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Endoscopy and cohomology growth on $U(3)$

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Abstract

We apply the endoscopic classification of automorphic forms on $U(3)$ to study the growth of the first Betti number of congruence covers of a Picard modular surface. As a consequence, we establish a case of a conjecture of Sarnak and Xue on cohomology growth.

1. Introduction

Let $U(p,q;\mathbb{R})$ denote the real unitary group of signature (p,q). Let $\mathbb{H}_\mathbb{C}$ be the globally symmetric space $U(2,1;\mathbb{R})/(U(2;\mathbb{R})\times U(1;\mathbb{R}))$. Let $\Gamma\subset U(2,1;\mathbb{R})$ be an arithmetic congruence lattice arising from a Hermitian form in three variables with respect to a CM extension E/F. If $\mathcal{O}$ is the ring of integers of F and $\mathfrak{n}\subseteq\mathcal{O}$ is an ideal, we may define principal congruence subgroups $\Gamma(\mathfrak{n})\subseteq\Gamma$, and let $Y(\mathfrak{n})$ be the arithmetic locally symmetric space $\Gamma(\mathfrak{n})\backslash\mathbb{H}_\mathbb{C}$. We give the precise definition of these objects, and the statement of Theorem 1 below, in § 2.3. Put $V(\mathfrak{n})=|\Gamma:\Gamma(\mathfrak{n})|$. It is asymptotically equal to the volume of $Y(\mathfrak{n})$. We let $H^1_{(2)}(Y(\mathfrak{n}),\mathbb{C})$ be the space of square integrable harmonic 1-forms on $Y(\mathfrak{n})$, and let $b^1_{(2)}(Y(\mathfrak{n}))$ be its dimension.

When Γ is cocompact, Sarnak and Xue [SX91] made a general conjecture on the asymptotic multiplicities of automorphic forms which implies that $b^1_{(2)}(Y(\mathfrak{n}))\ll_\epsilon V(\mathfrak{n})^{1/2+\epsilon}$. In the case of $U(2,1;\mathbb{R})$ they are able to prove the weaker bound $b^1_{(2)}(Y(\mathfrak{n}))\ll_\epsilon V(\mathfrak{n})^{7/12+\epsilon}$. This paper settles their conjecture in this case, by proving the following upper bound on $b^1_{(2)}(Y(\mathfrak{n}))$.

Theorem 1. *We have $b^1_{(2)}(Y(\mathfrak{n}))\ll V(\mathfrak{n})^{3/8}$, and there exists Γ such that $b^1_{(2)}(Y(\mathfrak{n}))\gg V(\mathfrak{n})^{3/8}$.*

The proof of Theorem 1 relies on the endoscopic classification of automorphic representations on $U(3)$ in [Rog90] (bearing in mind the remark below). The essential idea is that the automorphic forms contributing to $H^1_{(2)}(Y(\mathfrak{n}),\mathbb{C})$ in Matsushima's formula are nontempered, and Rogawski shows that they are all transfers of one-dimensional representations on the endoscopic group $U(2)\times U(1)$ of $U(3)$. Our work lies in making this result quantitative. Note that Rogawski also proves that $b^1_{(2)}(Y(\mathfrak{n}))=0$ if $Y(\mathfrak{n})$ arises from a nine-dimensional division algebra with involution over E, and when combined with Theorem 1 this provides an understanding of the growth of $b^1_{(2)}$ for all arithmetic congruence lattices in $U(2,1;\mathbb{R})$.

We note that when $F=\mathbb{Q}$, one should be able to obtain the upper bound in Theorem 1 using the results of Mok [Mok12]. Moreover, if $\mathfrak{n}=\mathfrak{a}\mathfrak{p}^k$ with $\mathfrak{a}$ and $\mathfrak{p}$ fixed, $\mathfrak{p}$ prime, and k growing, the lower bound in Theorem 1 is proven in [CM12], while the upper bound follows by combining [CM12] with [GR91] or [BMM13, Proposition 13.8].

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Remark. There is a question of priority of the work on $U(3)$, see [Fli06, III. 6, pp. 392–396], and [Rog90, p. xii]. Flicker has identified an error in his work, specifically in the proof of the global multiplicity one theorem; see [Fli04] and [CF09, Remark (ii), p. 1250]. In [Fli06], Flicker states that this error appears also in [Rog90], and he gives a plan for an alternative proof of multiplicity one which relies on a local multiplicity one theorem at all places. He only establishes this local theorem when the residual characteristic is not 2 (see [Fli04], first line of the introduction). He states in [Fli04, p. 6] that he believes it should be possible to carry out the proof in residual characteristic 2 in a similar way to the proof at the other places, but this has not been done as yet. However, it is proven in [Fli06] that the multiplicity one theorem for $U(3)$ holds for any automorphic representation each of whose dyadic local components lies in the packet of a constituent of a parabolically induced representation (see also [CF09, Remark (ii), p. 1250]). The automorphic representations considered in this paper are of precisely this form, as they are Saito-Kurokawa lifts, and so we are able to use the multiplicity formula of [Rog92] and [Fli06, p. 218] in our case.

