ON THE SOLUTION OF SOME AXISYMMETRIC BOUNDARY VALUE PROBLEMS BY MEANS OF INTEGRAL EQUATIONS

VIII. POTENTIAL PROBLEMS FOR A CIRCULAR ANNULUS †

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1. Introduction

This paper concludes a series of papers (**1**) on a group of axisymmetric boundary value problems in potential and diffraction theory by considering some potential problems for a circular annulus. The Dirichlet problem for an annulus has recently been considered by Gubenko and Mossakovskiĭ (**2**), who, by a somewhat complicated method, show it to be governed by either one of two Fredholm integral equations of the second kind. The purpose of the present paper is to show how the method developed in previous papers, by which certain integral representations of the potentials in problems for circular disks and spherical caps are used to reduce such problems to the solutions of either single Abel integral equations or Abel and Fredholm equations, can be applied to both the Dirichlet and Neumann problems for the annulus to give reasonably straightforward derivations of the governing Fredholm equations.

We first show in Section **2** that the Dirichlet problem for the annulus is essentially identical with the Neumann problem for a plane screen containing a circular aperture coplanar with a concentric circular disk, whose radius is less than that of the aperture. We then construct an integral representation of the potential for this latter problem by means of previous results and, using this, derive a Fredholm equation governing the problem. This equation is suitable for iteration when the inner radius of the annulus is small compared with the outer radius, that is, when the annulus is approximately a circular disk with a small concentric aperture. Next, in Section **3** we consider the indentation of an elastic half-space by a flat-ended annular punch, a problem which reduces to the Dirichlet problem for an annulus, the annulus being maintained at a constant potential, and obtain the iterative solution of the Fredholm equation of Section **2** for this case. Finally, in Section **4** we give

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corresponding results to those of Section **2** for the Neumann problem for the annulus.

2. The Dirichlet Problem for a Circular Annulus

We consider a thin, circular annulus of inner and outer radii a and b respectively and use cylindrical polar coordinates (ϖ, θ, z) with the centre and axis of the annulus as origin and z-axis respectively. The annulus is thus given by $z = 0 (a < \varpi < b)$. In the axisymmetric Dirichlet problem we require a potential $V(\varpi, z)$ which takes given values $f(\varpi)$ on the annulus, so that

$$V(\varpi, 0) = f(\varpi), \quad z = 0 (a<\varpi<b), \qquad (2.1)$$

$f(\varpi)$ being a continuously differentiable function. Further, $V(\varpi, z)$ is continuously differentiable at all points except those on the annulus, while V and $\partial V/\partial z$ are continuous for approach to the annulus except that for points on either rim ($z = 0$, $\varpi = a, b$) $\partial V/\partial z$ tends to infinity as the inverse square root of the distance from the rim. Finally, $V(\varpi, z)$ is $0(r^{-1})$ at a large distance r from the origin.

Since $V(\varpi, z)$ is symmetric about the plane $z = 0$, it follows that

$$\frac{\partial V}{\partial z} = 0, \quad z = 0 (0 \leqq \varpi < a,\ b < \varpi), \qquad (2.2)$$

and hence we can regard the problem as that of determining a function $V(\varpi, z)$ in the half-space $z>0$ given the conditions (2.1) and (2.2) on $z = 0$, $V(\varpi, z)$ being equal to $V(\varpi, -z)$ for $z<0$.

Since $f(\varpi)$ is continuously differentiable for $(a<\varpi<b)$, we follow Gubenko and Mossakovskiĭ (**2**) in expanding it as

$$f(\varpi) = \sum_{n=-\infty}^{\infty} a_n \varpi^n$$

and define

$$f_0(\varpi) = \sum_{n=0}^{\infty} a_n \varpi^n, \quad f_1(\varpi) = \sum_{n=-\infty}^{-1} a_n \varpi^n,$$

so that

$$f(\varpi) = f_0(\varpi) + f_1(\varpi).$$

This decomposition of $f(\varpi)$ is unique and the definitions of $f_0(\varpi)$ and $f_1(\varpi)$ may be extended to all ϖ in the intervals $(0 \leqq \varpi < b)$ and $(a < \varpi < \infty)$ respectively.

