LINEAR OPERATORS WHICH COMMUTE WITH TRANSLATIONS

PART II: APPLICATIONS OF THE REPRESENTATION THEOREMS

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This paper comprises a number of applications of the results of Part I. We use essentially the same notation as in Part I with a few additions necessary for the problems at hand.

The first section deals for the most part with a problem which one of the authors has treated elsewhere [3] and [4] in different settings.[2] In the present case, it takes the form: Suppose T is an averaging operator on $C(X)$, where X is a compact group. Under what conditions is X representable as a direct product of groups S_1 and S_2 such that

$$Tf(x_1, x_2) = \int_{S_1} f(s_1, x_2) ds_1,$$

where ds_1 is the Haar measure on S_1. In the process of solving this problem, we also characterize those averaging operators T which (for X locally compact) map $C_c(X)$ into $C(X)$ and commute with translations.

In Section 2, we discuss the normalizers of various topological algebras. The problem of finding the normalizer of various partially ordered algebras has been condidered in [5] and [6]. In commutative algebras the normalizer coincides with the algebra of multipliers. The problem of finding the multiplier algebra for various Banach algebras and convolution algebras has been discussed in [2], [7], [14], [15]. We show here that the normalizer of $L^1(X)$ coincides with $M_{bd}(X)$ and the normalizers of $C_c(X)$ and $L_c^1(X)$ coincide with $M_c(X)$. In addition various new general results about normalizers are found.

In Section 3 the results of 2.8 and Section 5 in Part I are applied to problems regarding the division of measures and of distributions. Our results are analogous to those of Wells in [13].

Finally, in Section 4 we comment on the problem of isomorphisms of convolution algebras in relation to representation theorems for multipliers.

[1] This paper was prepared while the first-named author was a Visiting Fellow at the Institute of Advanced Studies of the Australian National University, during 1963.

[2] The numbers in square brackets refer to bibliographical items listed at the end of Part II.

1. Averaging operators which commute with translations

Let X be a locally compact topological group, and let $C_c(X)$ and $C(X)$ be given their natural topologies.

For the purposes of this paper, an *averaging operator* T is a non-zero continuous linear mapping from $C_c(X)$ to $C(X)$ such that

$$T(fTg) = TfTg. \tag{1.1}$$

From Theorem 1.2 of Part I, it is clear that if T commutes with the right translations ρ_a [resp. with the left translations τ_a], then there is a $\mu \in M(X)$ such that

$$Tf = \mu * f \text{ [resp. } f * \mu]$$

for $f \in C_c(X)$.

Using this result, we prove the following theorem which is a generalization of a theorem of Birkhoff [1] and is related to the results of Kelley [12].

1.1 Theorem. *Let T be a continuous linear mapping of C_c into C. If T is an averaging operator which commutes with the ρ_a [resp. τ_a], then there exist a closed subgroup S of X and a constant c such that*

$$Tf(x) = c\int_S f(s^{-1}x)d\sigma(s), \quad \text{[resp. } Tf(x) = c\int_S f(xs^{-1})d\sigma(s)] \tag{1.2}$$

where σ is the normalized left [resp. right] Haar measure on S. The converse is also valid.

Proof. Since $Tf = \mu * f$ [resp. $f * \mu$] for all $f \in C_c$, we can write equation (1.1) as follows:

$$\int_X f(y^{-1}x)d\mu(y)\int_X g(z^{-1}y^{-1}x)d\mu(z) = \int_X f(y^{-1}x)d\mu(y)\int_X g(z^{-1}x)d\mu(z).$$

$$\text{[resp. } \int_X f(xy^{-1})\Delta(y)d\mu(y)\int_X f(xy^{-1}z^{-1})\Delta(z)d\mu(z) = \int_X f(xy^{-1})\Delta(y)d\mu(y)\int_X g(xz^{-1})\Delta(z)d\mu(z)].$$

Let $x = e$, and replace f by $\check{f}$ to obtain

$$\int_X \check{f}(y)d\mu(y)\int_X g(z^{-1}y^{-1})d\mu(z) = \int_X \check{f}(y)d\mu(y)\int_X g(z^{-1})d\mu(z)$$

$$\text{[resp. } \int_X \check{f}(y)\Delta(y)d\mu(y)\int_X g(y^{-1}z^{-1})\Delta(z)d\mu(z) = \int_X \check{f}(y)\Delta(y)d\mu(y)\int_X g(z^{-1})\Delta(z)d\mu(z)]$$

for all $f \in C_c(X)$. Thus it follows that

$$\int_X g(z^{-1}y^{-1})d\mu(z) = \int_X g(z^{-1})d\mu(z)$$
$$[\text{resp. } \int_X g(y^{-1}z^{-1})\Delta(z)d\mu(z) = \int_X g(z^{-1})\Delta(z)d\mu(z)]$$

for all $g \in C_c(X)$ and all y in S the support of μ. Therefore,

$$\int_X g(yz)d\mu(z) = \int_X g(z)d\mu(z)$$
$$[\text{resp. } \int_X g(zy)\Delta(x)d\mu(z) = \int_X g(z)\Delta(z)d\mu(z)]$$

for all $y \in S$ and $g \in C_c(X)$. In the case where T commutes with the τ_a and $Tf = f * \mu$, let ν be the measure on X defined by the equation $\nu(A) = \int_A \Delta(z)d\mu(z)$. Thus

$$\mu(yA) = \mu(a) \ [\text{resp. } \nu(Ay) = \nu(A)] \tag{1.3}$$

for all $y \in S$ and all Borel sets A in X. Since $A \to y^{-1}A$ is a one-to-one mapping of the set of Borel sets of X onto itself, it follows that equations (1.3) are valid if $y \in S \cup S^{-1}$. A short argument involving the definition of $|\mu|$ shows that equations (1.3) are valid when μ is replaced by $|\mu|$. If A is a neighbourhood of $s \in S$, then yA is a neighbourhood of ys and since $|\mu|(A) > 0$, $|\mu|(yA) > 0$, so $ys \in S$ for $y \in S \cup S^{-1}$. Therefore

$$(S \cup S^{-1}) \cdot S \subseteq S,$$

and hence S is a subgroup X. It is closed because it is a support. Because μ is invariant under ρ_a for each $a \in S$, it is clear that $\mu = c\sigma$, where c is complex number and σ is the normalized left Haar measure on S. Thus the first of equations (1.2) is valid.

A similar argument shows that ν is a right invariant measure on S which is a closed subgroup of X. Thus, in this case,

$$\begin{aligned} Tf(x) &= \int_S f(xs^{-1})\Delta(s)d\mu(z) \\ &= c\int_S f(xs^{-1})d\sigma(z), \end{aligned}$$

where σ is the normalized right Haar measure on S and $\nu = c\sigma$.

Conversely, if S is a closed subgroup of X, the equation (1.2) defines an operator which commutes with ρ_a [resp. τ_a]. Equation (1.1) is easily verifiable if one uses the fact that ds is the left [resp. right] Haar measure on S and is invariant.

For the remainder of this section we use ds to stand for the left [resp. right] normalized Haar measure on S.

1.2 Corollary. $T^2 = cT$.

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PROOF. Indeed,

$$T^2f(x) = c^2 \int_S ds_1 \int_S f(s^{-1}s_1^{-1}x)ds$$

$$[\text{resp. } T^2f(x) = c^2 \int_S ds_1 \int_S f(s^{-1}s_1^{-1}x)ds]$$

where ds_1 [resp. ds] stands for the normalized left [resp. right] Haar measure on S.