2. Notation

2.1 Number fields

Let E/F be a CM extension of number fields, with $\mathcal{O}_E$ and $\mathcal{O} = \mathcal{O}_F$ their rings of integers and $\mathbb{A}_E$ and $\mathbb{A} = \mathbb{A}_F$ their rings of adèles. We denote the maximal compact subrings of the finite adèles $\mathbb{A}_{E,f}$ and $\mathbb{A}_f$ by $\widehat{\mathcal{O}}_E$ and $\widehat{\mathcal{O}}$. Let N be the norm map from E to F, $\mathbb{A}_E^1$ the group of norm 1 idèles of E, and $I_E^1 = \mathbb{A}_E^1/E^1$. We shall denote places of E and F by w and v respectively, with corresponding completions E_w and F_v, and define $E_v = E \otimes_F F_v$.

Fix a character μ of $\mathbb{A}_E^\times/E^\times$ whose restriction to $\mathbb{A}^\times/F^\times$ is the character associated to E/F by class field theory. Let S_f be a set of finite places of F containing all the places at which E/F is ramified, all places below those at which μ is ramified, all places dividing a rational prime p that satisfies $p \leqslant 9[F:\mathbb{Q}]+1$, and at least one place that is nonsplit in E. Let S_∞ be the set of infinite places of F, and let $S = S_\infty \cup S_f$.

2.2 Unitary groups

Let $\Phi_n = (\Phi_{ij})$, where $\Phi_{ij} = (-1)^{i-1}\delta_{i,n+1-j}$ and $\delta_{a,b}$ is the Kronecker delta function. The matrix Φ_n defines a Hermitian form with respect to E/F if n is odd, and if $x \in E$ satisfies $\mathrm{tr}_{E/F}(x) = 0$ then $x\Phi_n$ is Hermitian if n is even. We let $U(n)$ be the unitary group of this Hermitian form. It is a quasi-split F-group, and its group of F-points is

$$U(n,F) = \{g \in \mathrm{GL}(n,E) \mid g\Phi_n{}^t\overline{g} = \Phi_n\}.$$

For any ideal $\mathfrak{n} \subseteq \mathcal{O}$, we define the compact subgroup $U(n,\mathfrak{n}) \subset U(n,\mathbb{A}_f)$ by

$$U(n,\mathfrak{n}) = \{g \in U(n,\widehat{\mathcal{O}}) \subset \mathrm{GL}(n,\widehat{\mathcal{O}}_E) \mid g \equiv I_n(\mathfrak{n}\widehat{\mathcal{O}}_E)\}.$$

We shall denote $U(3)$ by G^*. If $\mathfrak{n} \subseteq \mathcal{O}$ is an ideal, define the compact subgroup $K^*(\mathfrak{n}) = U(3,\mathfrak{n}) \otimes_{v|\infty} K_v^*(\mathfrak{n})$ of $G^*(\mathbb{A})$ by setting $K_v^*(\mathfrak{n}) = U(2;\mathbb{R}) \times U(1;\mathbb{R})$ if $v|\infty$. For $v \nmid \infty$, we put $K_v^*(\mathfrak{n}) = U(3,\mathfrak{n}) \cap G_v^*$.

Choose a place $v_0 \in S_\infty$. Let Φ be a Hermitian form on E^3 with respect to E/F that is indefinite at v_0 and definite at all other real places of F, and let G be the unitary group of Φ. It is known, see for instance [PY07, § 1.2], that the isomorphism class of G over F depends only on the extension E/F and the place v_0. In particular, G is quasi-split if and only if $F = \mathbb{Q}$.

If v is a finite place of F that splits in E, then there are isomorphisms from G_v and G_v^* to $\mathrm{GL}(3, F_v)$ that are canonical up to inner automorphism. If v is finite and nonsplit in E/F, it follows from a theorem of Landherr [Lan36] that there is a unique Hermitian form on E_v^3 with respect to E_v/F_v. This gives an isomorphism from G_v to G_v^* that is canonical up to inner automorphism. If we let $K = \otimes_v K_v$ be a compact open subgroup of $G(\mathbb{A})$ such that $K_{v_0} = U(2;\mathbb{R}) \times U(1;\mathbb{R})$, $K_v = U(3;\mathbb{R})$ when $v_0 \neq v|\infty$, and K_v is hyperspecial whenever $v \notin S$, we may then fix isomorphisms $\phi_v : G_v \xrightarrow{\sim} G_v^*$ for all finite v such that $\phi_v K_v = K_v^*$ for $v \notin S$.