We now write $V(\varpi, z)$ as the sum of four potentials

$$V(\varpi, z) = V_0(\varpi, z) + V_1(\varpi, z) + V_2(\varpi, z) + U(\varpi, z), \qquad (2.3)$$

where we define $V_i(\varpi, z)$, $(i = 0, 1, 2)$, as the potentials in the half-space $z>0$, which on $z = 0$ satisfy the conditions

$$V_0 = f_0(\varpi) \quad (0 \leqq \varpi < b), \quad \frac{\partial V_0}{\partial z} = 0 \quad (b<\varpi), \qquad (2.4)$$

$$\frac{\partial V_1}{\partial z} = 0 \quad (0 \leqq \varpi < a), \quad V_1 = f_1(\varpi) \quad (a<\varpi), \qquad (2.5)$$

$$V_2 = 0 \qquad (0 \leqq \varpi < b), \quad \frac{\partial V_2}{\partial z} = -\frac{\partial V_1}{\partial z} \quad (b < \varpi). \qquad \text{(2.6)}$$

The potential $U(\varpi, z)$ defined for $z > 0$ thus satisfies the conditions

$$\frac{\partial U}{\partial z} = -\frac{\partial}{\partial z}(V_0 + V_2), \quad z = 0 (0 \leqq \varpi < a), \qquad \text{(2.7)}$$

$$U = 0, \quad z = 0 (a < \varpi < b), \quad \frac{\partial U}{\partial z} = 0, \quad z = 0 (b < \varpi). \qquad \text{(2.8)}$$

For $z > 0$ the functions V_0 and V_1 are identical with the Dirichlet potentials for a disk of radius b and a plane screen containing a circular aperture of radius a respectively. Further, for $z > 0$ the function $(V_2 + U)$ is identical with the Neumann potential for a screen containing a circular aperture of radius b concentric with a coplanar disk of radius a, $\partial/\partial z(V_2 + U)$ taking given non-zero values on the screen and the disk. We write this function $(V_2 + U)$ as the sum of two functions, V_2 and U, where V_2 is identical for $z > 0$ with the Neumann potential for a screen containing an aperture of radius b and is introduced in order that, for $z > 0$, U be identical with the Neumann potential for the screen and the concentric disk with $\partial U/\partial z$ equal to zero on the screen. This function U is now the potential to be determined.

We may note that, since each potential must be symmetric about $z = 0$,

$$V_i(\varpi, z) = V_i(\varpi, -z), \quad (i = 0, 1, 2), \quad U(\varpi, z) = U(\varpi, -z), \quad z < 0,$$

and thus V_0 and V_1 are identical with the appropriate Dirichlet potentials for all z, while V_2 and U are identical with the negatives of the appropriate Neumann potentials for $z < 0$.

Integral representations of the functions V_i, which enable these functions to be found explicitly by single applications of the known solution of Abel's integral equation, have been given by Green and Zerna (3) and myself (4). The potential $V_0(\varpi, z)$ for the Dirichlet problem (2.4) for a circular disk of radius b is found as (3)

$$V_0(\varpi, z) = \tfrac{1}{2} \int_{-b}^{b} \frac{g_0(t)\,dt}{(\varpi^2 + (z - it)^2)^{\frac{1}{2}}}, \qquad \text{(2.9)}$$

where

$$(\varpi^2 + (z - it)^2)^{\frac{1}{2}} = \rho e^{-i\tau}, \qquad \text{(2.10)}$$

with

$$0 \leqq \tau \leqq \pi/2 \quad \textit{for} \quad 0 \leqq t < a,$$

$$-\pi/2 \leqq \tau < 0 \quad \textit{for} \quad -a \leqq t < 0.$$

The function $g_0(t)$ is real, even, and satisfies the Abel integral equation

$$\int_0^{\varpi} \frac{g_0(t)\,dt}{(\varpi^2 - t^2)^{\frac{1}{2}}} = f_0(\varpi) \quad (0 \leqq \varpi < b),$$

the solution of which is

$$g_0(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\varpi f_0(\varpi)d\varpi}{(t^2-\varpi^2)^{\frac{1}{2}}} \quad (0 \leqq t < b).$$

