with $|x| > 1/(1 - |q|)^2$. By Lemma 1, we have the assertion.

COROLLARY 1.   *For $x \in \mathbf{C}$ with $|x| > 1/(1 - |q|)^2$,*

$$g'(x, u, q) = \frac{u}{u-1} (\log x + 1) + \frac{u}{u-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(u, q) \frac{1}{x^k} ,$$

where $g'(x, u, q)$ is the derivative $\dfrac{d}{dx} g(x, u, q)$.

2.   The function $G(x, q)$

In [5], the q-Riemann ζ-function was defined by

$$(2.1) \qquad \zeta_q(s) = \frac{2-s}{s-1} (1-q) \sum_{n=1}^{\infty} \frac{q^n}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n}{[n]^s} .$$

LEMMA 2 (Satoh).   *$\zeta_q(s)$ is analytic in the whole complex plane. For $k \in \mathbf{Z}$ with $k \geq 1$,*

$$\zeta_q(1-k) = \begin{cases} q\beta_1(q) & (k=1) \\ -\dfrac{\beta_k(q)}{k} & (k \geq 2). \end{cases}$$

Proof.   See [5].

By Lemma 1, Lemma 2 and (2.1), we obtain the relation

$$(2.2) \qquad -\frac{\beta_k(q)}{k} = -\frac{k+1}{k} H_k(q^{-1}, q) + \frac{1}{1-q} H_{k-1}(q^{-1}, q)$$

for $k \in \mathbf{Z}$ with $k \geq 2$.

Now we define the function $G(x, q)$ by

$$(2.3) \quad G(x, q) = (qx - x - 1)g'(x, q^{-1}, q) + 2(1-q)g(x, q^{-1}, q) + \frac{1}{1+q} .$$

PROPOSITION 2. *For $x \in \mathbf{C}$ with $|x| > 1/(1 - |q|)^2$,*

$$G(x, q) = \left(x - \frac{1}{1+q}\right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} \beta_{k+1}(q) \frac{1}{x^k} \, .$$

Proof. By Proposition 1 and Corollary 1, the left hand side of (2.3) equals

$$\{(q-1)x - 1\} \frac{1}{1-q} (\log x + 1) + 2\left(x \log x + \frac{q}{1-q^2} (\log x + 1)\right) + \frac{1}{1+q}$$

$$+ \{(q-1)x - 1\} \frac{1}{1-q} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} H_k(q^{-1}, q) \frac{1}{x^k}$$

$$+ 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} H_{k+1}(q^{-1}, q) \frac{1}{x^k}$$

$$= \left(x - \frac{1}{1+q}\right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left\{\frac{k+2}{k+1} H_{k+1}(q^{-1}, q) - \frac{1}{1-q} H_k(q^{-1}, q)\right\} \frac{1}{x^k} \, .$$

By (2.2), we have the assertion.

Remark. The formula in Proposition 2 is a q-analogue of the classical asymptotic series (see [9]).

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} \sim \left(x - \frac{1}{2}\right) \log x - x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} B_{k+1} \frac{1}{x^k} \, ,$$

where B_n is the ordinary Bernoulli number.

3. Values of $L_q(s, \chi)$

For $r \in \mathbf{Z}$ with $r \geq 1$ and a function $f(x)$, let $f^{(r)}(x)$ be the r-th derivative $\frac{d^r}{dx^r} f(x)$.

LEMMA 3. *For $r \in \mathbf{Z}$ with $r \geq 2$,*

$$\frac{(-1)^r}{(r-2)!} g^{(r)}(x, u, q) = \sum_{n=0}^{\infty} \frac{u^{-n}}{(x + [n])^{r-1}} \, .$$

Proof. We can see that

$$g^{(1)}(x, u, q) = \sum_{n=0}^{\infty} u^{-n} \{\log(x + [n]) + 1\},$$

and

$$g^{(2)}(x, u, q) = \sum_{n=0}^{\infty} \frac{u^{-n}}{(x + [n])}.$$

Inductively we can see that

$$g^{(r)}(x, u, q) = (-1)^r (r - 2)! \sum_{n=0}^{\infty} \frac{u^{-n}}{(x + [n])^{r-1}}$$

for $r \geq 2$. So we have the assertion.

The q-L-series was defined by

$$(3.1) \qquad L_q(s, \chi) = \frac{2-s}{s-1} (1-q) \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n \chi(n)}{[n]^s},$$

where χ is a primitive Dirichlet character with conductor f. $L_q(s, \chi)$ interpolates the generalized q-Bernoulli numbers $\beta_{n,\chi}(q)$ (defined in [4]) at non positive integers. So we can regard $L_q(s, \chi)$ as the function which the Koblitz problem (1) requires (see [4],[5]).