Let $s_1 s = \xi$ [resp. $s\, s_1 = \xi$]; then

$$T^2f(x) = c^2 \int_S ds_1 \int_S f(\xi^{-1}x)d(s_1^{-1}\xi) = c^2 \int_S ds_1 \int_S f(\xi^{-1}x)d\xi$$

$$[\text{resp. } T^2f(x) = c^2 \int_S ds_1 \int_S f(x\xi^{-1})d(\xi s_1^{-1}) = c^2 \int_S ds_1 \int_S f(x\xi^{-1})d\xi].$$

Hence it follows in both cases that

$$T^2f(x) = c \int_S Tf(x)ds_1 = cTf(x).$$

Now we determine some necessary and sufficient conditions for S to be a normal subgroup of X. The following definition is needed to carry out this programme:

1.3 Let T be an averaging operator which commutes with the ρ_a [resp. τ_a], then let

$$x \equiv y \text{ if and only if } f(x) = f(y) \text{ for all } f \in TC_c.$$

LEMMA. *If T is an averaging operator which commutes with the ρ_a [resp. τ_a], then $x \equiv y$ if and only if $y \in Sx$ [resp. $y \in xS$].*

PROOF. If $x \equiv y$, then, since TC_c is closed with respect to right [resp. left] translations, for any $a \in X$ we have $xa \equiv ya$. [resp. $ax \equiv ay$].

Thus $x \equiv y$ if and only if $e \equiv yx^{-1}$ [resp. $x^{-1}y \equiv e$]. If $x \equiv e$, then $Tf(x) = Tf(e)$ for all $f \in C_c$ and so

$$\int_S f(s^{-1}x)ds = \int_S f(s^{-1})ds.$$

$$[\text{resp. } \int_S f(xs^{-1})ds = \int_S f(s^{-1})ds.]$$

Then we have

$$\int_S f(x^{-1}s)ds = \int_S f(s)ds \tag{1.4}$$

$$[\text{resp. } \int_S f(sx^{-1})ds = \int_S f(s)ds]$$

for all $f \in C_c$. Thus if σ stands for the normalized left [resp. right] Haar measure on S, then from equations (1.4) it follows that

$$\sigma(x^{-1}A) = \sigma(A) \ \ [\text{resp. } \sigma(Ax^{-1}) = \sigma(A)],$$

where A is any neighbourhood in S of $e \in X$. Therefore $x^{-1} \in S$, and so $x \in S$. Hence if $x \equiv y$, then $y \in Sx$ [resp. $y \in xS$].

Conversely, suppose $y \in Sx$ [resp. $y \in xS$]. Then $yx^{-1} \in S$ [resp. $xy^{-1} \in S$]. Now for $s_0 \in S$ and $f \in TC_c$, there is a $g \in C_c$ such that $Tg = f$, and

$$f(s_0 x) = c\int_S g(s^{-1}s_0 x)dx.$$
$$[\text{resp. } f(xs_0) = \int_S f(xs_0 s^{-1})ds_1].$$

Let $s_0^{-1}s = \xi$ [resp. $s\ s_0^{-1} = \xi$]; then

$$(1.5)\qquad \begin{aligned} f(s_0 x) &= c\int_S g(\xi^{-1}x)d\xi = f(x). \\ [\text{resp. } f(xs_0) &= c\int_S y(x\xi^{-1})d\xi = f(x)]. \end{aligned}$$

From equation (1.5) it follows that for $x \in X$ and $s \in S$, $sx = x$ [resp. $xs \equiv x$]. Hence $y = yx^{-1}x \equiv x$ [(resp. $xx^{-1}y = y \equiv x$].

1.4 THEOREM. *If T is an averaging operator which commutes with the ρ_a [resp. τ_a], and if S is the closed subgroup of X corresponding to T, then the following statements are equivalent*:

(1) *S is a normal subgroup of X.*
(2) *$(Tf)^{\vee} = T\check{f}$.*
(3) *$f \in TC_c \Rightarrow \check{f} \in TC_c$.*
(4) *For every $f \in TC_c$, every $x \in X$, and every $s \in S$, $f(x^{-1}sx) = f(s)$.*

PROOF. Since the proofs for the two cases, commutation with the ρ_a and with the τ_a, are entirely analogous we present only the proof for the case when T commutes with the ρ_a.

(1) $\Rightarrow$ (2): If S is normal, then $xS = Sx$ for all $x \in X$, and so $xs \equiv x$ and $sx \equiv x$ for all $x \in X$ and $s \in S$. In general, we have from Corollary 1.2

$$(1.6)\qquad T[(T\check{f})^{\vee}] = c(T\check{f})^{\vee}.$$

Since S is normal,

$$(1.7)\qquad \begin{aligned} T[(T\check{f})^{\vee}](x) &= c\int_S d\xi\, c\int_S f(\xi^{-1}xs)ds \\ &= c\int_S ds\, c\int_S f(\xi^{-1}xs)d\xi \\ &= c\int_S Tf(xs)ds = cTf(x). \end{aligned}$$

The equations (1.6) and (1.7) can be combined to yield

$$c(T\check{f})^{\vee} = cTf(x).$$

Thus $T\check{f} = (Tf^{\vee})$.

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(2) $\Rightarrow$ (3): If $f = Tg$, then $\check{f} = (Tg)^{\vee} = T\check{g} \in TC_c$ and so $f \in TC_c \Rightarrow \check{f} \in TC_c$.

(3) $\Rightarrow$ (4): By (3), if $f \in TC_c$, then $\check{f} \in TC_c$. Now if $\rho'_a = 1/\Delta(a)\,\rho_a$, it is easily shown that if $f \in TC_c$ then $\rho'_a f \in TC_c$. Thus $h = (\rho'_{x_0}\rho'_{s_0}\check{f})^{\vee} \in TC_c$ for each $f \in TC_c$, each $x_0 \in X$, and each $s_0 \in S$, and so

$$h(x) = (\rho'_{x_0}\rho'_{s_0}\check{f})^{\vee}(x) = f(x_0 s_0 x) \tag{1.8}$$

for each $x \in X$.

By Lemma 1.3, if $f \in TC_c$ then

$$\int_S f(sx)ds = f(x),$$

and so

$$\begin{aligned} f(x_0 s_0 x) &= \int_S h(sx)ds = \int_S f(x_0 s_0 sx)ds = \int_S f(x_0 sx)ds \\ &= f(x_0 x) \end{aligned}$$

for each $f \in TC_c$, $x_1 x_0 \in X$, and $s_0 \in S$. Choose $x = x_0^{-1}$; then

$$f(x_0 s_0 x_0^{-1}) = f(e)$$

for all $x_0 \in X$ and $s_0 \in S$. Equivalently,

$$f(x^{-1}sx) = f(e) = f(s)$$

for all $x \in X$, $s \in S$.

(4) $\Rightarrow$ (1): By (4) and Lemma 1.3,

$$x^{-1}sx \in S$$

for every $s \in S$. Thus S is normal.

Now we derive the product theorem mentioned in the Introduction. Assume for the remainder of this section that X is a compact group and that T_1 and T_2 are averaging operators on $C(X)$ which commute with ρ_a for each $a \in X$. We will work entirely with the ρ_a, but it is clear from the symmetry of Theorem 1.1 that analogous results for operators which commute with the τ_a are also provable.

1.5 By theorem 1.1, there is a closed subgroup S_i and a constant c_i $(i = 1, 2)$ such that

$$T_i f(x) = c_i \int_{S_i} f(s_i^{-1}x)ds_i,$$

where ds_i is the normalized Haar measure on S_i $(i = 1, 2)$. T_1 and T_2 are said to be *complementary* if

$$T_1 T_2 f = L(f) \cdot 1,$$

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where 1 stands for the function defined on X with constant value 1, and L stands for a non-trivial continuous linear functional on $C(X)$.

1.6 THEOREM. *Let σ signify the normalized Haar measure on X, and σ_i the normalized Haar measure on S_i $(i = 1, 2)$. If T_1 and T_2 are complementary, then*

(1) $\sigma_1 * \sigma_2 = \sigma_2 * \sigma_1 = \sigma$,
(2) $T_1 T_2 = T_2 T_1$.

