ON POLYHEDRAL REALIZABILITY OF CERTAIN SEQUENCES

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A finite sequence $(p_k) = (p_3, p_4, \ldots)$ of non-negative integers shall be called <u>realizable</u> provided there exists a 3-valent 3-polytope P which has p_i i-gonal faces for every i. P is called a <u>realization</u> of (p_k).

For realizability of a sequence (p_k), from Euler's formula follows

$$\sum_{k \geq 3} (6-k)p_k = 12 \qquad (*)$$

as a necessary condition. However, there are no general sufficient conditions. Furthermore (*) places no restriction on the number p_6. Considering only sequences (p_k) with $p_k = 0$ for all $k \geq 7$, there are 19 triads (p_3, p_4, p_5) satisfying (*) and we have a natural problem: For what values of p_6 is in case of a fixed triad (p_3, p_4, p_5) the sequence (p_3, p_4, p_5, p_6) realizable? In Grünbaum-Motzkin [2], the problem is solved for the sequences $(4, 0, 0, p_6)$, $(0, 6, 0, p_6)$, $(0, 0, 12, p_6)$, in Grünbaum [3] also for the sequence $(3, 1, 1, p_6)$.

The aim of this little note is to show how it is possible by slight modifications of the graphs used by Grünbaum-Motzkin [2], to answer the question of realizability of some other sequences (p_3, p_4, p_5, p_6).

THEOREM. <u>The sequences</u> $(0, 2, 8, p_6)$, $(0, 3, 6, p_6)$, $(0, 4, 4, p_6)$, $(2, 2, 2, p_6)$, $(1, 3, 3, p_6)$ <u>are realizable for all values of</u> p_6. <u>The sequence</u> $(1, 2, 5, p_6)$ <u>is realizable if and only if</u> $p_6 \neq 0$. <u>The sequences</u> $(2, 0, 6, p_6)$, $(0, 5, 2, p_6)$ <u>are realizable if and only if</u> $p_6 \neq 1$. <u>The sequence</u> $(1, 1, 7, p_6)$ <u>is realizable if and only if</u> $p_6 > 1$.

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The realizability of the sequences will be demonstrated by construction. We shall use also the catalogue of trivalent polytopes in Brückner [1]. The non-realizability of certain sequences follows from the non-existence of these in Brückner [1], whose catalogue of trivalent polytopes is supposed to be complete.

For briefness' sake we denote the (graph of the) realization of the sequences (1, 1, 7, n), (0, 2, 8, n), (0, 3, 6, n), (2, 0, 6, n) (0, 4, 4, n), (1, 2, 5, n), (0, 5, 2, n), (2, 2, 2, n), (1, 3, 3, n) by A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n, K_n respectively (n is the value of p_6).

1. (1, 1, 7, n).

Let us draw each of the graphs a and b in Fig. 1 on a hemisphere with the heavy line as equator. Connect them in such a way as to make the 2-valent vertices on the heavy line of one identical with the 3-valent vertices of the other. We get A_5. Analogously, combining a and c, we obtain A_6; a and d, A_7; g and c, A_8; g and d, A_9. For remaining $n = j + 5i$, $5 \leqq j \leqq 9$, $i = 1, 2, \ldots,$ we proceed similarly as described above, only the relevant two graphs should be separated by i "belts" ρ (Fig. 3) each consisting of five hexagons. A_2 is on Fig. 4; A_3, A_4 arise from A_2 by successive splitting of the indicated faces by edges.

The procedure being similar in other cases, we shall briefly introduce only the corresponding graphs represented in the figures.

2. (0, 2, 8, n)

b and b yields B_4
b and c B_5
c and c B_6
c and d yields B_7
d and d B_8.

For $n \geqq 9$ we use "belts" ρ. B_0 is no. X 85 in [1]. B_1 is Fig. 5; B_2, B_3 arise from B_1 by successive splitting of faces as indicated.

