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A NOTE ON ISOMETRIC IMMERSIONS

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Abstract

Let N be a complete connected Riemannian manifold with sectional curvatures bounded from below. Let M be a complete simply connected Riemannian manifold with sectional curvatures $K_M(\sigma) \leqslant -a^2$ ($a \geqslant 0$) and with dimension $< 2\dim N$. Suppose that N is isometrically immersed in M and that its image lies in a closed ball of radius ρ. Then $\sup(K_N(\sigma) - K_M(\sigma)) \geqslant \mu^2(a\rho)/\rho^2$ where the function μ is defined by $\mu(x) = x\coth x$ for $x > 0$, $\mu(0) = 1$ and the supremum is taken over all sections tangent to N.

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This is a generalisation of previous results by Jacobowitz [4], Moore [6], Baikoussis and Koufogiorgos [1] and Ishihara [3]. To prove the main theorem we need the following

LEMMA 1. *Let M be a Riemannian manifold with sectional curvatures $\leqslant -a^2$. Suppose that $\gamma\colon [0,1] \to M$ is a geodesic and put $T = \gamma'(t)$. Let V be a Jacobi field along γ which is zero to $t = 0$ and is everywhere perpendicular to γ. Then at $t = 1$.*

$$\frac{\langle \nabla_T V, V\rangle}{\langle V, V\rangle} \geqslant \mu(a\lambda) \tag{1}$$

where λ is the length of γ.

PROOF. This can be extracted from the proof of the Rauch Comparison Theorem given in [2]. On page 32 of this reference the inequality

$$\frac{\langle \nabla_T V, V\rangle}{\langle V, V\rangle} \geqslant \frac{\langle \nabla_T V_0, V_0\rangle}{\langle V_0, V_0\rangle}$$

This work was done during the time when the first author was a visitor at the University of Southampton.

occurs where the subscript zero refers to a comparison manifold M_0. Our lemma follows by choosing an appropriate Jacobi field in the complete simply connected space of constant sectional curvature $-a^2$.

The following algebraic lemma is proved on pages 28–29 of [5].

LEMMA 2. *Let* $S: R^n \times R^n \to R^p$ *be a symmetric bilinear function such that, for all* $X \neq 0$, $S(X, X) \neq 0$. *Then if* $p < n$ *we can find non-zero vectors* X, Y *such that* $S(X, Y) = 0$, $S(X, X) = S(Y, Y)$.

THEOREM. *Let* N *be a complete connected Riemannian manifold with sectional curvature bounded from below. Let* M *be a complete simply connected Riemannian manifold with sectional curvatures* $K_M(\sigma) \leqslant -a^2$ $(a \geqslant 0)$ *and with* $\dim M < 2 \dim N$. *Suppose that* N *is isometrically immersed in* M *and that its image lies in a closed ball of radius* ρ. *Then*

$$(*) \qquad \sup(K_N(\sigma) - K_M(\sigma)) \geqslant \mu^2(a\rho)/\rho^2.$$

PROOF. In order to simplify the notation we shall assume that N is imbedded in M. The Riemannian connections ∇ and ∇' of M and N respectively are related by the Gauss formula

$$(2) \qquad \nabla_X Y = \nabla'_X Y + S(X, Y)$$

where S is the second fundamental form of the immersion. The corresponding sectional curvatures are related by the Gauss equation

$$(3) \qquad K_N(X \wedge Y) - K_M(X \wedge Y) = \Delta(X, Y)$$

where

$$(4) \qquad \Delta(X, Y) = \frac{\langle S(X, X), S(Y, Y)\rangle - \|S(X, Y)\|^2}{\|X \wedge Y\|^2}$$

Now let 0 be the centre of the ball in M and consider the function Φ defined on N by $\Phi(P) = \frac{1}{2}\{d(0, P)\}^2$ where d is the distance function on M. Our theorem follows from an application of Theorem A′ in [7] to the function Φ but we have first to do some calculations.