PROOF. Since

$$T_1 T_2 f(x) = c_1 c_2 \sigma_1 * \big(\sigma_2 * f(x)\big) = L(f) \cdot 1$$

for all $x \in X$, and since T_i commutes with right translations,

$$L(f) = L(\rho_a f)$$

for all $a \in X$ and $f \in C(X)$. Thus the measure L is a Haar measure, and $L = k\sigma$ for some complex number k. Since $T_1 T_2(1) = c_1 c_2 = L(1) = k\sigma(1) = k$, it follows that $\sigma_1 * \sigma_2 = \sigma$, and hence

$$\sigma = \check{\sigma} = \check{\sigma}_2 * \check{\sigma}_1 = \sigma_2 * \sigma_1$$

by the compactness and hence unimodularity of X and S_i. Therefore (1) is valid; (2) follows immediately from (1).

1.7 THEOREM. *The set $T_1 C \cdot T_2 C = \{f_1 f_2 | f_i \in TC_i\}$ separates points in X if and only if $S_1 \cap S_2 = \{e\}$.*

PROOF. If $T_1 C \cdot T_2 C$ separates points, then for $x_0 \neq e$ with $x_0 \in S_1 \cap S_2$ there is $f_i \in TC_i$ $(i = 1, 2)$ such that

$$f_1(x_0) f_2(x_0) = 0 \quad \text{and} \quad f_1(e) f_2(e) = 1.$$

Either $f_1(x_0) = 0$ or $f_2(x_0) = 0$. In the former case, it follows from Lemma 1.3 that x_0 cannot be a member of S_1. A similar argument involving f_2 and S_2 holds when $f_1(x_0) \neq 0$. Therefore, $S_1 \cap S_2 = \{e\}$.

On the other hand, if $S_1 \cap S_2 = \{e\}$ and $x_1 \neq x_2$ in X, then for $i = 1, 2$,

$$x_i(S_1 \cap S_2) = x_i S_1 \cap x_i S_2 = x_i.$$

If for example $x_1 S_1 = x_2 S_1$, then $x_1 S_2 \neq x_2 S_2$, and by Lemma 1.3 there is an $f_2 \in T_2 C$ such that $f_2(x_1) \neq f_2(x_2)$. A similar argument holds if $x_1 S_1 \neq x_2 S_1$.

Now the main theorem of the section can be stated.

1.8 THEOREM. *If T_1 and T_2 are complementary averaging operators on $C(X)$ which commute with right translations, if $T_1 C \cdot T_2 C$ separates points in X, and if, for $i = 1, 2$, $T_i C$ is closed under the operation $f \to \check{f}$, then there exist normal subgroups S_1 and S_2 of X such that*

(1) *$T_i f = c_i \sigma_i * f$, where σ_i is the normalized Haar measure on S_i and c_i is a complex number,*

(2) *$S_1 \times S_2 = X$,*

(3) *Every $x \in X$ can be uniquely written in the form $x = x_1 x_2$, where $x_i \in S_i$, and*

$$T_1 f(x) = c_1 \int_{S_1} f(s_1 x_2) ds_1, \tag{1.9}$$

$$T_2 f(x) = c_2 \int_{S_2} f(x_1 s_2) ds_2. \tag{1.10}$$

Proof. Part (1) is the substance of Theorem 1.1. From Theorems 1.4 and 1.7, it follows that S_1 and S_2 are normal and that $S_1 \cap S_2 = \{e\}$. Therefore in order to show that $X = S_1 \times S_2$, we need only show that $X = S_1 S_2$. Suppose $x_0 \in X \backslash S_1 S_2$. Both $S_1 S_2$ and $x_0 S_1 S_2$ are closed subsets of X. Since S_1 and S_2 are normal,

$$S_1 S_2 = S_2 S_1, \; x_0 S_1 S_2 = S_1 x_0 S_2 = S_1 S_2 x_0 = \cdots, \tag{1.11}$$

and $x_0 S_1 S_2 \cap S_1 S_2 = \emptyset$. The topological normality of X ensures the existence of a continuous function f defined on X into $[0, 1]$ such that

$$f(x_0 S_1 S_2) = 0 \quad \text{and} \quad f(S_1 S_2) = 1.$$

Since

$$T_2 f(x) = c_2 \int_{S_2} f(s_2^{-1} x) ds_2,$$

it follows that for $s_1' \in S_1$ and $s_2' \in S_2$, we have

$$T_2 f(s_1' s_2') = c_2 \int_{S_2} f(s_2^{-1} s_1' s_2') ds_2 = c_2$$

because $s_2^{-1} s_1' s_2' \in S_1 S_2$ by equations (1.11). Similarly,

$$T_2 f(x_0 s_1' s_2') = c_2 \int_{S_2} f(s_2^{-1} x_0 s_1' s_2') ds = 0$$

because $s_2^{-1} x_0 s_1' s_2' \in S_2 x_0 S_1 S_2 = x_0 S_1 S_2$ by equations (1.11). Let

$$f_2(x) = \frac{1}{c_2} T_2 f(x);$$

then $f_2 \in T_2 C$ and $f_2(x_0 S_1 S_2) = 0$ while $f(S_1 S_2) = 1$. Now

$$T_1 T_2(e) = T_1 f_2(e) = c_1 \int_{S_1} f_2(s_1^{-1}) ds_1 = c_1,$$

and so

$$T_1 T_2 f(x_0) = T_1 f_2(x_0) = c_1 \int_{S_1} f(s_1^{-1} x_0) ds = 0$$

because $s_1^{-1} x_0 \in S_1 x_0 = x_0 S_1 \subseteq x_0 S_1 S_2$. Thus the hypothesis that $x_0 \notin S_1 S_2$ is incompatible with the complementary nature of T_1 and T_2. Therefore $X = S_1 S_2$, and so $X = S_1 \times S_2$. Thus Part (2) is proved.

To prove Part (3) note that if $x = x_1 x_2$, where $x_i \in S_i$ $(i = 1, 2)$, then

$$T_1 f(x_1 x_2) = c_1 \int_{S_1} f(s_1 x_1 x_2) ds_1,$$

and so

$$T_1 f(x_1 x_2) = c_1 \int_{S_1} f(s_1 x_2) ds_1.$$

In addition,

$$\begin{aligned} T_2 f(x_1 x_2) &= c_2 \int_{S_2} f(s_2 x_1 x_2) ds_2 \\ &= c_2 \int_{S_2} f(x_1 s_2 x_2) ds_2 \end{aligned}$$

by the permutability of elements from S_1 and S_2. Since the group is unimodular ds_2 is both right and left invariant, and hence

$$T_2 f(x_1 x_2) = c_2 \int_{S_2} f(x_1 s_2) ds_2.$$

REMARK. Conversely, if $X = S_1 \times S_2$ and $T_i f = \sigma_i * f$ $(i = 1, 2)$, where σ_i is the Haar measure on S_i, then it is clear that T_1 and T_2 are complementary averaging operators, and by Theorem 1.7, $T_1 C \cdot T_2 C$ separates points in X.

2. Normalizers

Normalizers, which are defined in 2.2, have been discussed by R. E. Johnson [11] and by one of the authors in [6]. In addition, the results of B. E. Johnson [10] concerning the double centralizers are also relevant to the study of normalizers.

Here we outline some of the general properties of normalizers and extend some of the results of Choda and Nakamura [7] to normalizers. Finally, we find the normalizers of $L^1(X)$, $L^1_c(X)$, and $C_c(X)$, where X is a locally compact group.