3. (0, 3, 6, n)

b and f yields C_3
c and f C_4
d and f C_5
b and e yields C_6
c and e C_7.

For $n > 7$ we use "belts" ρ. C_0, C_1 are nos. IX 33, X 84 in [1]. C_2 is on Fig. 6.

4. (2, 0, 6, n)

h and l yields D_3
m and m D_{10}
k and m D_{11}
k and k yields D_{12}
k and n D_{13}
n and n D_{14}.

For $n = 9$ and all $n > 14$ we use "belts" γ. D_0 and D_2 are nos. VIII 9 and X 63 in [1]. D_4, D_5 arise from D_3 by successive splitting of h as indicated. D_6 results from a and a. D_7 is on Fig. 7; D_8 results from D_7 by splitting of a face.

5. (0, 4, 4, n)

o and r yields E_2
p and r E_3
o and o yields E_4
o and p E_5

r and r yields E_0; E_1 is no. IX 32 in [1]; for $n > 5$ we use "belts" σ.

6. (1, 2, 5, n)

r and s yields F_2
o and s F_4
p and s yields F_5
p and t F_7.

F_1 is no. IX 28 in [1]; F_3 is on Fig. 8; for $n = 6$ and $n > 7$ we use "belts" σ.

7. (0, 5, 2, n)

a' and d' yields G_6; a' and e', G_7; a' and b', G_8; a' and c', G_9 (Fig. 2). $G_{10}, \ldots, G_{13}$ are obtained by using in the preceding constructions, instead of the graph a', another graph which is constructed by adding to a' four disjoint hexagons in such a way that two vertices of each hexagon remain 2-valent. All other G_i, $i > 13$, are obtained by successive adding to the graphs constructed above of quadruples of hexagons in a manner analogous to that just mentioned. G_0, G_2, G_3 are in Brückner [1, no. VII 5, IX 31, X 82]. G_4 is on Fig. 9; G_5 arises from G_4 by the indicated splitting of a face.

8. (2, 2, 2, n)

q and r yields H_1; f' and g', H_2; o and q, H_3. H_o is no. VI 1

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in [1]. For even $n > 2$, we insert successive pairs of disjoint hexagons as in 7. For odd $n > 3$, we use "belts" σ.

9. (1, 3, 3, n)

h' and k' yields K_1; u and r, K_2; o and u, K_4. K_o is no. VII 4 in [1]. For odd $n > 1$, we insert pairs of disjoint hexagons as in 7; for even $n > 4$ we use "belts" σ.

In conclusion we remark that all the graphs we have constructed are planar and 3-connected and therefore realizable as vertices and edges of 3-polytopes (cf. Grünbaum [3, p. 235]).

Conjecture. The remaining sequences $(3, 0, 3, p_6)$, $(2, 3, 0, p_6)$, $(2, 1, 4, p_6)$, $(1, 4, 1, p_6)$, $(1, 0, 9, p_6)$, $(0, 1, 10, p_6)$ are realizable, for all except possibly a finite number of values of p_6.

For each of these sequences we know an infinite number of odd and an infinite number of even values of p_6 rendering the sequences realizable (cf. Conjecture 2 in Grünbaum [4] : Given a sequence $(p_3, p_4, \ldots, p_n)$ of non-negative integers satisfying (*) there exists a constant c such that either for each even, or else for each odd, p_6 with $p_6 \geqq c$ there exists a trivalent 3-polytope P having p_i i-gonal faces for all $i \geqq 3$).