In a previous paper (5) I have shown that the representation (2.9) can be obtained from the representation of $V_0(\varpi, z)$ as the potential due to a distribution of charge on the disk. For our present purposes it is, however, more convenient to regard the function $v(\varpi, z;\ t)$, defined for $z > 0$ as

$$v(\varpi, z;\ t) = (\varpi^2+(z-it)^2)^{-\frac{1}{2}} \quad (t>0),$$

as the potential of a " complex disk charge " associated with the disk of radius t and centre the origin in the plane $z = 0$. The sum $2v_0(\varpi, z;\ t)$ of this potential and the potential

$$v(\varpi, z;\ -t) = (\varpi^2+(z+it)^2)^{-\frac{1}{2}} \quad (z>0)$$

of the " image disk charge " in the plane $z = 0$ is such that $\partial v_0/\partial z$ is zero for $(z = 0,\ \varpi > t)$. Hence, if we sum such potentials $v_0(\varpi, z;\ t)$ with a density function $g_0(t)$ for $0<t<b$, we obtain

$$V_0(\varpi, z) = \int_0^b g_0(t)v_0(\varpi, z;\ t)dt$$

as a potential satisfying the second of (2.4). This is, however, exactly the representation (2.9) provided $g_0(t)$ is real and even.

The functions $V_1(\varpi, z)$ and $V_2(\varpi, z)$ satisfying (2.5) and (2.6) are found as (see (4))

$$V_1(\varpi, z) = \frac{1}{2i}\int_a^\infty g_1(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} - \frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right]dt \quad (z>0), \ldots(2.11)$$

where $g_1(t)$ is a real odd function given by

$$g_1(t) = -\frac{2}{\pi}\frac{d}{dt}\int_t^\infty \frac{\varpi f_1(\varpi)d\varpi}{(\varpi^2-t^2)^{\frac{1}{2}}} \quad (a<t<\infty),$$

and

$$V_2(\varpi, z) = \tfrac{1}{2}\int_b^\infty g_2(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right]dt \quad (z>0), \ldots(2.12)$$

where $g_2(t)$ is a real even function given by

$$g_2(t) = -\frac{2}{\pi}\int_t^\infty \frac{1}{(\varpi^2-t^2)^{\frac{1}{2}}}\frac{d}{d\varpi}\int_a^\varpi \frac{sg_1(s)ds}{(\varpi^2-s^2)^{\frac{1}{2}}}\,d\varpi \quad (b<t<\infty)\ldots\ldots(2.13)$$

Thus, since each of the functions $g_i(t)$ can be determined, the functions $V_i(\varpi, z)$, $(i = 0, 1, 2)$, are known explicitly.

We now construct a representation of the potential $U(\varpi, z)$. We begin by determining the potential $u(\varpi, z;\ t)$ for a " complex disk charge " of radius $t(0<t<a)$ in the plane $z = 0$ in the presence of a screen $z = 0(b<\varpi)$, $\partial u/\partial z$ being zero on the screen. By combining this with the image potential $u(\varpi, z;\ -t)$ we obtain a function satisfying conditions (2.8), and hence by suitably summing such functions for $0<t<a$ we obtain a representation of $U(\varpi, z)$.

We write the potential $u(\varpi, z; t)$ for $z>0$ as

$$u(\varpi, z;\ t) = v(\varpi, z;\ t)+u_1(\varpi, z;\ t) \quad (0<t<a),$$

where as before $v(\varpi, z; t)$ is equal to $(\varpi^2+(z-it)^2)^{-\frac{1}{2}}$, and require $u_1(\varpi, z; t)$ to satisfy the conditions

$$u_1 = 0, \quad z = 0(0 \leqq \varpi < b), \quad \frac{\partial u_1}{\partial z} = -\frac{\partial v}{\partial z}, \quad z = 0(b<\varpi),$$

u_1 being the potential of a Neumann problem for a screen with an aperture of radius b. Hence (see (4)) we can represent $u_1(\varpi, z; t)$ as

$$u_1(\varpi, z;\ t) = \tfrac{1}{2}\int_b^\infty h(s)\left[\frac{1}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}}+\frac{1}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds \quad (z>0),$$

and find that the second condition on $z = 0$ is satisfied provided (4)

$$\int_\varpi^\infty \frac{sh(s)ds}{(s^2-\varpi^2)^{\frac{1}{2}}} = \frac{it}{(\varpi^2-t^2)^{\frac{1}{2}}} \quad (b<\varpi).$$