Let A be an algebra over the complex field (other fields are of course possible with analogous results.) Assume that A contains no non-zero right or left annihilators, that is, A is *faithful*. Let $E_l(A)$ and $E_r(A)$ be the algebras of endomorphisms written on the left and right respectively of the linear space A. Now consider $M_l(A) \subseteq E_l(A)$ [resp. $M_r(A) \subseteq E_r(A)$] defined as follows:

$$\text{(2.1)} \qquad \begin{aligned} M_l(A) &= \{T \in E_l : T(fg) = (Tf)g \text{ for all } f, g \in A\}, \\ \text{[resp. } M_r(A) &= \{T \in E_r : (fg)T = f(gT) \text{ for all } f, g \in A\}]. \end{aligned}$$

2.1 The mapping $f \to \hat{f}$ [resp. $f \to \tilde{f}$], where $\hat{f}(g) = fg$ [resp. $(g)\tilde{f} = gf$] is an injection of A into M_l. [resp. A into M_r]. The algebra $M_l(A)$ [resp. $M_r(A)$] is called the *left* [resp. *right*] *multiplier algebra* of A.

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If we identify A with $\hat{A}$ in M_l [resp. $\tilde{A}$ in M_r], then the following lemma is easily seen to be valid.

LEMMA. *$M_l(A)$ [resp. $M_r(A)$] contains A as a left [resp. right] ideal.*

2.2 The set

$$N_l(A) = \{T \in M_l\colon \text{For each } f \in A, \text{ there is } f_T \in A \text{ such that } fT(g) = f_T g \text{ for each } g \in A\}$$

$$[\text{resp. } N_r(A) = \{T \in M_r\colon \text{For each } g \in A, \text{ there is } g^T \in A \text{ such that } (f)Tg = fg^T \text{ for every } f \in A\}]$$

is called the left [resp. *right*] *normalizer* of A.

2.3 LEMMA. (1) *$\hat{A}$ [resp. $\tilde{A}$] is an ideal of $N_l(A)$ [resp. $N_r(A)$].*

(2) *The identity mapping belongs to $N_l(A)$ [resp. $N_r(A)$].*

(3) *If S is an algebra which contains A as an ideal, contains an identity, and contains no non-zero left or right annihilators of A, then there is an injection of S into $N_l(A)$ [resp. $N_r(A)$].*

PROOF. Statements (1) and (2) are immediate from the definitions of N_l and N_r.

(3): Consider the mapping $s \to \hat{s}$ where $\hat{s}(f) = sf$ for $s \in S$ and $f \in A$. First, $\hat{s} \in M_l(A)$, and if $f, g \in A$, then in S we have

$$f\hat{s}(g) = fsg = (fs)g$$

where $fs = f_{\hat{s}} \in A$. Therefore $\hat{s} \in N_l(A)$. Now $s \to \hat{s}$ is clearly a homomorphism. If $s_1 = s_2$, then

$$(s_1 - s_2)f = 0$$

for all $f \in A$ and hence $s_1 = s_2$. Therefore $s \to \hat{s}$ is an injection of S into $N(A)$. An analogous result is clearly valid for $N_r(A)$.

2.3 For $T \in N_l(A)$ consider the mapping

$$(\cdot)T : f \to f_T,$$

where f_T is the unique (because of the lack of annihilators) element of A which satisfies the equation

$$fT(g) = f_T g \tag{2.2}$$

for all $g \in A$. Since $(hf)_T = h(f_T)$ and equation (2.2) holds, it follows that $(\cdot)T \in N_r(A)$. In [6] it is verified that

$$T \to (\cdot)T \tag{2.3}$$

is an isomorphism between $N_l(A)$ and $N_r(A)$.

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In order to distinguish whether a given T manifests itself as an element of N_l or N_r we adopt the convention that $T(\cdot)$ [resp. $(\cdot)T$] is the manifestation of T as an element of $N_l(A)$. [resp. $N_r(A)$].

2.4 Let A be a *quasi topological* algebra, that is, a topological linear space in which the mappings $x \to xy$ and $x \to yx$ are continuous for each $y \in A$. Then let $\mathfrak{M}_l(A)$ [resp. $\mathfrak{M}_r(A)$] be the subring of M_l [resp. M_r] composed of continuous endomorphisms of A. Let $\mathfrak{E}_l$, $\mathfrak{E}_r$, $\mathfrak{N}_l$, $\mathfrak{N}_r$ be defined analogously.

If $\mathfrak{E}_l$ [resp. $\mathfrak{E}_r$] is given the strong operator topology as a ring of operators on A, that is, the topology of pointwise convergence on A, then it is easily verified that $\mathfrak{E}_l$ [resp. $\mathfrak{E}_r$] is Hausdorff if A is Hausdorff.

2.5 LEMMA. Let A be Hausdorff.

(1) *$\mathfrak{M}_l(A)$ [resp. $\mathfrak{M}_r(A)$] is closed in $\mathfrak{E}_l(A)$ [resp. $\mathfrak{E}_r(A)$].*

(2) *The injection $f \to \hat{f}$ [resp. $f \to \tilde{f}$] of A into $\mathfrak{M}_l(a)$ [resp. $\mathfrak{M}_r(A)$] is continuous.*

PROOF. (1) If $\{\varphi_\alpha\}$ is a net in $\mathfrak{M}_l$ such that $\lim \varphi_\alpha = \varphi \in \mathfrak{E}_l$, then

$$\varphi_\alpha(f)g \to \varphi(f)g$$

for all $f, g \in A$, because A is a topological algebra. However $\varphi_\alpha(f)g = \varphi_\alpha(fg)$, and so

$$\varphi(fg) = \lim \varphi_\alpha(fg) = \lim \varphi_\alpha(f)g = \varphi(f)g.$$

Therefore $\varphi \in \mathfrak{M}_l$.

(2) If $\{f_\alpha\}$ is a net in A which converges to $f \in A$, then

$$f_\alpha g \to fg$$

for all $g \in A$, and so $\hat{f}_\alpha \to \hat{f}$ in the strong operator topology. Therefore $f \to \hat{f}$ is continuous.

The proofs of (1) and (2) are entirely analogous for operators on the other side.

2.6 REMARK. The inverse mapping $\hat{f} \to f$ [resp. $\tilde{f} \to f$] is not continuous unless A contains no left [resp. right] topological divisors of zero, that is, no nets $\{f_\alpha\}$ such that $f_\alpha g \to 0$ [resp. $gf_\alpha \to 0$] for all $g \in R$ while $\{f_\alpha\}$ does not converge to zero.

2.7 THEOREM. *If A satisfies the closed graph theorem and is Hausdorff, then $N_l(A)$ [resp. $N_r(A)$] is composed entirely of continuous functions, that is,*

$$N_l(A) \subseteq \mathfrak{M}_l(A) \quad [\text{resp. } N_r(A) \subseteq \mathfrak{M}_r(A)].$$

PROOF. Suppose $T \in N_l(A)$. It is continuous if it has a closed graph.

Assume that $f_\alpha \to 0$ in A and that $T(f_\alpha) \to h$; then since multiplication is continuous

$$gT(f_\alpha) \to gh$$

for any $g \in A$. However $T \in N_l(A)$, and so

$$(g)Tf_\alpha = gT(f_\alpha).$$

Therefore, since

$$(g)Tf_\alpha \to 0,$$

it follows that

$$gh = 0$$

for all $g \in A$. Since the only right annihilator of A is zero, $h = 0$ and T has a closed graph.

2.8 Remark. The convolution algebras $C_c(X)$, $L^1_c(X)$, and $L^1(X)$ all satisfy the closed graph theorem, as also do Banach algebras in general. From Theorem 2.7 the normalizers of these algebras contain only continuous operators.

2.9 Remark. Note that if A is commutative and satisfies the closed graph theorem, then $N_l = N_r = \mathfrak{M}_l = \mathfrak{M}_r = M_l = M_r$, so there is no need to distinguish between $\mathfrak{M}$ and M.