2.3 Adelic quotients

If $\mathfrak{n} \subseteq \mathcal{O}$ is relatively prime to S_f, we define $K(\mathfrak{n}) = \otimes_v K_v(\mathfrak{n})$ by setting $K_v(\mathfrak{n}) = K_v$ for $v \in S$, and $K_v(\mathfrak{n}) = \phi_v(K_v^*(\mathfrak{n}))$ for $v \notin S$. We define $Y(\mathfrak{n})$ to be the adelic quotient $G(F)\backslash G(\mathbb{A})/K(\mathfrak{n})Z(\mathbb{A})$. It is a finite union of finite volume quotients of the globally symmetric space $\mathbb{H}_\mathbb{C}$, and it is compact if and only if $F \neq \mathbb{Q}$. If we fix a translation-invariant volume form on $\mathbb{H}_\mathbb{C}$ and let $\mathrm{Vol}(Y(\mathfrak{n}))$ be the volume of $Y(\mathfrak{n})$ with respect to this form then we have $\mathrm{Vol}(Y(\mathfrak{n})) = c(\mathfrak{n})V(\mathfrak{n})$, where

$$V(\mathfrak{n}) = |U(3, \mathcal{O})Z(\mathbb{A}_f) : U(3, \mathfrak{n})Z(\mathbb{A}_f)| \tag{1}$$

and $c(\mathfrak{n}) \in \mathbb{R}^+$ has the property that $|\log c(\mathfrak{n})|$ is bounded in terms of our choice of K_v for $v \in S_f$. Note that the formulas for the orders of $\mathrm{GL}(3)$ and $U(3)$ over a finite field (see [Art55]) imply that $N\mathfrak{n}^8 \ll V(\mathfrak{n}) \ll N\mathfrak{n}^8$.

With this notation, the precise statement of Theorem 1 is that $b^1_{(2)}(Y(\mathfrak{n})) \ll V(\mathfrak{n})^{3/8}$, and that $b^1_{(2)}(Y(\mathfrak{n})) \gg V(\mathfrak{n})^{3/8}$ if K_v are chosen small enough for all $v \in S_f$.

2.4 Endoscopic groups

Let $H \simeq U(2) \times U(1)$ be the unique elliptic endoscopic group of G^*, which we consider to be embedded in the quasi-split group G^* as

$$\begin{pmatrix} * & & * \\ & * & \\ * & & * \end{pmatrix}.$$

We let $\det_0 : H \to U(1)$ and $\lambda : H \to U(1)$ be the maps given by the determinant on the $U(2)$ factor and projection onto the $U(1)$ factor. We fix an embedding of L-groups ${}^LH \to {}^LG^*$ associated to the character μ as in [Rog90, § 4.8.1] and [Fli06, p. 208]. The centers of G and G^* will both be denoted by $Z \simeq U(1)$. We identify Z with the diagonal subgroup of H. As $Z(F)\backslash Z(\mathbb{A}) \simeq I_E^1$, μ defines a character of $Z(F)\backslash Z(\mathbb{A})$ by restriction. It will also be denoted by μ. We shall denote the restriction of μ to E_w by μ_w, and its restriction to Z_v by μ_v.

2.5 Measures and function spaces

Choose Haar measures $dg = \otimes dg_v$, $dg^* = \otimes dg_v^*$, and $dh = \otimes dh_v$ on $G(\mathbb{A})$, $G^*(\mathbb{A})$ and $H(\mathbb{A})$ respectively, where dg_v and dg_v^* match under the isomorphism $\phi_v : G_v \xrightarrow{\sim} G_v^*$ at all finite places. We assume that the local measures give mass 1 to the hyperspecial maximal compacts for all $v \notin S$. Let $dz = \otimes_v dz_v$ be the Haar measure on $Z(\mathbb{A})$ that gives the maximal compact mass 1 everywhere. Let $d\overline{g} = \otimes_v d\overline{g}_v$ be the measure on $G(\mathbb{A})/Z(\mathbb{A})$ given by $d\overline{g}_v = dg_v/dz_v$.

For any place v and a character ω of $E_v^1 \simeq Z_v$, we define $C(G_v, \omega)$ to be the space of smooth complex-valued functions f on G_v such that f is compactly supported modulo Z_v, $f(zg) = \omega(z)^{-1}f(g)$, and if v is infinite then f is K_v-finite. If ω is a character of I_E^1, we define $C(G, \omega)$ to be the analogous space in the global case. The spaces $C(G^*, \omega)$ and $C(H, \omega)$ are defined similarly.