Now we evaluate the values of $L_q(s, \chi)$ at positive integers.

LEMMA 4.   *For $k \in \mathbf{Z}$ with $k \geq 2$,*

$$L_q(k, \chi) = \frac{2-k}{(k-1)!} (1-q) \frac{(-1)^k}{[f]^{k-1}} \sum_{a=1}^{f} q^{a(2-k)} \chi(a) g^{(k)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f\right)$$

$$+ \frac{1}{(k-1)!} \frac{(-1)^{k+1}}{[f]^k} \sum_{a=1}^{f} q^{a(1-k)} \chi(a) g^{(k+1)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f\right)$$

Proof.   By (3.1)

$$(3.2) \qquad L_q(k, \chi) = \frac{2-k}{k-1} (1-q) \sum_{a=1}^{f} \sum_{j=0}^{\infty} \frac{q^{a+fj} \chi(a+fj)}{[a+fj]^{k-1}}$$

$$+ \sum_{a=1}^{f} \sum_{j=0}^{\infty} \frac{q^{a+fj} \chi(a+fj)}{[a+fj]^k}.$$

By using the relation

$$[a + fj] = \frac{1 - q^{a+fj}}{1 - q} = \frac{1 - q^a}{1 - q} + q^a \frac{1 - q^f}{1 - q} \frac{1 - q^{fi}}{1 - q^f}$$

$$= [a] + q^a [f][j \, ; q^f],$$

the first term of (3.2) equals

$$\frac{2 - k}{k - 1} (1 - q) \frac{1}{[f]^{k-1}} \sum_{a=1}^{f} q^{a(2-k)} \chi(a) \sum_{j=0}^{\infty} \frac{q^{fj}}{\left(q^{-a} \frac{[a]}{[f]} + [j \, ; q^f] \right)^{k-1}},$$

and the second term of (3.2) equals

$$\frac{1}{[f]^k} \sum_{a=1}^{f} q^{a(1-k)} \chi(a) \sum_{j=0}^{\infty} \frac{q^{fj}}{\left(q^{-a} \frac{[a]}{[f]} + [j \, ; q^f] \right)^k}.$$

By Lemma 3, we have the assertion.

PROPOSITION 3. *For $k \in \mathbf{Z}$ with $k \geq 2$,*

$$L_q(k, \chi) = \frac{(-1)^k}{(k-1)!} \frac{1}{[f]^k} \sum_{a=1}^{f} q^{a(2-k)} \chi(a) G^{(k)} \left(q^{-a} \frac{[a]}{[f]}, q^f \right).$$

Proof.   By (2.3), we can inductively see that

(3.3) $G^{(k)}(x, q) = (qx - x - 1) g^{(k+1)}(x, q^{-1}, q) + (1 - q)(2 - k) g^{(k)}(x, q^{-1}, q)$

for $k \in \mathbf{Z}$ with $k \geq 2$. So we obtain

$$G^{(k)}\left(q^{-a} \frac{[a]}{[f]}, q^f \right) = \left\{ (q^f - 1) q^{-a} \frac{1 - q^a}{1 - q^f} - 1 \right\} g^{(k+1)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f \right)$$

$$+ (1 - q^f)(2 - k) g^{(k)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f \right)$$

$$= -q^{-a} g^{(k+1)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f \right) + [f](1 - q)(2 - k) g^{(k)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f \right).$$

By Lemma 4,

$$L_q(k, \chi) = \frac{(-1)^k}{(k-1)!} \frac{1}{[f]^k} \sum_{a=1}^{f} q^{a(2-k)} \chi(a)$$

$$\times \left\{ [f](1 - q)(2 - k) g^{(k)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f \right) - q^{-a} g^{(k+1)}\left(q^{-a} \frac{[a]}{[f]}, q^{-f}, q^f \right) \right\}$$

So we have the assertion.

Remark. By Lemma 3 and (3.3), we can see that

$$(3.4) \qquad \frac{(-1)^k}{(k-1)!} G^{(k)}(x, q) = \frac{2-k}{k-1} (1-q) \sum_{n=0}^{\infty} \frac{q^n}{(x+[n])^{k-1}}$$

$$+ (1 + x - qx) \sum_{n=0}^{\infty} \frac{q^n}{(x+[n])^k}$$

for $k \in \mathbf{Z}$ with $k \geq 2$. This relation is a q-analogue of the classical one

$$\frac{(-1)^k}{(k-1)!} \frac{d^k}{dx^k} \log \Gamma(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^k}.$$

By considering the result in Proposition 2, Proposition 3 and (3.4), we can regard $G(x, q)$ as the q-$\log$-Γ-function which the Koblitz problem (2) in the complex case requires.

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