The solution of this Abel equation is

$$h(s) = \frac{2it}{\pi(s^2-t^2)},$$

so that

$$u(\varpi, z;\ t) = \frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} + \frac{it}{\pi}\int_b^\infty \frac{1}{(s^2-t^2)}\left[\frac{1}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds \quad (z>0). \qquad \text{......(2.14)}$$

Similarly, we find the corresponding potential $u(\varpi, z; -t)$ for the "image disk charge" is given by

$$u(\varpi, z;\ -t) = \frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}} - \frac{it}{\pi}\int_b^\infty \frac{1}{(s^2-t^2)}\left[\frac{1}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds \quad (z>0), \qquad \text{......(2.15)}$$

so that

$$u(\varpi, z;\ -t) = \overline{u(\varpi, z;\ t)},$$

the bar denoting complex conjugate. The integrals in (2.14) and (2.15) can be evaluated, but, as will be seen later, it is unnecessary to do this.

The potential

$$\tfrac{1}{2}i[u(\varpi, z;\ t)-u(\varpi, z;\ -t)]$$

is thus a real function which vanishes on $z = 0(a<\varpi<b)$ since $t<a$ and which has its normal derivative zero on the screen $z = 0(b<\varpi)$. Thus it satisfies conditions (2.8) and hence, if we sum such potentials for $0<t<a$ with a density

function $g(t)$, we obtain, as a representation for $U(\varpi, z)$,

$$U(\varpi, z) = \frac{1}{2i}\int_0^a g(t)[u(\varpi, z;\ t) - u(\varpi, z;\ -t)]dt$$

$$= \frac{1}{2i}\int_0^a g(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} - \frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right.$$

$$\left. - \frac{2it}{\pi}\int_b^\infty \frac{1}{(t^2-s^2)}\left[\frac{1}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds\right]dt$$

$$= \frac{1}{2i}\int_{-a}^a g(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}}\right.$$

$$\left. - \frac{i}{\pi}\int_b^\infty \frac{1}{(t+s)}\left[\frac{1}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds\right]dt \quad (z>0), \ldots(2.16)$$

where $g(t)$ is to be chosen so that (2.7) is satisfied. We suppose $g(t)$ is real and odd. This potential $U(\varpi, z)$ then satisfies all the required conditions.

We now show that (2.7) is satisfied provided $g(t)$ is the solution of a certain Fredholm integral equation of the second kind. At any point in the half-space $z>0$ we have

$$\frac{\partial U}{\partial z} = \frac{1}{2i\varpi}\frac{\partial}{\partial \varpi}\int_0^a g(t)\left[\frac{(z-it)}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} - \frac{(z+it)}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right.$$

$$\left. - \frac{2it}{\pi}\int_b^\infty \frac{1}{(t^2-s^2)}\left[\frac{(z-is)}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}} + \frac{(z+is)}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds\right]dt,$$

and hence, as such a point approaches a point on $z = 0 (0 \leqq \varpi < a)$, the limit of $\partial U/\partial z$ is found using (2.10) as

$$\frac{\partial U}{\partial z} = -\frac{1}{\varpi}\frac{d}{d\varpi}\left[\int_0^\varpi \frac{tg(t)dt}{(\varpi^2-t^2)^{\frac{1}{2}}} - \frac{2}{\pi}\int_0^a tg(t)\int_b^\infty \frac{s}{(s^2-t^2)(s^2-\varpi^2)^{\frac{1}{2}}}ds\,dt\right],$$

$$(z=0, \quad 0 \leqq \varpi < a).$$

Similarly, from (2.9) and (2.12) we find that

$$\frac{\partial V_0}{\partial z} = \frac{1}{\varpi}\frac{d}{d\varpi}\int_\varpi^b \frac{tg_0(t)dt}{(t^2-\varpi^2)^{\frac{1}{2}}}, \quad \frac{\partial V_2}{\partial z} = \frac{1}{\varpi}\frac{d}{d\varpi}\int_b^\infty \frac{tg_2(t)dt}{(t^2-\varpi^2)^{\frac{1}{2}}}, \quad (z=0, \quad 0 \leqq \varpi < a).$$