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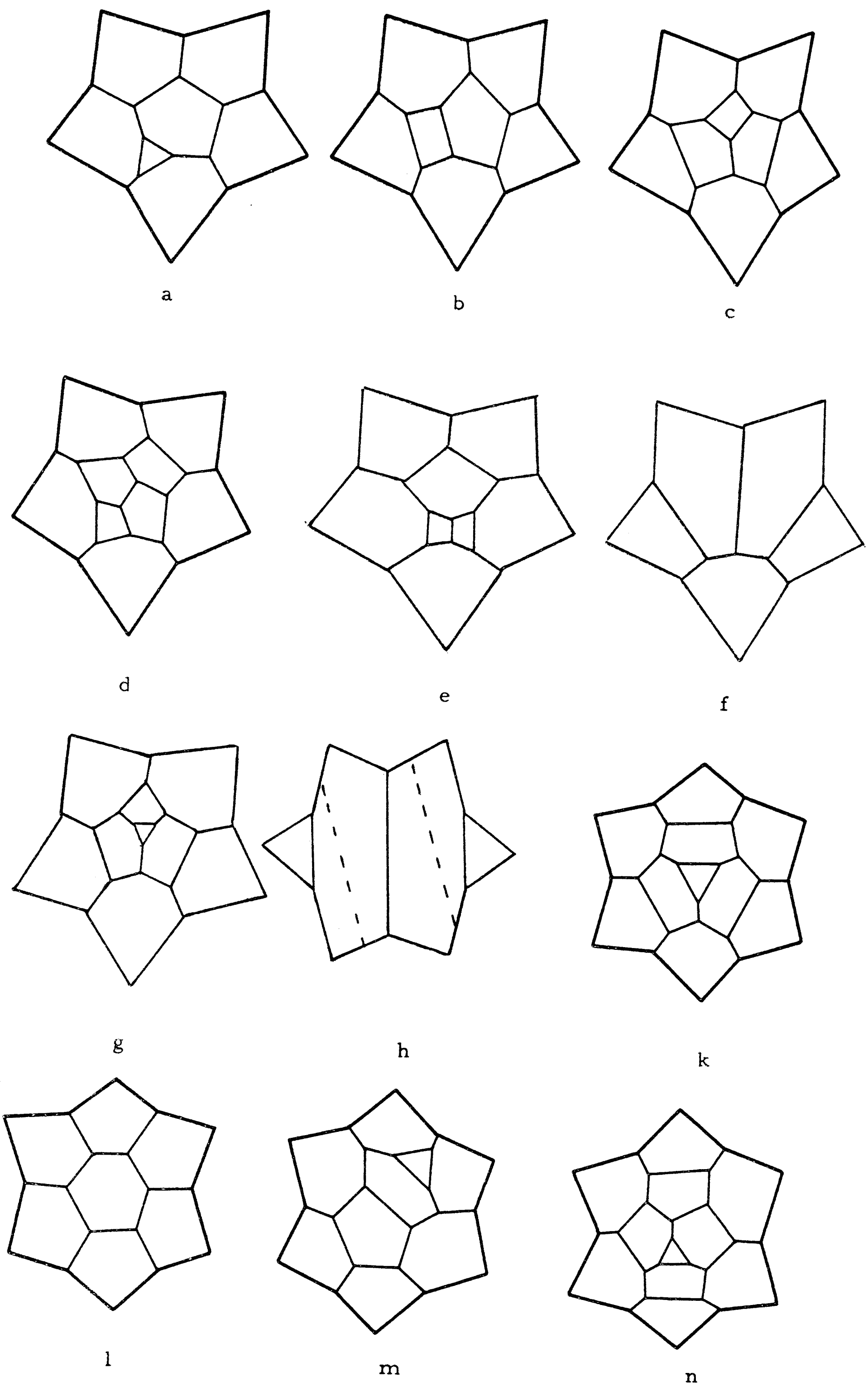

Figure 1 (page 1)