Suppose for the remainder of this section that A is Hausdorff and satisfies the closed graph theorem.

It may be of interest to consider topologies for N_l and N_r such that the (algebraic) isomorphism

$$T(\cdot) \to (\cdot)T \tag{2.3}$$

is a homeomorphism.

2.10 Proposition. *Let $\mathfrak{E}_l(A)$ [resp. $\mathfrak{E}_r(A)$] be given the topology for which, if $\{T_\alpha\}$ is a net,*

$$T_\alpha \to T \text{ if and only if } fT_\alpha(g) \to fT(g) \text{ [resp. } (f)T_\alpha g \to (f)Tg]$$

for all $f, g \in A$. If A is Hausdorff, then $\mathfrak{M}_l(A)$ [resp. $\mathfrak{M}_r(A)$] is closed in $\mathfrak{E}_l(A)$ [resp. $\mathfrak{E}_l(A)$]; $f \to \hat{f}$ [resp. $f \to \check{f}$] is a continuous injection; and if N_l [resp. N_r] is endowed with the restriction of this topology, then

$$T(\cdot) \to (\cdot)T \tag{2.3}$$

is a homomorphism of N_l onto N_r.

Proof. The first two assertions of the conclusion are easily verified. To verify that $T(\cdot) \to (\cdot)T$ is a homeomorphism from N_l onto N_r, consider $T_\alpha(\cdot) \to T(\cdot)$ in $N_l(A)$. Then for all $f, g \in A$

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$$fT_\alpha(g) \to fT(g),$$

and since T_α and $T \in N_l(A)$, $fT_\alpha(g) = (f)T_\alpha(g)$ and $fT(g) = (f)T(g)$. Therefore $(\cdot)T_\alpha \to (\cdot)T$ and the mapping $T(\cdot) \to (\cdot)T$ is continuous. The reverse argument is identical.

Now we prove some theorems for normalizers which are analogous to results found by Choda and Nakamura [7] for multiplier algebras.

2.11 PROPOSITION. *Let A be complete and $\{T_\alpha(\cdot)\}$ be a net in $N_l(A)$. If $\{T_\alpha(\cdot)\}$ is a Cauchy net in the strong operator topology for $\mathfrak{E}_l(A)$ and if $\{(\cdot)T_\alpha\}$ is a Cauchy net in the strong operator topology of $\mathfrak{E}_r(A)$, then there is a $T \in N_l(A)$ such that*

$$\lim T_\alpha(\cdot) = T(\cdot)$$

and

$$\lim (\cdot)T_\alpha = (\cdot)T$$

in the strong operator topology of $\mathfrak{E}_l(A)$ and $\mathfrak{E}_r(A)$ respectively.

PROOF. Let $T_\alpha(f) \to T_0(f)$ and $(f)T_\alpha \to (f)T_1$ define the operators $T_0 \in E_l(A)$ and $T_1 \in E_r(A)$ respectively. Since $T_\alpha(fg) = T_\alpha(f)g$, and since A is Hausdorff, it follows that $T_0(f)g = T_0(fg)$ and so $T_0 \in M_l(A)$. Analogously $T_1 \in M_r(A)$. For $f, g \in A$,

$$(g)T_\alpha f = gT_\alpha(f),$$

so

$$(g)T_1 f = \lim (g)T_\alpha f = \lim gT_\alpha(f) = gT_0(f).$$

Thus $T_0 \in N_l(A)$ and $(\cdot)T_0 = (\cdot)T_1$.

2.12 REMARK. Since C_c, L^1_c, L^1 are all complete, Proposition 2.11 can be applied to them.

2.13 COROLLARY. *$N_l(A)$ [resp. $N_r(A)$] is complete relative to the topology with convergent nets defined as follows: If $\{T_\alpha\}$ is a net in $N_l(A)$ [resp. $N_r(A)$], then $T_\alpha \to T$ if and only if both*

$$(g)T_\alpha \to (g)T \quad \textit{and} \quad T_\alpha(f) \to T(f)$$

for all $f, g \in A$. In addition, the mapping

$$T(\cdot) \to (\cdot)T$$

is a homeomorphism in this topology.

PROOF. If $\{T_\alpha\}$ is a Cauchy net in this topology, then $\{T_\alpha(f)\}$ and $\{(f)T_\alpha\}$ are Cauchy nets in A. By Proposition 2.10, there is a $T \in N_l(A)$ such that

$$T_\alpha(f) \to T(f) \quad \text{and} \quad (f)T_\alpha \to (f)T$$

for all $f \in A$. Therefore $N_l(A)$ [resp. $N_r(A)$] is complete.

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Since the topology of this corollary is entirely symmetric with respect to $N_l(A)$ and $N_r(A)$, the mapping $T(\cdot) \to (\cdot)T$ is a homeomorphism.

2.14 The topology of Corollary 2.13 can be called the *two-sided strong topology*. It is clear that $f \to \hat{f}$ [resp. $f \to \tilde{f}$] is a continuous injection of A into $N_l(A)$ [resp. $N_r(A)$] in this topology.

2.15 A net $\{f_\alpha\}$ in A is called a *two-sided approximate identity* if for every $g \in A$

$$f_\alpha g \to g \quad \text{and} \quad g f_\alpha \to g.$$

2.16 THEOREM. *$\hat{A}$ [resp. $\tilde{A}$] is dense in $N_l(A)$ [resp. $N_r(A)$], with respect to the two-sided strong topology, if and only if A possesses a two-sided approximate identity.*

PROOF. If $\hat{A}$ is dense in $N_l(A)$, then there is a net $\{\hat{f}_\alpha\}$ in $\hat{A}$ such that $\hat{f}_\alpha \to 1$, the identity operator, and so for each $g \in A$, $f_\alpha g \to g$ and $g f_\alpha \to g$. Therefore $\{f_\alpha\}$ is a two-sided approximate identity.

Conversely, if A has a two-sided approximate identity $\{f_\alpha\}$, then for $T \in N_l(A)$ and $g \in A$,

$$T \cdot \hat{f}_\alpha(g) = T(f_\alpha g) = T(f_\alpha)g = T(f_\alpha)\hat{\ } g,$$

and since $f_\alpha g \to g$,

$$T(f_\alpha)\hat{\ }(g) \to T(g) \tag{2.4}$$

for all $g \in A$. On the other hand,

$$(g)T(f_\alpha) = gT(f_\alpha) = (g)Tf_\alpha,$$

and so for all, $g \in A$,

$$(g)T(f_\alpha)\tilde{\ } \to (g)T. \tag{2.5}$$

Equations (2.4) and (2.5) together imply that A is dense in $N_l(A)$ in the two-side strong topology. The proof is analogous for operators on the right.

2.17 Assume that A is a Banach algebra. Then $\mathfrak{E}_l(A)$ [resp. $\mathfrak{E}_r(A)$] becomes a Banach algebra when it is endowed with the norm

$$||T|| = \sup_{||f|| \leqq 1} ||T(f)|| \tag{2.6}$$

$$\text{[resp. } ||T|| = \sup_{||f|| \leqq 1} ||(f)T||.\text{]} \tag{2.7}$$

In both cases the norm topology is stronger than the strong operator topology. Thus $\mathfrak{M}_l(A)$ [resp. $\mathfrak{M}_r(A)$] is closed in this norm topology. Similarly the injection $f \to \hat{f}$ [resp. $f \to \tilde{f}$] defined in 2.1, is continuous in this norm topology. Since the closed graph theorem is valid in A, $N_l(A) \subseteq \mathfrak{M}_l(A)$

[resp. $N_r(A) \subseteq \mathfrak{M}_r(A)$]. Neither $M_l(A)$ nor $M_r(A)$ need be closed in its appropriate norm topology. However, if $N_l(A)$ [resp. $N_r(A)$] is endowed with either of the equivalent norms

$$(2.8) \qquad |T| = ||T(\cdot)|| + ||(\cdot)T||$$

or

$$(2.9) \qquad |||T||| = \text{Max}\,\{||T(\cdot)||, ||(\cdot)T||\},$$

then the following result can be proved.