Thus, if we integrate equation (2.7) with respect to ϖ from 0 to ϖ $(0 \leqq \varpi < a)$ and use the expressions just found for the three derivatives, we obtain

$$\int_0^\varpi \frac{tg(t)dt}{(\varpi^2-t^2)^{\frac{1}{2}}} - \frac{2}{\pi}\int_0^a tg(t)\int_b^\infty \frac{1}{(s^2-t^2)}\left[\frac{s}{(s^2-\varpi^2)^{\frac{1}{2}}} - 1\right]ds\,dt$$

$$= \int_\varpi^b \frac{tg_0(t)dt}{(t^2-\varpi^2)^{\frac{1}{2}}} - \int_0^b g_0(t)dt + \int_b^\infty g_2(t)\left[\frac{t}{(t^2-\varpi^2)^{\frac{1}{2}}} - 1\right]dt$$

$$(0 \leqq \varpi < a). \quad \ldots\ldots(2.17)$$

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If we regard (2.17) as an Abel equation for $g(t)$ in which the first term on the left-hand side is equal to a known function, we obtain on solving this equation the Fredholm equation satisfied by $g(t)$ as

$$g(t)+\frac{1}{\pi^2}\int_0^a \frac{g(s)}{(t^2-s^2)}\left[2s\log\left(\frac{b-t}{b+t}\right)-2t\log\left(\frac{b-s}{b+s}\right)\right]ds$$

$$=h_0(t)-\frac{2t}{\pi}\int_b^\infty \frac{g_2(s)ds}{(t^2-s^2)} \quad (0\leqq t<a), \ldots\ldots(2.18)$$

where

$$h_0(t)=\frac{2}{\pi}\int_0^t \frac{1}{(t^2-\varpi^2)^{\frac{1}{2}}}\frac{d}{d\varpi}\int_\varpi^b \frac{sg_0(s)ds}{(s^2-\varpi^2)^{\frac{1}{2}}}d\varpi \quad (0\leqq t<a)$$

and $g_2(t)$ is given by (2.13). Since $g_0(t)$ is known, $h_0(t)$ can be found, and so the right-hand side of (2.18) is known. We may note that the kernel of (2.18) is continuous at $s = t$.

Equation (2.18) is a Fredholm integral equation of the second kind for $g(t)$, which can be solved numerically. Alternatively, we may note that the kernel is vanishingly small when b is very much greater than a, and thus for such a and b the equation can be solved by iteration. The solution so obtained perturbs on a corresponding solution for a disk of radius a and is that for an annulus whose inner radius a is small compared with its outer radius b. Once $g(t)$ is known, the potential U can be found at any point from (2.16).

In order to evaluate U and other quantities of interest it is convenient to use an alternative representation of U obtained from (2.16) by an interchange in the order of integration as

$$U(\varpi, z)=\frac{1}{2i}\int_0^a g(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}}-\frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right]dt$$

$$+\tfrac{1}{2}\int_b^\infty j(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}}+\frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right]dt, \ldots\ldots(2.19)$$

where

$$j(t)=\frac{2}{\pi}\int_0^a \frac{sg(s)ds}{t^2-s^2}=\frac{1}{\pi}\int_{-a}^a \frac{g(s)ds}{t-s} \quad (b<t), \ \ldots\ldots\ldots\ldots\ldots(2.20)$$

$j(t)$ being an even function. Once $g(t)$ has been found by iteration, $j(t)$ can be calculated to the same order of approximation, and the integrals in (2.19) evaluated. This is a more practical method of calculating U than first evaluating the inner integral in (2.16) and then attempting the evaluation of U. It may be noted that the two integrals in (2.19) are the appropriate representations (4) for the Neumann potentials for a disk of radius a and a screen with an aperture of radius b.

An alternative method of deriving (2.18) is to start with (2.19) and use the conditions (2.7) and (2.8) on $z = 0(0\leqq\varpi<a)$ and $z = 0(b<\varpi)$ to obtain two integral equations for $g(t)$ and $j(t)$, one of which is (2.20). Elimination of $j(t)$ then leads to (2.18). Alternatively, $g(t)$ can be eliminated to obtain a

Fredholm equation for $j(t)$ holding over the infinite interval $(b<t<\infty)$. The functions $g(t)$ and $j(t)$ are essentially the functions $u'_{1y}(x, 0)$ and $u'_{2x}(x, 0)$ introduced by Gubenko and Mossakovskiĭ (2), differing from them only by known functions. Equation (2.18) for $g(t)$ corresponds to the equation for $u'_{1y}(x, 0)$ obtained by eliminating $u'_{2x}(x, 0)$ between the equations (4.5) given by Gubenko and Mossakovskiĭ, while the equation for $j(t)$ mentioned above corresponds to their equation (6.1) used to obtain approximate solutions. Since this latter equation is over the infinite interval (b, ∞), it seems preferable to use (2.18), which is over the finite interval $(0, a)$.