If π is an admissible representation of G_v with central character ω, and $f \in C(G_v, \omega)$, we define $\pi(f)$ to be

$$\pi(f) = \int_{G_v/Z_v} f(g)\pi(g)\, d\overline{g}.$$

2.6 Automorphic forms

If ω is a unitary character of $Z(F)\backslash Z(\mathbb{A}) \simeq I_E^1$, we let $L^2(G, \omega)$ be the space of square integrable complex-valued functions ϕ on $G(F)\backslash G(\mathbb{A})$ that satisfy $\phi(zg) = \omega(z)\phi(g)$. We let $L_d^2(G, \omega)$ be the subspace that decomposes discretely under the action of $G(\mathbb{A})$. We define $L_d^2(H, \omega)$ similarly, recording only the action of the subgroup Z of $Z(H)$. We denote the set of discrete L-packets on G and H by $\Pi(G)$ and $\Pi(H)$; see [Rog90, §§ 12 and 13.3], and [Fli06, p. 217], for the definition and description of these sets.

3. The packets $\Pi(\xi)$

In [Rog90, §§ 13 and 14], and [Fli06, pp. 211–218], Rogawski and Flicker define an L-packet $\Pi(\xi) \in \Pi(G)$ for every one-dimensional representation $\xi \in L_d^2(H, \omega)$ satisfying certain conditions. In this section we recall the definition and important properties of these packets.

3.1 Split finite places

Let v be a finite place that splits in E/F, so that $E_v = E_w \oplus E_{w'}$. We identify E_w with $E_{w'}$. Put $\Phi = \Phi_3$. We have

$$G_v = \{(g, h) \mid g, h \in \mathrm{GL}(3, E_w), h = \Phi^t g^{-1} \Phi^{-1}\},$$

and

$$Z_v = \{(xI, x^{-1}I) \mid x \in E_w^\times\} \simeq E_v^1 \simeq E_w^\times.$$

Note that under the identification $Z_v \simeq E_w^\times$, we have $\mu_v(x) = \mu_w(x)^2$.

Let ξ be a unitary character of $H_v \simeq \mathrm{GL}(2, E_w) \times \mathrm{GL}(1, E_w)$. Let ω denote the restriction of ξ to Z_v. If P is a parabolic subgroup of G_v with Levi H_v, the local packet $\Pi_v(\xi)$ is the unitarily induced representation $I(\xi \otimes \det_0 \circ \mu_w)$ from P to G_v [Fli06, Proposition 4, p. 279]. It has central character $\omega \otimes \mu_v$, and we shall denote it by $\pi^n(\xi)$ as in [Rog90]; it is denoted by $\pi_\xi^\times$ in [Fli06].

3.2 Nonsplit finite places

If v is a finite place that does not split in E/F and ξ is a unitary character of H_v, the local packet $\Pi_v(\xi)$ contains two representations $\pi^n(\xi)$ and $\pi^s(\xi)$. The representation $\pi^n(\xi)$ is nontempered, and unramified whenever all data are unramified, while $\pi^s(\xi)$ is cuspidal. If the restriction of ξ to Z_v is ω, both representations in $\Pi_v(\xi)$ have central character $\omega \otimes \mu_v$.

3.3 Real places

We take the following results from [Rog90, § 12.3] and [Fli06, I.5]. For any real place v, let $t_v \in \mathbb{Z}$ be such that $\mu_v(z) = (z/\overline{z})^{t_v+1/2}$.

To describe $\Pi(\xi)$ at the place v_0, we recall the classification of cohomological representations of $U(2,1;\mathbb{R})$ ([Rog90, Proposition 15.2.1], [Fli06, I.5, p. 293], and [BW00, Theorem 4.11]). If π is an irreducible unitary G_{v_0}-module, we have $H^1(\mathfrak{g}, K; \pi) = 0$ unless $\pi \in \{J^+, J^-\}$, where J^+ and J^- are nontempered. When $\pi = J^\pm$, we have $H^1(\mathfrak{g}, K; \pi) = \mathbb{C}$ with Hodge types $(1,0)$ and $(0,1)$ respectively. In addition, $H^2(\mathfrak{g}, K; \pi) = 0$ unless $\pi \in \{1, D, D^+, D^-\}$, where 1 is the trivial representation, and D, D^+, and D^- are discrete series representations with Hodge types $(1,1)$, $(2,0)$ and $(0,2)$ respectively.