Consider a unit vector X tangent to N at P and choose a curve $\beta(u)$ in N with $\beta(0) = P$, $\beta'(0) = X$. Let $a(t, u)$, $0 \leqslant t \leqslant 1$, be a constant speed parametrisation of the (unique) geodesic in M from 0 to $\beta(u)$ and define vector fields T, $\tilde{X}$ along a by

$$T = a_* \frac{\partial}{\partial t}, \qquad \tilde{X} = a_* \frac{\partial}{\partial u};$$

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we have the formulas

(5) $$\langle \operatorname{grad} \Phi, X \rangle = \langle T, X \rangle$$

(6) $$\nabla'^2\Phi(X, X) = \langle \nabla_T \tilde{X}, X \rangle + \langle T, S(X, X) \rangle,$$

where $\nabla'^2\Phi$ is the Hessian of Φ. The first of these is a straightforward calculation involving the first variation. The second one can be derived as follows. Put $\bar{X} = \tilde{X}(1, u)$ so that $\bar{X}$ is a vector field along β which is tangent to N. Then

$$\nabla'^2\Phi(X, X) = \langle \nabla'_X \operatorname{grad} \Phi, X \rangle = X(\langle \operatorname{grad} \Phi, \bar{X} \rangle) - \langle \operatorname{grad} \Phi, \nabla'_X \bar{X} \rangle.$$

Now use (5), (2) and the fact that, because $[T, \tilde{X}] = 0$, $\nabla_{\tilde{X}} T = \nabla_T \tilde{X}$ to obtain

$$\begin{aligned} \nabla'^2\Phi(X, X) &= X(\langle T, \tilde{X} \rangle) - \langle T, \nabla'_X \bar{X} \rangle \\ &= \langle \nabla_X T, \bar{X} \rangle + \langle T, \nabla_X \bar{X} - \nabla'_X \bar{X} \rangle \\ &= \langle \nabla_T \tilde{X}, X \rangle + \langle T, S(X, X) \rangle. \end{aligned}$$

For our next calculations we restrict the vector fields to the geodesic $\gamma\colon t \to a(t, 0)$. Because $\tilde{X}$ is a Jacobi field it follows that

$$T^2(\langle T, \tilde{X} \rangle) = 0 \quad \text{and consequently} \quad \langle T, \tilde{X} \rangle = kt$$

where $k = \langle T, X \rangle$. The vector field

$$\hat{X} = \tilde{X} - \frac{\langle T, \tilde{X} \rangle T}{\lambda^2} = \tilde{X} - \frac{ktT}{\lambda^2},$$

where $\lambda = \|T\|$ is the length of γ, is thus a Jacobi field which is everywhere perpendicular to γ. A calculation gives the relations

(7) $$\langle \tilde{X}, \tilde{X} \rangle = \langle \hat{X}, \hat{X} \rangle + k^2t^2/\lambda^2$$

(8) $$\langle \nabla_T \tilde{X}, \tilde{X} \rangle = \langle \nabla_T \hat{X}, \hat{X} \rangle + k^2t^2/\lambda^2.$$

Choose a point P_0 on N different from 0 and put $\lambda_0 = d(0, P_0)$. According to Theorem A′ of [7], for any $\varepsilon' > 0$, $\varepsilon > 0$, there exists a point P on N at which

$$d(0, P) \geqslant \lambda_0, \qquad \|\operatorname{grad} \Phi\| < \varepsilon', \qquad \nabla'^2\Phi(X, X) < \varepsilon,$$

where X is any unit vector tangent to N at P. We will work out the implications of these inequalities using the notation we have already introduced but restricting our vector fields to their values at P.