Equation (2.18) can also be derived by formulating the problem of determining $V(\varpi, z)$ in the half-space $z>0$ as a triple integral equations problem (6) and treating these equations by a method similar to that given recently by me for triple series equations (7). Another integral equation method for this problem has also been given by J. C. Cooke (8).

The method given in this section is essentially one for an annulus which can be regarded as a circular disk with a small concentric aperture. While equation (2.18) holds for all values of a and b, approximate solutions can only readily be obtained when $a \ll b$. The case of the narrow annulus for which a and b are approximately equal, however, has been treated by Grinberg and Kuritsyn (9) using another integral equation method which gives solutions perturbing on corresponding two-dimensional solutions for an infinitely long strip.

3. The Indentation of an Elastic Half-space by an Annular Punch

As an example of a Dirichlet problem for an annulus we consider the indentation of an elastic half-space $z>0$ by a rigid flat-ended annular punch pressed normally against the surface $z = 0$ of the half-space, the centre of the annulus being the origin for cylindrical polar coordinates (ϖ, θ, z). If the shearing stress $\sigma_{\varpi z}$ is zero at all points of $z = 0$, the normal stress σ_{zz} vanishes at those points of $z = 0$ not in contact with the punch, and the z-component w of the displacement is constant over the region of contact, then on $z = 0$ we have

$$w = \varepsilon \quad (a<\varpi<b), \quad \sigma_{zz} = 0 \quad (0 \leqq \varpi < a,\ b<\varpi<\infty), \quad \sigma_{\varpi z} = 0 \quad (0 \leqq \varpi < \infty), \qquad \ldots\ldots(3.1)$$

where ε is the depth of penetration of the punch. Also, all components of stress and displacement tend to zero at large distances from the origin.

Green and Zerna (3) show that, if the displacement $\boldsymbol{D}$ is taken as

$$2\mu \boldsymbol{D} = (1-2\eta)\ \text{grad}\ \phi + z\ \text{grad}\frac{\partial \phi}{\partial z} - (3-4\eta)\frac{\partial \phi}{\partial z}\,\boldsymbol{k},$$

where $\phi(\varpi, z)$ is a potential function, μ the shear modulus, η Poisson's ratio, and $\boldsymbol{k}$ the unit vector in the z-direction, the last of (3.1) is satisfied, while the first two conditions give on $z = 0$

$$V = \varepsilon \quad (a<\varpi<b),$$

$$\frac{\partial V}{\partial z} = 0 \quad (0 \leqq \varpi<a,\ b<\varpi<\infty), \qquad \text{(3.2)}$$

where

$$V = -\frac{(1-\eta)}{\mu}\frac{\partial \phi}{\partial z}.$$

Thus $V(\varpi, z)$ is a potential in the half-space $z>0$ which satisfies the boundary conditions (2.1) and (2.2) of the Dirichlet problem for an annulus, being identical with the electrostatic potential for the annulus maintained at a constant potential ε. It is thus given by (2.3).

Since $f(\varpi) = \varepsilon$, we have

$$f_0(\varpi) = \varepsilon, \quad f_1(\varpi) = 0;$$

so the functions V_1 and V_2 satisfying conditions (2.5) and (2.6) are identically zero at all points, while V_0 is given by (2.9) with

$$g_0(t) = \frac{2\varepsilon}{\pi}.$$

The function U is then given by (2.16), $g(t)$ satisfying the integral equation (2.18) with

$$h_0(t) = \frac{2\varepsilon}{\pi^2}\log\left(\frac{b-t}{b+t}\right), \quad h_2(t) = 0.$$

When $a \ll b$, we obtain the iterative solution of (2.18) as

$$g(t) = -\frac{4\varepsilon t}{\pi^2 b}\left[1+\frac{t^2}{3b^2}+\frac{4a^3}{9\pi^2 b^3}+\frac{t^4}{5b^4}+\frac{4a^3}{15\pi^2 b^5}\left(t^2+\frac{14a^2}{15}\right)+0\left(\frac{a^6}{b^6}\right)\right].$$

From (2.20) we then find that

$$j(t) = -\frac{8\varepsilon a^3}{3\pi^3 b t^2}\left[1+\frac{a^2}{5}\left(\frac{3}{t^2}+\frac{1}{b^2}\right)+\frac{4a^3}{9\pi^2 b^3}+0\left(\frac{a^4}{b^4}\right)\right],$$

and can thus calculate $U(\varpi, z)$ at any point from (2.19).