For any one-dimensional representation ξ of H_{v_0}, the local packet $\Pi_{v_0}(\xi)$ is disjoint from $\{J^{\pm}\}$ unless $\xi = (\det_0)^{-t_{v_0}-1}\lambda$ (case 1) or $\xi = (\det_0)^{-t_{v_0}}\lambda^{-1}$ (case 2). In the remaining two cases, we have

$$\Pi_{v_0}(\xi) = \begin{cases} \{J^+, D^-\} & \text{in case 1,} \\ \{J^-, D^+\} & \text{in case 2.} \end{cases}$$

We will denote the nontempered member of $\Pi(\xi)$ by $\pi^n(\xi)$, and the tempered member by $\pi^s(\xi)$.

At the remaining places, we have $G_v = U(3;\mathbb{R})$. The packet $\Pi_v(\xi)$ is only defined for ξ of the form $(\det_0)^{p-t_v}\lambda^q$ with $p-q \geqslant 1$ or $q-p \geqslant 2$, and when it is, it consists of one irreducible representation of G_v which we denote $\pi^s(\xi)$. The packet $\Pi_v(\xi)$ consists of the trivial representation exactly when ξ is either $(\det_0)^{-t_v-1}\lambda$ or $(\det_0)^{-t_v}\lambda^{-1}$.

3.4 Global packets

Let $\xi \in L^2_d(H,\omega)$ be a one-dimensional representation. Define the global L-packet $\Pi(\xi)$ to be $\otimes_v \Pi_v(\xi_v)$. It is proven that $\Pi(\xi) \in \Pi(G)$ ([Rog90, Theorem 13.3.2 and § 14], and [Fli06, p. 218]), and that any representation $\pi = \otimes_v \pi_v \in L^2_d(G,\omega)$ satisfying $\pi_{v_0} \simeq J^{\pm}$ must lie in a packet $\Pi(\xi)$ for some ξ ([Rog90, Theorem 13.3.6], and [Fli06, p. 219]). If $\pi = \otimes_v \pi_v \in \Pi(\xi)$, define $n(\pi)$ to be the number of places at which $\pi_v = \pi^s(\xi_v)$. By [Rog92] and [Fli06, p. 218], there is a global factor $\varepsilon(\xi,\mu) = \pm 1$ such that

$$m(\pi) = \tfrac{1}{2}(1 + \varepsilon(\xi,\mu)(-1)^{n(\pi)}).$$

In particular, $m(\pi)$ is either 0 or 1.

3.5 Transfers and character identities

Suppose that v is finite and $f \in C(G_v,\omega)$. There exists a function $f^H \in C(H,\omega\mu_v^{-1})$, called a transfer of f, such that the unstable orbital integrals of f match the stable integrals of f^H; see [Rog90, § 4.9], and [Fli06, I.2] for details. Note that we define this transfer in the non-quasi-split case by applying the identification $\phi_v : G_v \xrightarrow{\sim} G_v^*$ defined in § 2.2 followed by the usual transfer for G^*. When ξ is a character of H_v such that the restriction of ξ to Z_v is $\omega\mu_v^{-1}$ and v is split, we have [Rog90, Lemma 4.13.1]

$$\mathrm{tr}(\pi^n(\xi))(f) = \xi(f^H),$$

and when v is nonsplit we have (see [Rog90, Corollary 12.7.4] and [Fli06, p. 215])

$$\mathrm{tr}(\pi^n(\xi))(f) + \mathrm{tr}(\pi^s(\xi))(f) = \xi(f^H). \tag{2}$$

4. Proof of Theorem 1

4.1 The upper bound

We modify our notation sightly, and now define $J^{\pm}$ to be the representation of $G_\infty = \otimes_{v|\infty} G_v$ that is equal to $J^{\pm}$ at G_{v_0} and trivial at all other places. We also define Ξ_∞ to be the set of characters of H_∞ that are equal to either $(\det_0)^{-t_v-1}\lambda$ or $(\det_0)^{-t_v}\lambda^{-1}$ at each place v. By Matsushima's formula, we have

$$b^1_{(2)}(Y(\mathfrak{n})) = \sum_{\substack{\pi \in L^2_d(G,1) \\ \pi_\infty \simeq J^{\pm}}} m(\pi) \dim(\pi_f^{K_f(\mathfrak{n})}).$$

The results recalled in § 3 allow us to rewrite this as

$$b^1_{(2)}(Y(\mathfrak{n})) = \sum_{\substack{\xi \in L^2_d(H,\mu^{-1}) \\ \xi_\infty \in \Xi_\infty}} \sum_{\substack{\pi \in \Pi(\xi) \\ \pi_\infty \simeq J^\pm}} m(\pi) \dim(\pi_f^{K_f(\mathfrak{n})}).$$