2.18 THEOREM. *If $N_l(A)$ [resp. $N_r(A)$] is endowed with either of the norms* (2.8) *or* (2.9), *then it is complete and the mapping*

$$T(\cdot) \to (\cdot)T$$

is an isometry of the Banach algebra $N_l(A)$ onto $N_r(A)$.

PROOF. Let $\{T_\alpha(\cdot)\}$ be a Cauchy net in $N_l(A)$ with regard to the norm $|T|$. Since

$$||T_\alpha(f) - T_\beta(f)|| \leqq |T_\alpha - T_\beta|\; ||f||,$$

$\{T_\alpha(\cdot)\}$ is a Cauchy net with respect to the two-sided strong topology on $N_r(A)$. By Corollary 2.13, it follows that there is a $T \in N_r(A)$ such that $T(\cdot) \to T(\cdot)$ in the two-sided topology.

Now $\mathfrak{M}_l(A)$ is complete with respect to the norm of equation (2.6), and so there exists $T_0 \in \mathfrak{M}_l(A)$ such that $T_\alpha(\cdot) \to T_0(\cdot)$ in norm. Similarly there is $T_1 \in \mathfrak{M}_r(A)$ such that $(\cdot)T_\alpha \to (\cdot)T_1$ in the norm of equation (2.7). Since in either case the strong operator topology is weaker than the norm topology,

$$T_1(\cdot) = T_0(\cdot) = T(\cdot), \text{ and } N_l(A) \text{ is complete.}$$

Finally, it is obvious that $T(\cdot) \to (\cdot)T$ is an isometry.

2.19 Now we come to use some of the results of Part I to find the normalizers of certain special convolution algebras of measures. Let X be a locally compact group. Recall that $L^1(X)$, $L^1_c(X)$, and $C_c(X)$ stand respectively for the set of integrable functions, the set of integrable functions with compact support, and the set of continuous functions with compact support. Each of these spaces forms a topological algebra if it is endowed with its natural topology and multiplication is taken to be convolution.

2.20 PROPOSITION. *Let A be taken to be either $L^1(X)$, $L^1_c(X)$, or $C_c(X)$. A continuous linear operator on A commutes with right [resp. left] translations if and only if it is a left [resp. right] multiplier.*

PROOF. We prove the result for right translations and left multipliers. The result for left translations and right multipliers follows analogously.

Suppose T commutes with the ρ_a's for $f, g \in A$. Then

$$T(f * g) = T \int_X \rho_s f g(s) ds$$

where ds stands for the left Haar measure on X. When $A = L^1$ we may use [8, p. 323] to show that T and $\int$ commute. Similarly, when $A = L^1_c$ the same theorem may be used on $L^1(\mathfrak{A})$, where $\mathfrak{A}$ is a compact set which contains the sum $S_f + S_g$ of the supports of f and g. Finally if $A = C_c$, the same theorem can be applied to $C(\mathfrak{A})$ under the usual norm. Thus in general we have

$$T(f * g) = \int_X T\rho_s f g(s) ds$$

for $f, g \in A$, and since T commutes with ρ_s,

$$\begin{aligned} T(f * g) &= \int_X \rho_s T f g(s) ds \\ &= (Tf) * g. \end{aligned}$$

Conversely, suppose T commutes with right convolution, that is, T is a left multiplier. For $f, g \in A$,

$$\begin{aligned} T(f * g) &= T \int_X \rho_s f g(s) ds \\ &= \int_X T\rho_s f g(s) ds \end{aligned}$$

by the continuity argument advanced in the previous paragraph. By hypothesis, $T(f * g) = (Tf) * g$, so

$$T(f * g) = \int_X T\rho_s f g(s) ds = \int_X \rho_s \rho_s T f g(s) ds$$

for all $f, g \in A$. Thus for all $f \in A$ and all $s \in X$,

$$T\rho_s f = \rho_s T f.$$

2.21 COROLLARY. *If* $\xi : \mu \to \mu * (\cdot)$ and $\eta : \mu \to (\cdot) * \mu$, then

(a) $\xi M_{bd}(X) = \mathfrak{M}_l(L^1)$, (a′) $\eta M_{bd}(X) = \mathfrak{M}_r(L^1)$

(b) $\xi M_c(X) = \mathfrak{M}_l(C_c)$ (b′) $\eta M_c(X) = \mathfrak{M}_r(C_c)$,

(c) $\xi M_c(X) = \mathfrak{M}_l(L^1_c)$, (c′) $\eta M_c(X) = \mathfrak{M}_r(L^1_c)$.

This corollary follows directly from Corollary 2.6.2, Theorem 1.2, and Corollary 2.6.1 of Part I combined with Proposition 2.20.

2.22 The normalizers of L^1_c, L^1, and C_c are characterized by the following theorem:

THEOREM. *The following statements are valid:*

(a) $\mathfrak{M}_l(L^1) = N_l(L^1)$, (a′) $\mathfrak{M}_r(L^1) = N_r(L^1)$,
(b) $\mathfrak{M}_l(L^1_c) = N_l(L^1_c)$, (b′) $\mathfrak{M}_r(L^1_c) = N_r(L^1_c)$,
(c) $\mathfrak{M}_l(C_c) = N_l(C_c)$, (c′) $\mathfrak{M}_r(C_c) = N_r(C_c)$.

PROOF. If $\mu \in M_{bd}(X)$, then

$$f * (\mu * g) = (f * \mu) * g$$

for f, g belonging to any of L^1, L^1_c, C_c. Thus $\mu * (\cdot) \in N_l(L^1)$ [resp. $(\cdot) * \mu \in N_r(L^1)$] for $\mu \in M_{bd}$, and further $\mu * (\cdot) \in N_l(L^1_c)$ [resp. $(\cdot) * \mu \in N_r(L^1_c)$] and $\mu * (\cdot) \in N_l(C_c)$ [resp. $(\cdot) * \mu \in N_r(C_c)$] for $\mu \in M_c$. Thus in (a) through (c′) the left-hand member is a subset of the right-hand member.

Since in L^1, L^1_c, C_c the closed graph theorem is valid, by Theorem 2.7 it follows that (a) through (c′) are all valid.

2.23 Theorem 2.22 together with Lemma 2.3 ensures that any algebra with identity which contains L^1 or L^1_c or C_c as an ideal and contains no right or left annihilator of this ideal except 0 can be injected (algebraically) into $\mathfrak{M}_l$ or $\mathfrak{M}_r$.

3. Certain division problems

We shall here apply the results of Part I, 2.8 and § 5 to some problems regarding the division of measures and of distributions.

3.1 THEOREM. *Suppose that*

$$\mu, \lambda_1, \lambda_2, \cdots, \lambda_s \in M_{bd}(X).$$

In order that

$$\mu * L^1 \subseteq \lambda_1 * L^1 + \cdots + \lambda_s * L^1 \quad [\textit{resp. } L^1 * \mu \subseteq L^1 * \lambda_1 + \cdots + L^1 * \lambda_s] \tag{3.1}$$

it is necessary and sufficient that there exist $\nu_1, \cdots, \nu_s \in M_{bd}(X)$ *such that*

$$\mu = \lambda_1 * \nu_1 + \cdots + \lambda_s * \nu_s \quad [\textit{resp. } \mu = \nu_1 * \lambda_1 + \cdots + \nu_s * \lambda_s]. \tag{3.2}$$

REMARK. The meaning of the first relation in (3.1) is, of course that each convolution $\mu * f$, where $f \in L^1$, is expressible as a sum $\lambda_1 * f_1 + \cdots + \lambda_s * f_s$, where $f_i \in L^1$ $(i = 1, \cdots, s)$ is suitably chosen. Similar conventions apply to the second relation in (3.1), and to subsequent analogous symbolism.