The total force P exerted by the punch on the half-space is found as

$$\begin{aligned} P &= -2\pi\int_a^b \varpi(\sigma_{zz})_{z=0}\,d\varpi \\ &= \frac{2\pi\mu}{(1-\eta)}\left[\int_0^b g_0(t)dt - \frac{1}{\pi}\int_0^a g(t)\log\left(\frac{b-t}{b+t}\right)dt\right] \\ &= \frac{4\varepsilon\mu b}{(1-\eta)}\left[1-\frac{4\sigma^3}{3\pi^2}-\frac{8\sigma^5}{15\pi^2}-\frac{16\sigma^6}{27\pi^4}+0(\sigma^7)\right], \end{aligned}$$

where $\sigma = a/b$. This expression agrees with that obtained by a limiting procedure in a previous paper (7).

4. The Neumann Problem for a Circular Annulus

We give without proof corresponding results to those of Section **2** for the Neumann problem for an annulus. Again using cylindrical polar coordinates (ϖ, θ, z) we require a potential $V(\varpi, z)$, whose normal derivative $\partial V/\partial z$ takes given values $f(\varpi)$ on the annulus $z = 0 (a<\varpi<b)$ and which satisfies continuity conditions similar to those for the corresponding potential of Section **2**. Further, $V(\varpi, z)$ is $0(r^{-3})$ at a large distance r from the origin. Since $V(\varpi, z)$ is anti-symmetric about $z = 0$, the problem is that of determining $V(\varpi, z)$ in the half-space $z>0$ subject to the conditions on $z = 0$

$$\left.\begin{aligned} \frac{\partial V}{\partial z} &= f(\varpi) \quad (a<\varpi<b), \\ V &= 0 \qquad (0 \leqq \varpi<a,\ b<\varpi), \end{aligned}\right\} \qquad \text{(4.1)}$$

$V(\varpi, z)$ being equal to $-V(\varpi, -z)$ for $z<0$.

We assume $f(\varpi)$ can be expanded as

$$f(\varpi) = f_0(\varpi)+f_1(\varpi) \quad (a<\varpi<b),$$

where

$$f_0(\varpi) = \sum_{n=0}^{\infty} a_n \varpi^n, \quad f_1(\varpi) = \sum_{n=-\infty}^{-3} a_n \varpi^n,$$

the definitions of these functions being extended to all ϖ in the intervals $(0 \leqq \varpi<b)$ and $(a<\varpi<\infty)$ respectively.

We now write $V(\varpi, z)$ as the sum of four potentials

$$V(\varpi, z) = V_0(\varpi, z)+V_1(\varpi, z)+V_2(\varpi, z)+U(\varpi, z),$$

where we define $V_i(\varpi, z)$, $(i = 0, 1, 2)$, as the potentials in the half-space $z>0$, which on $z = 0$ satisfy the conditions

$$\frac{\partial V_0}{\partial z} = f_0(\varpi) \quad (0 \leqq \varpi<b), \quad V_0 = 0 (b<\varpi),$$

$$V_1 = 0 (0 \leqq \varpi<a), \quad \frac{\partial V_1}{\partial z} = f_1(\varpi) \quad (a<\varpi),$$

$$\frac{\partial V_2}{\partial z} = 0 (0 \leqq \varpi<b), \quad V_2 = -V_1 \quad (b<\varpi).$$