Let $1_{K(\mathfrak{n})} \in C(G(\mathbb{A}_f), 1)$ be the characteristic function of $Z(\mathbb{A}_f)K_f(\mathfrak{n})$. We have

$$\int_{G(\mathbb{A}_f)/Z(\mathbb{A}_f)} 1_{K(\mathfrak{n})}\, d\overline{g} = cV(\mathfrak{n})^{-1},$$

where $V(\mathfrak{n})$ is as in (1) and c depends only on our choice of K_v for $v \in S_f$, and so applying the upper bound $m(\pi) \leqslant 1$ (see the remark in the introduction) gives

$$b^1_{(2)}(Y(\mathfrak{n})) \ll V(\mathfrak{n}) \sum_{\substack{\xi \in L^2_d(H,\mu^{-1}) \\ \xi_\infty \in \Xi_\infty}} \sum_{\pi \in \Pi(\xi)} \operatorname{tr}(\pi_f(1_{K(\mathfrak{n})})). \tag{3}$$

We now transfer $1_{K(\mathfrak{n})}$ to a function $1^T_{K(\mathfrak{n})} = \otimes_v 1^T_{K_v(\mathfrak{n})} \in C(H(\mathbb{A}_f), \mu^{-1})$. If $v \in S_f$, we let $1^H_{K_v(\mathfrak{n})} \in C(H_v, \mu_v^{-1})$ be any transfer of $1_{K_v(\mathfrak{n})}$, and set $1^T_{K_v(\mathfrak{n})} = 1^H_{K_v(\mathfrak{n})}$. When $v \notin S$, we let K_v^H be a hyperspecial maximal compact subgroup of H_v, and let $K_v^H(\mathfrak{p}^n)$ be its standard principal congruence subgroups. We define $1_{K_v^H(\mathfrak{n})} \in C(H_v, \mu_v^{-1})$ to be the function supported on $Z_v K_v^H(\mathfrak{n})$ and equal to 1 on $K_v^H(\mathfrak{n})$. This is well defined as μ_v was assumed to be unramified. Set $1^T_{K_v(\mathfrak{n})} = Nv^{-2 \operatorname{ord}_v \mathfrak{n}} 1_{K_v^H(\mathfrak{n})}$. When v is split, the character identity

$$\operatorname{tr}(\pi^n(\xi_v))(1_{K_v(\mathfrak{n})}) = \xi_v(1^T_{K_v(\mathfrak{n})}) \tag{4}$$

may be directly verified. When v is inert, the character identity

$$\operatorname{tr}(\pi^n(\xi_v))(1_{K_v(\mathfrak{n})}) + \operatorname{tr}(\pi^s(\xi_v))(1_{K_v(\mathfrak{n})}) = \xi_v(1^T_{K_v(\mathfrak{n})}) \tag{5}$$

follows from (2) and the following proposition of Ferarri [Fer07]. We are grateful to Sug Woo Shin for making us aware of this.

Proposition 2. *If $v \notin S$ is inert, the functions $1_{K_v(\mathfrak{n})}$ and $Nv^{-2 \operatorname{ord}_v \mathfrak{n}} 1_{K_v^H(\mathfrak{n})}$ are a transfer pair.*

Proof. This is an application of [Fer07, Theorem 3.2.3] in the case $G = U(3)$ and $H = U(2) \times U(1)$. The sign $\epsilon_{G,H}$ appearing in the theorem is 1 in our case because we may take the F-tori T and T_H appearing in the definition of the character $\chi_{G,H}$ on [Fer07, p. 372] to be isomorphic. The assumption that S_f contained all primes dividing a rational prime p with $p \leqslant 9[F : \mathbb{Q}] + 1$ implies that the residual characteristic of v is 'assez grande' in the sense of [Fer07, p. 371]. □

The identities (4) and (5) and our description of the packet $\Pi(\xi)$ imply that

$$\sum_{\pi \in \Pi(\xi)} \operatorname{tr} \pi_f(1_{K(\mathfrak{n})}) = 2\xi_f(1^T_{K(\mathfrak{n})}),$$

so that (3) becomes

$$b^1_{(2)}(Y(\mathfrak{n})) \ll V(\mathfrak{n}) \sum_{\substack{\xi \in L^2_d(H,\mu^{-1}) \\ \xi_\infty \in \Xi_\infty}} \xi_f(1^T_{K(\mathfrak{n})}). \tag{6}$$

Any $\xi \in L^2_d(H, \mu^{-1})$ is of the form $\xi_\theta = (\theta \circ \det_0) \otimes (\theta^{-2}\mu^{-1} \circ \lambda)$ for some character $\theta \in \widehat{I}^1_E$, and the condition that $(\xi_\theta)_\infty \in \Xi_\infty$ restricts θ_∞ to a finite set Θ_∞. We define the conductor $\mathfrak{f}_\theta$ of θ to be the largest ideal $\mathfrak{m}$ such that θ is trivial on $U(1, \mathfrak{m})$.