PROOF. The sufficiency is obvious, since

$$\nu * f \in L^1 \text{ and } f * \nu \in L^1 \text{ if } f \in L^1 \text{ and } \nu \in M_{bd}.$$

To prove necessity, assume that (3.1) holds. With the notation of Part I, 2.8, define $J \subset (L^1)^s$ as in 2.8.2 of Part I. Then (3.1) ensures that there exists a linear map $T : L^1 \to (L^1)^s/J$ such that $Tf = \boldsymbol{g}_J$ if and only if $\boldsymbol{g} = (g_1, \cdots, g_s)$ and

$$\mu * f = \lambda_1 * g_1 + \cdots + \lambda_s * g_s \quad \text{[resp. } f * \mu = g_1 * \lambda_1 + \cdots + g_s * \lambda_s\text{]}. \tag{3.3}$$

It is evident that this T commutes with right [resp. left] translations.

Let us verify that T has a closed graph, and is therefore continuous. Suppose indeed that $f^{(n)} \to 0$ in L^1 and $Tf^{(n)} \to \boldsymbol{g}_J$ in $(L^1)^s/J$; we have to show that $g_J = 0$, that is, that $\boldsymbol{g} \in J$. Now, without altering g_J, we may suppose that $Tf^{(n)} = \boldsymbol{g}_J^{(n)}$, where

$$\sum_{i=1}^{s} \|g_i^{(n)} - g_i\|_{L^1} \to 0.$$

For each n we have

$$\mu * f^{(n)} = \lambda_1 * g_1^{(n)} + \cdots + \lambda_s * g_s^{(n)}$$
$$\text{[resp. } f^{(n)} * \mu = g_1^{(n)} * \lambda_1 + \cdots + g_s^{(n)} * \lambda_s\text{]}.$$

Letting $n \to \infty$, it follows that

$$0 = \lambda_1 * g_1 + \cdots + \lambda_s * g_s$$
$$\text{[resp. } 0 = g_1 * \lambda_1 + \cdots + g_s * \lambda_s\text{]},$$

showing that $\boldsymbol{g} \in J$.

By 2.8.2 (c) and Theorem 2.9 in Part I, we infer that there exists $\boldsymbol{\nu} \in (M_{bd})^s$ such that

$$Tf = (\boldsymbol{\nu} * f)_J \quad \text{[resp. } (f * \boldsymbol{\nu})_J\text{]}. \tag{3.4}$$

In view of (3.3) this means that

$$\mu * f = \lambda_1 * \nu_1 * f + \cdots + \lambda_s * \nu_s * f$$
$$\text{[resp. } f * \mu = f * \nu_1 * \lambda_1 + \cdots + f * \nu_s * \lambda_s\text{]}$$

for all $f \in L^1$. Whence follows (3.2).

3.2 Remarks. For bounded measures on the half-line $(0, \infty)$, an analogous result is given by Wells [13]. We have been unable to decide whether the analogue of Theorem 3.1, in which L^1 is everywhere replaced by C_0, is valid. But see 3.11 below.

In case $s = 1$ and X is Abelian, Theorem 2.9 in Part I has the following application.

3.3 Theorem. *Let X be a locally compact Abelian group $\hat{X}$ the dual group, and S a subset of $\hat{X}$ such that $S \subset \overline{\mathrm{Int}\, S}$. Let φ be a function on S with*

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the property: to each $f \in L^1$ corresponds at least one $g \in L^1$ such that $\varphi(\hat{f}|S) = \hat{g}|S$. Then there exists $\nu \in M_{bd}(X)$ such that $\varphi = \hat{\nu}|S$. (The converse is true and trivial, whatever the subset S of $\hat{X}$.)

Proof. We now define T: $L^1 \to L^1/J$, where

$$J = \{f \in L^1 : \hat{f} = 0 \text{ on } S\},$$

by $Tf = g_J$ where $g \in L^1$ and $\hat{g}|S = \varphi(\hat{f}|S)$. The result will follow from Theorem 2.9 in Part I (with $s = 1$), as soon as it is shown that J satisfies the condition in Part I, 2.8.1. For this we must show that if $j_\alpha \in J$, $\|j_\alpha\|_1$ is bounded, and $j_\alpha \to j$ for $\sigma(L^1, C_0)$, then $j \in J$ too. Now let $\Omega = \operatorname{Int} S$. If $\mu \in L^1(X)$, we have by the Fubini-Tonelli theorem

$$\int u(\xi)\hat{j}_\alpha(\xi)d\xi = \int j_\alpha(x)v(x)dx,$$

where

$$v(x) = \int u(\xi)\overline{\xi(x)}d\xi$$

belongs to $C_0(X.)$. So it follows that

$$\int u(\xi)\hat{j}(\xi)d\xi = \lim_\alpha \int u(\xi)\hat{j}_\alpha(\xi)d\xi.$$

The right-hand side of this expression is zero whenever $u = 0$ on $\hat{X}\backslash S$, a fortiori if $u = 0$ on $\hat{X}\backslash\Omega$. From this it follows that $\hat{j}(\xi) = 0$ on Ω and, by continuity on $\bar{\Omega} \supseteqq S$. Thus $j \in J$.

As we shall now show, if $X = R^n$ and $s = 1$, the results of Part I, § 5 lead to partial analogues of Theorem 3.1 for distributions.

3.4 Theorem. *Suppose $X = R^n$, $\mu \in \mathscr{D}'$, $\lambda \in \mathscr{D}'$, and*

$$\mu * C_c^\infty \subseteq \lambda * \mathscr{D}'_c. \tag{3.5}$$

To each $\xi \in \mathscr{D}'_c$, which is the limit in $\mathscr{D}'$ of finite linear combinations of translates of λ, corresponds $\nu_\xi \in \mathscr{D}'_c$ such that

$$\mu * \xi = \lambda * \nu_\xi. \tag{3.6}$$

Proof. Given $f \in C_c^\infty$ there exists $g \in \mathscr{D}'_c$ such that

$$\mu * f = \lambda * g. \tag{3.7}$$

This g is in general not uniquely determined by f. However, if $\lambda * g = \lambda * g'$, then $\xi * g = \xi * g'$ for any $\xi \in \mathscr{D}'$ which is the limit of finite linear combinations of translates of λ. So $Tf = \xi * g$ is uniquely defined. It is evident that $T : C_c^\infty \to \mathscr{D}'_c$ is linear and commutes with translations, and that

$$T(f * \varphi) = Tf * \varphi \text{ for } f, \varphi \in C_c^\infty. \tag{3.8}$$

We now show that T is continuous from C_c^∞ into $\mathscr{D}'_c$, where the latter is given the topology $\sigma(\mathscr{D}'_c, C)$. To accomplish this, it suffices to show that for each $u \in C^\infty$, the map $T_u : C_c^\infty \to C^\infty$ defined by

$$T_u f = (Tf) * u \tag{3.9}$$

is continuous from C_c^∞ into C^∞. (This remark is based on the observation that $T_u f(0) = \langle u, Tf \rangle$.) For this it is in turn sufficient to show that $T_u | C_{c,K}$ has a graph closed in $C_{c,K}^\infty \times C^\infty$, K being any compact subset of R^n. Now for $f, \varphi \in C_c^\infty$ we have from (3.8) and (3.9)

$$\begin{aligned} T_u(f * \varphi) &= T(f * \varphi) * u = (Tf * \varphi) * u = (Tf * u) * \varphi \\ &= T_u f * \varphi. \end{aligned}$$

The convolutions are associative since Tf and φ have compact supports. Thus

$$T_u(f * \varphi) = T_u f * \varphi \qquad (f, \varphi \in C_c^\infty). \tag{3.10}$$

Supposing that $f_n \to 0$ in $C_{c,K}^\infty$ and $T_u f_n \to h$ in C^∞, then by (3.10) it follows that for any $\varphi \in C_c^\infty$ we have

$$T_u(f_n * \varphi) = T_u f_n * \varphi = T_u \varphi * f_n \to 0$$

in C^∞, and also

$$T_u f_n * \varphi \to h * \varphi$$

in C^∞. Thus $h * \varphi = 0$ for all $\varphi \in C_c^\infty$. Hence $h = 0$, and the graph is closed. With this, the continuity of T is established.