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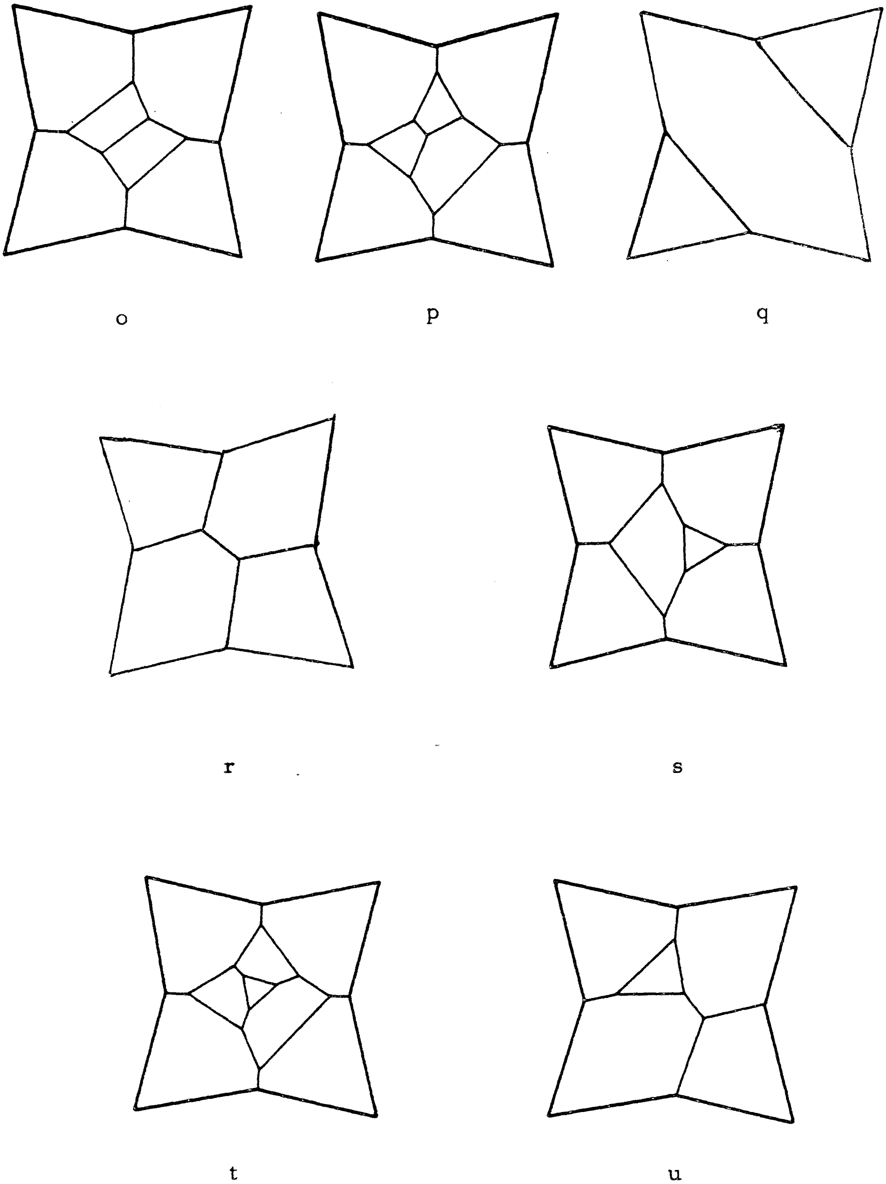

Figure 1 (page 2)