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Assume that $\theta \in \widehat{I}_E^1$ satisfies $\theta_\infty \in \Theta_\infty$ and $(\xi_\theta)_f(1_{K(\mathfrak{n})}^T) \neq 0$. For $v \in S_f$, the condition $(\xi_\theta)_v(1_{K_v(\mathfrak{n})}^T) \neq 0$ and the fact that $1_{K_v(\mathfrak{n})}^T$ is a smooth function that is independent of $\mathfrak{n}$ imply that $\mathrm{ord}_v \mathfrak{f}_\theta$ is bounded by a constant depending only on K_v. If $v \notin S$, it may be easily seen that $(\xi_\theta)_v(1_{K_v(\mathfrak{n})}^T) \neq 0$ if and only if $\mathrm{ord}_v \mathfrak{f}_\theta \leqslant \mathrm{ord}_v \mathfrak{n}$. Consequently, there exists an ideal $\mathfrak{a} \subseteq \mathcal{O}$ that is divisible only by primes in S_f such that $\mathfrak{f}_\theta | \mathfrak{a}\mathfrak{n}$. The number of characters with $\theta_\infty \in \Theta_\infty$ and $\mathfrak{f}_\theta | \mathfrak{a}\mathfrak{n}$ is $\sim |U(1,\mathcal{O}) : U(1,\mathfrak{n})|$, and for each θ we have

$$(\xi_\theta)_f(1_{K(\mathfrak{n})}^T) \ll N\mathfrak{n}^{-2} |U(2,\mathcal{O}) : U(2,\mathfrak{n})|^{-1}.$$

Combining these bounds with (6) and substituting the definition of $V(\mathfrak{n})$, we obtain

$$\begin{aligned} b^1_{(2)}(Y(\mathfrak{n})) &\ll \frac{|U(1,\mathcal{O}) : U(1,\mathfrak{n})||U(3,\mathcal{O})Z(\mathbb{A}_f) : U(3,\mathfrak{n})Z(\mathbb{A}_f)|}{N\mathfrak{n}^2 |U(2,\mathcal{O}) : U(2,\mathfrak{n})|} \\ &= \frac{|U(3,\mathcal{O}) : U(3,\mathfrak{n})|}{N\mathfrak{n}^2 |U(2,\mathcal{O}) : U(2,\mathfrak{n})|}. \end{aligned}$$

The formulas for the order of the groups $\mathrm{GL}(3)$ and $U(3)$ over a finite field [Art55] imply that this is $\ll N\mathfrak{n}^3$, which completes the proof. $\square$

4.2 The lower bound

Let $\xi_\infty^0 \in \Xi_\infty$ be the character that is equal to $(\det_0)^{-t_v-1}\lambda$ at every infinite place v, so that $\Pi_{v_0}(\xi_{v_0}^0) = \{J^+, D^-\}$. Define

$$\Theta(\mathfrak{n}) = \{\theta \in \widehat{I}_E \mid \mathfrak{f}_\theta = \mathfrak{n}, (\xi_\theta)_\infty = \xi_\infty^0\} \quad \text{and} \quad \Xi(\mathfrak{n}) = \{\xi_\theta \mid \theta \in \Theta(\mathfrak{n})\}.$$

As $\mathfrak{n}$ was assumed relatively prime to S_f, $\theta \in \Theta(\mathfrak{n})$ is unramified at S_f and hence trivial at all nonsplit $v \in S_f$. Because E/F is CM, the elements $x \in \mathcal{O}_E$ with $Nx = 1$ are exactly the roots of unity in E, and it follows that $|\Xi(\mathfrak{n})| = |\Theta(\mathfrak{n})| \gg N\mathfrak{n}$.

For nonsplit $v \in S_f$, choose K_v so that $\pi^n(1_v)^{K_v}$ and $\pi^s(1_v)^{K_v}$ are both nonzero. For split $v \in S_f$ and $\xi \in \Xi(\mathfrak{n})$, $\pi^n(\xi_v)$ is the principal series representation $I(\xi_v \otimes \det_0 \circ \mu_w)$. We see that we may choose K_v so that $\pi^n(\xi_v)^{K_v} \neq 0$ for all unramified ξ_v. Matsushima's formula and the results of § 3 once again imply that

$$b^1_{(2)}(Y(\mathfrak{n})) \geqslant \sum_{\xi \in \Xi(\mathfrak{n})} \sum_{\substack{\pi \in \Pi(\xi) \\ \pi_\infty = J^+}} m(\pi) \dim(\pi_f^{K_f(\mathfrak{n})}).$$