Appealing to Theorem 5.2 (c) in Part I, we conclude that there exists $\nu_\xi \in \mathscr{D}'_c$ such that

$$Tf = \nu_\xi * f. \tag{3.11}$$

Thus from (3.7) and (3.11) it follows that, for each $f \in C_c^\infty$, there exists $g \in \mathscr{D}'_c$ such that

$$\mu * f = \lambda * g \text{ and } \xi * g = \nu_\xi * f.$$

Since ξ, f, g all have compact supports,

$$\xi * \mu * f = \xi * \lambda * g = \lambda * \xi * g = \lambda * \nu_\xi * f,$$

which shows that (3.6) holds.

3.5 Corollary. *If $X = R^n$, $\mu \in \mathscr{D}'$, $\lambda \in \mathscr{D}'_c$, and*

$$\mu * C_c^\infty \subseteq \lambda * \mathscr{D}'_c, \tag{3.5}$$

there exists $\nu \in \mathscr{D}'_c$ such that

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(3.6) $$\mu = \lambda * \nu.$$

(The converse is true and trivial.)

PROOF. We may assume that $\lambda \neq 0$: if $\lambda = 0$, (3.5) entails that $\mu = 0$, so that (3.6) is true for *any* $\nu \in \mathscr{D}'_c$.

If $\lambda \neq 0$ and $\lambda \in \mathscr{D}'_c$, its translates span the whole of $\mathscr{D}'$, since use of the Fourier transform shows that the only $\varphi \in C_c^\infty$ orthogonal to all translates of λ is $\varphi = 0$. Hence we may apply Theorem 3.4, taking therein $\xi = \varepsilon$, the Dirac measure.

If we assume (as in Corollary 3.5) that $\lambda \in \mathscr{D}'_c$, there is an analogous result, as follows.

3.6 THEOREM. *Suppose that* $X = R^n$, $\mu \in \mathscr{D}'$, $\lambda \in \mathscr{D}'_c$, *and that*

(3.12) $$\mu * C_c^\infty \subseteq \lambda * \mathscr{D}'.$$

To each $\xi \in \mathscr{D}'_c$ *which is the weak limit in* $\mathscr{D}'_c$ *of finite linear combinations of translates of* λ *there corresponds* $\nu_\xi \in \mathscr{D}'$ *such that*

(3.13) $$\mu * \xi = \lambda * \nu_\xi.$$

PROOF. This proof is very similar to that of Theorem 3.5, appeal being made to Theorem 5.2 (a) of Part I. We omit the details.

3.7 COROLLARY. *Suppose that* $X = R^n$, $\mu \in \mathscr{D}'_c$, *and that*

(3.12) $$\mu * C_c^\infty \subseteq \lambda * \mathscr{D}'.$$

Suppose further that

(3.14) $$u \in C^\infty \text{ and } \lambda * u = 0 \text{ imply } u = 0.$$

Then there exists $\nu \in \mathscr{D}'$ *such that*

(3.15) $$\mu = \lambda * \nu.$$

PROOF. Condition (3.14) is precisely that required to ensure that the Dirac measure ε (and hence *every* member of $\mathscr{D}'_c$) is the weak limit in $\mathscr{D}'_c$ of finite linear combinations of translates of λ. Apply Theorem 3.6, taking $\xi = \varepsilon$.

Finally, by using Theorem 5.3 (b) of Part I, we may derive the following theorem.

3.8 THEOREM. *Suppose that* $X = R^n$, $\mu \in \mathscr{D}'$, $\lambda \in \mathscr{D}'$, *and that the translates of* λ *are total in* $\mathscr{D}'$ *(as happens if* $\lambda \neq 0$, $\lambda \in \mathscr{D}'_c$*). Suppose also that*

(3.16) $$\mu * C_c \subseteq \lambda * M_c.$$

Then there exists $\nu \in \mathscr{D}'_c$, *with* $\hat{\nu}$ *bounded on* R^n, *such that*

(3.17) $$\mu = \lambda * \nu.$$

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3.9 REMARK. Notice that if (3.17) holds with ν as specified, then not only is (3.16) true, but even

$$\mu * L_c^2 \subseteq \lambda * L_c^2. \tag{3.18}$$

3.10 *The relation* $\mu * C_0 \subseteq \lambda * C_c$. It is here supposed that μ and λ belong to $M_{bd}(X)$. As stated in 3.2, we do not know whether this entails that $\mu = \lambda * \nu$ for some $\nu \in M_{bd}$.

However, partial results, analogous to Theorems 3.4, 3.6 and 3.8, can be established. In particular one can show that, if

$$\mu * C_0 \subseteq \lambda * C_0, \tag{3.19}$$

and if $\xi \in M_{bd}$ satisfies the condition

$$g \in C_0 \text{ and } \lambda * g = 0 \text{ imply } \xi * g = 0, \tag{3.20}$$

then there exists $\nu_\xi \in M_{bd}$ such that

$$\mu * \xi = \lambda * \nu_\xi. \tag{3.21}$$

(The proof, which follows the customary pattern, is left to the reader.)

Now (3.19) itself implies that $\xi = \mu$ fulfills (3.20). For if we take a net $\{f_\alpha\}$ in C_0 converging for $\sigma(M_{bd}, C_0)$ to ε, we have $\mu * f_\alpha = \lambda * g_\alpha$ for certain $g_\alpha \in C_0$. Then $\mu = \lim \mu * f_\alpha$ weakly. Hence if $g \in C_0$ and $\lambda * g = 0$, we have

$$\begin{aligned} \mu * g &= \lim \mu * f_\alpha * g = \lim \lambda * g_\alpha * g \\ &= \lim (\lambda * g) * g_\alpha = \lim 0 = 0. \end{aligned}$$

So we infer that (3.19) entails

$$\mu * \mu = \lambda * \nu \text{ for some } \nu \in M_{bd}. \tag{3.22}$$

In particular, $C_0 = \lambda * C_0$ holds if and only if λ is inversible in M_{bd}.

Again, if $\lambda * C_0$ is dense in C_0, then (3.20) is valid for *any* $\xi \in M_{bd}$. So, taking $\xi = \varepsilon$, (3.19) is seen to imply the best-possible conclusion, namely

$$\mu = \lambda * \nu \text{ for some } \nu \in M_{bd}. \tag{3.23}$$

4. Isomorphisms of convolution algebras

Representation theorems for multipliers of L^1 (see Corollary 2.6.2 of Part I) were used by Wendel [14], [15] as the basis for a study of isometric and bipositive isomorphisms of L^1-group algebras. Theorems 1.2, Corollaries 2.6.1 and 2.6.2, and the results of Section 3, all in Part I, can likewise be used to show that if X and X' are two locally compact groups, and if A denotes any one of the convolution algebras L^1, L_c^1, C_c, or L^p $(1 < p < \infty)$

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over a compact group, then the existence of a bipositive isomorphism between $A(X)$ and $A(X')$ entails that X and X' are isomorphic topological groups. Details of the arguments appear in Edwards [9].

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