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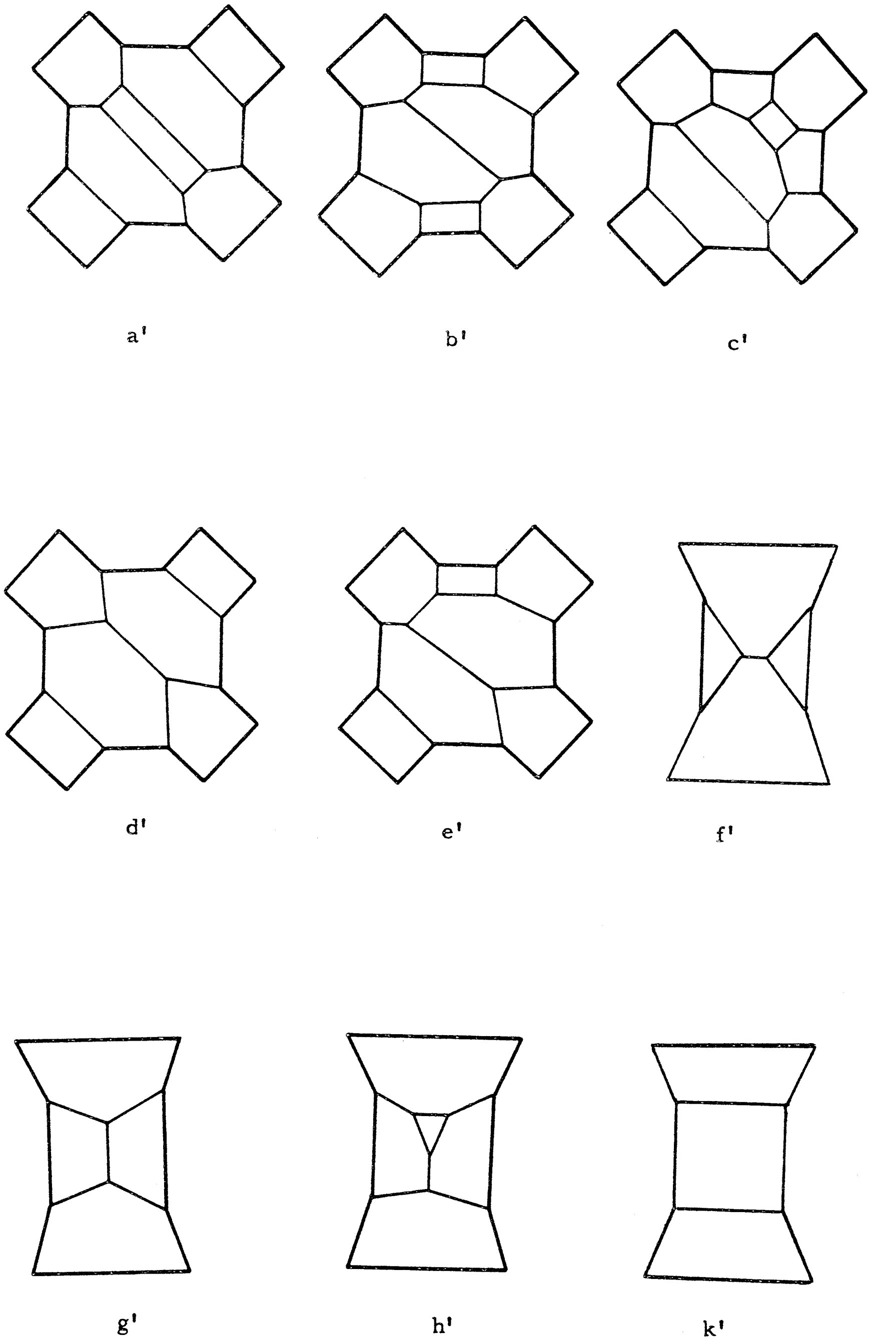

