NOTE ON QUASIGROUPS AND TREES

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If letters a, b, c, ... are used to denote points of a nondegenerate plane cubic curve, other than the singular point if any, and if the product ab is defined as the third point of the curve collinear with a and b, we obtain an algebraic system having nonassociative multiplication ($ab \cdot c \neq a \cdot bc$ in general). It is in fact a totally symmetric entropic quasigroup (these terms are defined below). This idea, which was put forward at a meeting of the Edinburgh Mathematical Society a few years ago, will be exploited in a forthcoming paper. Such quasigroups have many properties which can be interpreted geometrically. Or, conversely, known properties of cubic curves suggest theorems about totally symmetric entropic quasigroups, which if established will involve a gain in generality, since not every such quasigroup can be " placed " on a cubic.

The purpose of the present note is to mention certain properties (" symmetries ") possessed by any totally symmetric entropic quasigroup, which are most conveniently expressed generally in terms of trees. The relevance to geometry on the cubic may be indicated briefly by an example. Suppose that a, b, c, d, e, f are the six points where some conic cuts the cubic. It can be shown that the condition for this may be written $ab \cdot cd = ef$, or equally $(ab \cdot cd)e = f$. Either of these must therefore be a relation which involves the six elements a, b, c, d, e, f symmetrically. It follows also that $(ab \cdot cd)e$ is a formula for the sixth point in which the conic through five given points of the cubic meets the cubic; it must therefore be a symmetric function of its five arguments. It is with such symmetries of equations and functions that this note is concerned, in a general totally symmetric entropic quasigroup. Postulates defining such a quasigroup will be introduced one at a time.

Since nonassociative multiplication is to be considered, to avoid writing many brackets, let it be understood that a dot separates factors whose multiplication is to be delayed, e.g. $(ab \cdot cd)e \cdot fg$ means $[\{(ab)(cd)\}e](fg)$. The word *product* is to be understood as meaning the product of a number of factors suitably bracketed (or perhaps a single element), and an *equation* means an equality between two products.

With any product we can associate a tree. The tree may be thought of as a pedigree in which the product is regarded as being descended from the partial products and ultimately from the separate factors (example: fig. 1). This is a tree in which one vertex is distinguished as the *root* of the tree, and proceeding from the root the tree bifurcates repeatedly.

If two products are equated, let us represent this by drawing the two trees and joining their roots (example: fig. 2). We get thus a single tree of a different kind; it has the property that every vertex lies on either one or three edges, and now one edge is distinguished—let us call it the *trunk* of the tree—representing the equality sign.

Now suppose that we have a groupoid G (i.e. a set closed under multiplication) which obeys the *demisymmetric* law (so called by A. Sade (5))

$$ab \cdot a = b \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

for all elements a, b of G. Then $ab = c$ implies $ca = b$, and conversely, if this is so, then (1) holds. Then likewise $ca = b$ implies $bc = a$, which in turn implies $ab = c$. Thus (1) is equivalent to the statement that each of the equations

$$ab = c, \quad ca = b, \quad bc = a \quad \dots\dots\dots\dots\dots\dots\dots(1a)$$

implies the other two. Hence (1) is equivalent to the identity $b \cdot ab = a$, as observed by Sade ((5), p. 91). We observe that G is necessarily a quasigroup
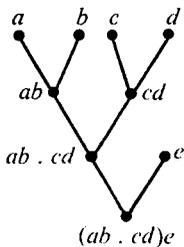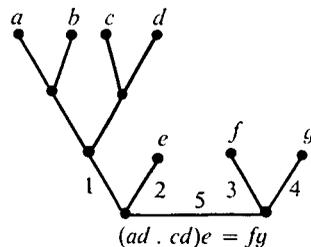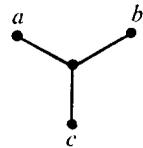
| FIG. 1. | FIG. 2. | FIG. 3. |

(i.e. a groupoid in which an equation $ab = c$ can always be solved uniquely for a or b given the other two).

Let us express this result in terms of trees. The tree for $ab = c$ is shown in fig. 3, c being represented by the trivial tree which consists of a single vertex only; and the demisymmetric law asserts that it does not matter which edge we regard as the trunk.