These potentials are found as **(4)**

$$V_0(\varpi, z) = \frac{1}{2i}\int_{-b}^{b} \frac{g_0(t)dt}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}},$$

where $g_0(t)$ is a real odd function given by

$$g_0(t) = \frac{2}{\pi}\int_0^t \frac{\varpi f_0(\varpi)d\varpi}{(t^2-\varpi^2)^{\frac{1}{2}}} \quad (0 \leqq t<b),$$

$$V_1(\varpi, z) = \tfrac{1}{2}\int_a^{\infty} g_1(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right] dt,$$

where $g_1(t)$ is a real even function given by

$$g_1(t) = \frac{-2}{\pi}\int_t^\infty \frac{\varpi f_1(\varpi)d\varpi}{(\varpi^2-t^2)^{\frac{1}{2}}} \quad (a<t<\infty),$$

and

$$V_2(\varpi, z) = \frac{1}{2i}\int_b^\infty g_2(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} - \frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}}\right]dt,$$

where $g_2(t)$ is a real odd function given by

$$g_2(t) = \frac{2}{\pi}\frac{d}{dt}\int_t^\infty \frac{\varpi}{(\varpi^2-t^2)^{\frac{1}{2}}}\int_a^\varpi \frac{g_1(s)ds}{(\varpi^2-s^2)^{\frac{1}{2}}}d\varpi \quad (b<t<\infty). \quad \text{.........(4.2)}$$

The potential $U(\varpi, z)(z>0)$ satisfies the conditions on $z = 0$

$$U = -(V_0+V_2) \quad (0 \leqq \varpi < a),$$

$$\frac{\partial U}{\partial z} = 0 \quad (a<\varpi<b), \quad U = 0 \quad (b<\varpi),$$

and is identical with the Dirichlet potential for a screen containing a circular aperture of radius b concentric with a coplanar circular disk of radius a, the potential being zero on the screen and taking given values on the disk. We construct a suitable representation of $U(\varpi, z)$ by a method similar to that employed in Section **2** and obtain

$$U(\varpi, z) = \tfrac{1}{2}\int_0^a g(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2+(z+it)^2)^{\frac{1}{2}}} + \frac{2i}{\pi}\int_b^\infty \frac{s}{s^2-t^2}\left[\frac{1}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}} - \frac{1}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds\right]dt$$

$$= \tfrac{1}{2}\int_{-a}^a g(t)\left[\frac{1}{(\varpi^2+(z-it)^2)^{\frac{1}{2}}} + \frac{i}{\pi}\int_b^\infty \frac{1}{(t+s)}\left[\frac{1}{(\varpi^2+(z-is)^2)^{\frac{1}{2}}} - \frac{1}{(\varpi^2+(z+is)^2)^{\frac{1}{2}}}\right]ds\right]dt$$

$$(z>0). \quad \text{......(4.3)}$$

The conditions on $z = 0$ are satisfied provided $g(t)$ is the solution of the Fredholm integral equation

$$g(t)+\frac{1}{\pi^2}\int_0^a \frac{g(s)}{(t^2-s^2)}\left[2t\log\left(\frac{b-t}{b+t}\right) - 2s\log\left(\frac{b-s}{b+s}\right)\right]ds$$

$$= h_0(t)+\frac{2}{\pi}\int_b^\infty \frac{sg_2(s)ds}{(t^2-s^2)} \quad (0 \leqq t < a),$$

where

$$h_0(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\varpi}{(t^2-\varpi^2)^{\frac{1}{2}}}\int_\varpi^b \frac{g_0(s)ds}{(s^2-\varpi^2)^{\frac{1}{2}}}d\varpi$$

and $g_2(t)$ is given by (4.2). When $a \ll b$, this equation can be solved by iteration to give a solution for an annulus whose inner radius a is small compared with its outer radius b, this solution perturbing on a corresponding solution for a disk of radius a.

An alternative representation for $U(\varpi, z)$ is found from (4.3) as

$$U(\varpi, z) = \tfrac{1}{2} \int_0^a g(t) \left[\frac{1}{(\varpi^2 + (z - it)^2)^{\frac{1}{2}}} + \frac{1}{(\varpi^2 + (z + it)^2)^{\frac{1}{2}}} \right] dt + \frac{1}{2i} \int_b^\infty j(t) \left[\frac{1}{(\varpi^2 + (z - it)^2)^{\frac{1}{2}}} - \frac{1}{(\varpi^2 + (z + it)^2)^{\frac{1}{2}}} \right] dt,$$

where

$$j(t) = \frac{2t}{\pi} \int_0^a \frac{g(s)ds}{s^2 - t^2} \quad (b < t < \infty),$$

$j(t)$ being an even function. The two integrals in this expression are the appropriate representations (4) of the Dirichlet potentials for a disk of radius a and a screen with an aperture of radius b.

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