Figure 2

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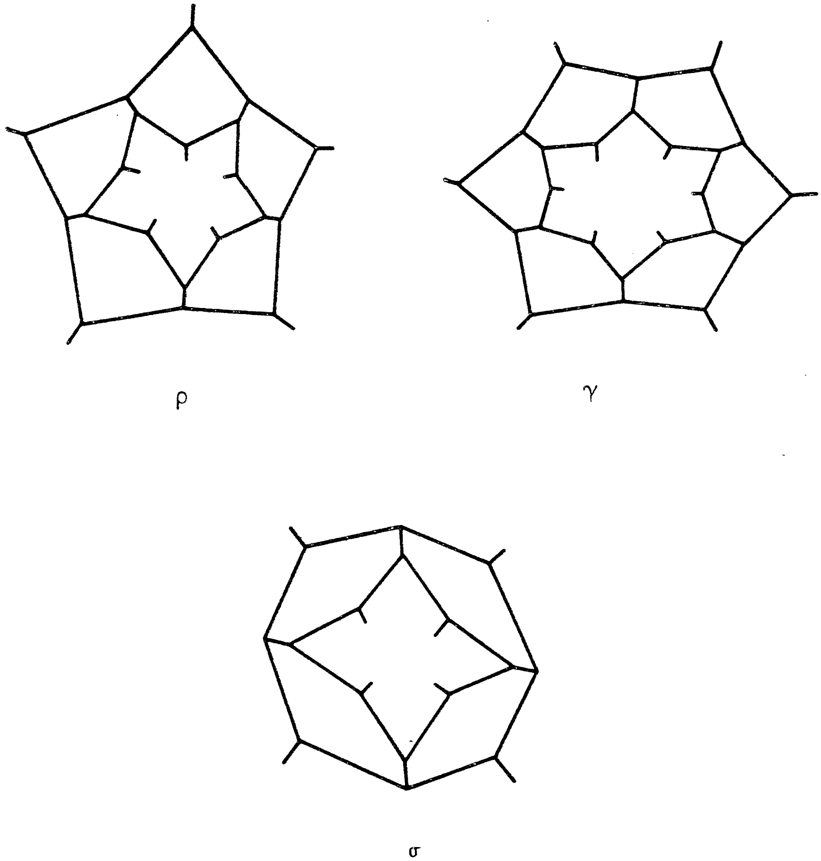

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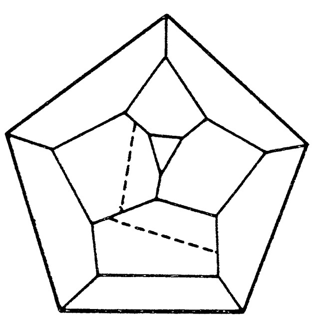

Figure 4

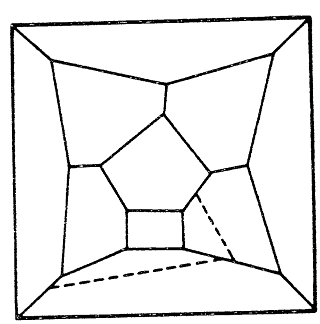

Figure 5

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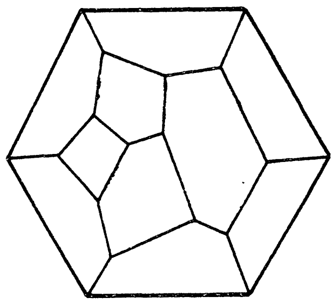

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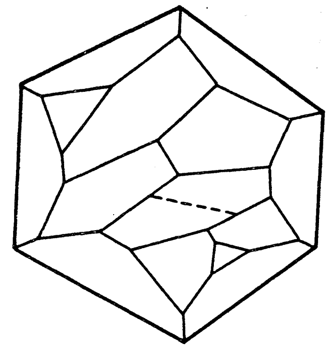

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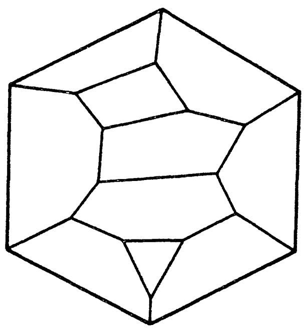

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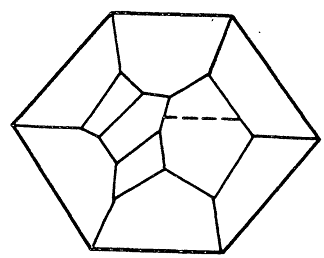

Figure 9

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