It follows from (7) that

(9) $$\langle \hat{X}, \hat{X} \rangle = 1 - k^2/\lambda^2.$$

Further, (5) leads to the inequality $|k| \leqslant \|\operatorname{grad} \Phi\| < \varepsilon'$ and, as $\lambda \geqslant \lambda_0$,

(10) $$\|\hat{X}\|^2 > 1 - \varepsilon'^2/\lambda_0^2.$$

The argument

$$\begin{aligned}\langle \nabla_T \tilde{X}, X\rangle &= \langle \nabla_T \hat{X}, \hat{X}\rangle + k^2/\lambda^2\\ &= \langle \nabla_T \hat{X}, \hat{X}\rangle + 1 - \|\hat{X}\|^2\\ &\geqslant 1 + (\mu(a\lambda) - 1)\|\hat{X}\|^2\\ &\geqslant 1 + (\mu(a\lambda) - 1)(1 - \varepsilon'^2/\lambda_0^2)\end{aligned}$$

uses (8), (9), (1) and (10). It then follows from (6) that

$$\varepsilon > 1 + (\mu(a\lambda) - 1)(1 - \varepsilon'^2/\lambda_0^2) + \langle T, S(X, X)\rangle.$$

Given any positive integer m we can take $\varepsilon = 1/m$, $\varepsilon'^2 = \lambda_0^2/m$ and the above inequality implies that, at some point P_m,

$$\langle T, S(X, X)\rangle + (1 - 1/m)\mu(a\lambda_m) < 0$$

where $\lambda_m = d(0, P_m)$. Consequently

$$\|S(X, X)\| > (1 - 1/m)\mu(a\lambda_m)/\lambda_m,$$

an inequality which we can also express as

$$\|S(X, X)\|/\langle X, X\rangle > (1 - 1/m)\mu(a\lambda_m)/\lambda_m \tag{11}$$

for all non-zero vectors X tangent to N at P_m.

The inequality (11) shows that Lemma 2 is applicable to the function S. Thus, using (3) and (4), there are non-zero vectors X and Y tangent to N at P_m such that

$$K_N(X \wedge Y) - K_m(X \wedge Y) = \langle S(X, X), S(Y, Y)\rangle > /\|X \wedge Y\|^2.$$

Because $\|X \wedge Y\|^2 \leqslant \|X\|^2\|Y\|^2$ the inequality (11) gives

$$K_N(X \wedge Y) - K_M(X \wedge Y) > \left(1 - \frac{1}{m}\right)^2 \frac{\mu^2(a\lambda_m)}{\lambda_m^2} \geqslant \left(1 - \frac{1}{m}\right)^2 \frac{\mu^2(a\rho)}{\rho^2}.$$

The fact that this is true for all m proves the theorem.

We note that the inequality (*) is sharp in the sense that if M is a complete simply connected space of constant sectional curvature $-a^2$, $a \geqslant 0$ and N is the boundary of a closed ball of radius ρ, then we obtain the equality $\sup(K_N(\sigma) - K_M(\sigma)) = \mu^2(a\rho)/\rho^2$. In fact, we have $S(X, X) = (\mu(a\rho)/\rho^2)T$. So from (3) and (4) we obtain the above equality.

References

[1] C. Baikoussis and T. Koufogiorgos, 'Isometric immersions of complete Riemannian manifolds into euclidean space', *Proc. Amer. Math. Soc.* **79** (1980), 87–88.

[2] J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry* (North-Holland, Amsterdam and Oxford, 1975).

[3] T. Ishihara, 'Radii of immersed manifolds and nonexistence of immersions', *Proc. Amer. Math. Soc.* **78** (1980), 276–279.

[4] H. Jacobowitz, 'Isometric embedding of a compact Riemannian manifold into euclidean space', *Proc. Amer. Math. Soc.* **40** (1973), 245–246.
[5] S. Kobayashi and N. Nomizu, *Foundations of differential geometry*, Vol. II (Interscience, New York, 1969).
[6] J. D. Moore, 'An application of second variation to submanifold theory', *Duke Math. J.* **42** (1975), 191–193.
[7] H. Omori, 'Isometric immersions of Riemannian manifolds', *J. Math. Soc. Japan* **19** (1967), 205–214.

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