This can be generalised. The equivalence of equations (1a) shows that if the left side of an equation is a product of two factors we can transfer either factor to the other side of the equation, the left factor to the right of it or the right factor to the left of it. Also if the right side is a product (as in $c = ab$), the same statement applies. These four possibilities mean that in any tree representing a given equation, instead of the given trunk we can take any edge adjacent to it as trunk. Thus in fig. 2, instead of the given trunk 5, we could take 1, 2, 3 or 4 as trunk and read the tree as asserting respectively

$$ab \cdot cd = e \cdot fg, \quad e = fg \cdot (ab \cdot cd), \quad g \cdot (ab \cdot cd)e = f, \quad (ab \cdot cd)e \cdot f = g.$$

Since this can be applied repeatedly we have the following result:

If the demisymmetric law holds, then in any tree representing an equation ANY *edge can be taken as the trunk.*

(Of course it may be convenient to redraw the tree to facilitate rereading it, placing the trunk horizontal and the other branches above it.) From this point of view (1) and $b \cdot ab = a$ are rereadings of the tree for $ab = ab$.

Now let us impose on G two further postulates: firstly the commutative law

$$ab = ba; \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

then the quasigroup G becomes *totally symmetric* (2, 6), i.e. an equation $ab = c$ implies all six equations derived by permuting a, b, c; and finally the *entropic* law

$$ab \cdot cd = ac \cdot bd. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(3)$$

Quasigroups with this property were originally called *abelian* by Murdoch (3, 4) and Bruck (2). They are a generalisation of abelian groups, and are also isotopic to abelian groups (2). However, *abelian* is elsewhere understood as meaning " commutative ". Etymologically *entropic* refers to " inner turning ", meaning that the two inner factors in $ab \cdot cd$ can be exchanged. Other names for the property (3) have been proposed, e.g. *bisymmetric* (Aczél) and *medial* (Stein).

(2) and (3) imply that $ab \cdot cd$ is a symmetric function (equal to each expression obtainable by permuting the letters). So is $(ab \cdot cd)(ef \cdot gh)$, since we can interchange cd with ef and also apply the entropic law within each parenthesis. In fact we have a sequence of symmetric functions

$$ab, \quad ab \cdot cd, \quad (ab \cdot cd)(ef \cdot gh), \quad \dots$$

(as noted by Aczél (1)). If here any of the partial products are replaced by single factors we obtain a function which is symmetric in the unaffected factors. For example, $(ab \cdot c)(e \cdot gh)$ is symmetric in a, b, g, h (for in a quasigroup c can always be represented as $c_1 d$ and e as $e_1 f$); and this expression, since it is of the form $\alpha c \cdot e\beta$, is also symmetric in c and e.

Expressed in terms of trees the argument shows that *in a commutative entropic quasigroup any product is symmetric in all those of its factors which appear at the same altitude in the tree.* $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(4)$

In this statement the *altitude* of a vertex in a rooted tree is the number of edges which separate that vertex from the root; the altitude of the tree itself is the greatest of the altitudes of its vertices. Thus in the tree for $(ab \cdot c)(e \cdot gh)$, the factors a, b, g, h are at altitude 3, while c, e are at altitude 2; the altitude of the tree is 3.

If (1) also holds we can prove a result stronger than (4). I am indebted to I. R. Porteous for pointing out the algebraic result which I now express in terms of trees.

A factor a can always be replaced by $xa \cdot x$ and thus have its altitude raised by 2. Hence *in a totally symmetric entropic quasigroup any product is symmetric in all those of its factors which are at even altitude in the tree, and is also symmetric in all the rest (those at odd altitude).*

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Another way of expressing this result is the following. In a totally symmetric entropic quasigroup any two factors of a product can be interchanged if in the tree which represents the product they are separated by an even number of edges. In this form the same result applies to equations; for when the equation is represented by a tree we can always pick as trunk an edge which will put the two factors in question on the same side of the equation.

The final conclusion which we have reached is that *in a totally symmetric entropic quasigroup, in any product or equation, if we divide the factors involved into two sets, each set corresponding to vertices in the tree separated by even numbers of edges, then the product or equation is symmetric in each set.* For example, the equation $(ab \cdot cd) \cdot ef = gh \cdot i$ expresses a relation which is symmetric in a, b, c, d, g, h, and also in e, f, i; we can also regard gh as a single element and interchange it in the equation with e or f, and other such exchanges are possible, as will be seen on drawing the tree.

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