Let $\xi \in \Xi(\mathfrak{n})$, and let I be a finite set of inert places disjoint from S. Then, because we assumed there was at least one nonsplit $v \in S_f$, there exists $\pi_I \in \Pi(\xi)$ with $\pi_{I,\infty} = J^+$ and $m(\pi_I) = 1$, and such that the set of $v \notin S$ with $\pi_{I,v} = \pi^s(\xi_v)$ is exactly I. We have assumed that $\pi_{I,v}^{K_v} \neq 0$ for all $v \in S_f$, and so π_I makes a contribution of at least

$$\prod_{v \in I} \dim(\pi^s(\xi_v)^{K_v(\mathfrak{n})}) \prod_{v \notin S \cup I} \dim(\pi^n(\xi_v)^{K_v(\mathfrak{n})})$$

to $b^1_{(2)}(Y(\mathfrak{n}))$. Summing over I, we obtain

$$b^1_{(2)}(Y(\mathfrak{n})) \geqslant \prod_{\substack{v \notin S \\ v \text{ split}}} \dim(\pi^n(\xi_v)^{K_v(\mathfrak{n})}) \prod_{\substack{v \notin S \\ v \text{ inert}}} (\dim(\pi^n(\xi_v)^{K_v(\mathfrak{n})}) + \dim(\pi^s(\xi_v)^{K_v(\mathfrak{n})})).$$

We now define $1_{K(\mathfrak{n})}^S \in C(G(\mathbb{A}^S), 1)$ to be the characteristic function of $\otimes_{v \notin S} K_v(\mathfrak{n}) Z(F_v)$, and let $1_{K(\mathfrak{n})}^{S,T} \in C(H(\mathbb{A}^S), \mu^{-1})$ be the product over the places $v \notin S$ of the transfers defined in § 4.1.

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Applying the character identities (4) and (5) and summing over $\Xi(\mathfrak{n})$ gives

$$b^1_{(2)}(Y(\mathfrak{n})) \gg V(\mathfrak{n}) \sum_{\xi \in \Xi(\mathfrak{n})} \xi^S(1^{S,T}_{K(\mathfrak{n})}).$$

We have

$$\xi^S(1^{S,T}_{K(\mathfrak{n})}) \gg N\mathfrak{n}^{-2}|U(2,\mathcal{O}) : U(2,\mathfrak{n})|^{-1}$$

when $\xi \in \Xi(\mathfrak{n})$, and reasoning as in the case of the upper bound gives $b^1_{(2)}(Y(\mathfrak{n})) \gg N\mathfrak{n}^3$.

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References

Art55 E. Artin, *The orders of the classical simple groups*, Comm. Pure Appl. Math. **8** (1955), 455–472.

BMM13 N. Bergeron, J. Millson and C. Moeglin, *The Hodge conjecture and arithmetic quotients of complex balls*, Preprint (2013), arXiv:1306.1515.

BW00 A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Mathematical Surveys and Monographs, vol 67 (American Mathematical Society, Providence, RI, 2000).

CF09 P.-S. Chan and Y. Flicker, *Cyclic odd degree base change lifting for unitary groups in three variables*, Int. J. Number Theory **5** (2009), 1247–1309.

CM12 M. Cossutta and S. Marshall, *Theta lifting and cohomology growth in p-adic towers*, Int. Math. Res. Not. IMRN (2012), doi:10.1093/imrn/rns139.

Fer07 A. Ferrari, *Théorème de l'indice et formule des traces*, Manuscripta Math. **124** (2007), 363–390.

Fli04 Y. Flicker, *Characters, genericity, and multiplicity one for U(3)*, J. Anal. Math. **94** (2004), 1–15.

Fli06 Y. Flicker, *Automorphic representations of low rank groups* (World Scientific, Hackensack, NJ, 2006).

GR91 S. Gelbart and J. Rogawski, *L-functions and Fourier–Jacobi coefficients for the unitary group* U(3), Invent. Math. **105** (1991), 445–472.

Lan36 W. Landherr, *Äquivalenze Hermitscher Formen über einem beliebigen algebraischen Zahlkörper*, Abh. Math. Sem. in Univ. Hambg. **11** (1936), 245–248.

Mok12 C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc., to appear, arXiv:1206.0882v5.

PY07 G. Prasad and S.-K. Yeung, *Fake projective planes*, Invent. Math. **168** (2007), 321–370.

Rog90 J. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123 (Princeton University Press, Princeton, NJ, 1990).

Rog92 J. Rogawski, *The multiplicity formula for A-packets*, in *The zeta functions of Picard modular surfaces* (Université de Montréal, QC, 1992), 395–419.

SX91 P. Sarnak and X. Xue, *Bounds for multiplicities of automorphic representations*, Duke Math. J. **64** (1991), 207–227.

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