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HARMONIC MORPHISMS IN NONLINEAR POTENTIAL THEORY[1]

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§ 1. Introduction

This article concerns the following problem: given a family of partial differential operators with similar structure and given a continuous mapping f from an open set Ω in $\mathbf{R}^n$ into $\mathbf{R}^n$, then when does f pull back the solutions of one equation in the family to solutions of another equation in that family? This problem is typical in the theory of differential equations when one wants to use a coordinate change to study solutions in a different environment.

To describe our objective more precisely, fix a number $1 < p < \infty$ and let $\mathbf{A}_p$ denote the family of all mappings $\mathscr{A}: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfying assumptions (1.5)–(1.9) below. In particular, we impose the growth condition $\mathscr{A}(x, h) \cdot h \approx |h|^p$. A function u defined in an open subset Ω of $\mathbf{R}^n$ is said to be *$\mathscr{A}$-harmonic* in Ω if it is a continuous solution in Ω to the quasilinear elliptic equation

$$-\operatorname{div} \mathscr{A}(x, \nabla u) = 0 . \tag{1.1}$$

That is, u belongs to the local Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega)$ and satisfies

$$\int_\Omega \mathscr{A}(x, \nabla u(x)) \cdot \nabla \varphi(x)\, dx = 0$$

for all test functions $\varphi \in C_0^\infty(\Omega)$. Equations of the form (1.1) can be viewed as measurable perturbations of the *p-Laplace* equation

$$-\operatorname{div}(|\nabla u(x)|^{p-2} \nabla u(x)) = 0 , \tag{1.2}$$

which naturally reduces to the Laplace equation $\Delta u = 0$ when $p = 2$.

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1.3. Definition. Let $\mathscr{A}^*$ and $\mathscr{A}$ belong to $\mathbf{A}_p$. A continuous mapping $f\colon \Omega \to \mathbf{R}^n$ is an *$(\mathscr{A}^*, \mathscr{A})$-harmonic morphism* if $u \circ f$ is $\mathscr{A}^*$-harmonic in $f^{-1}(\Omega')$ whenever u is $\mathscr{A}$-harmonic in Ω'. Further, f is an *A_p-harmonic morphism* if f is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism for some $\mathscr{A}^*$ and $\mathscr{A}$ in $\mathbf{A}_p$.

In recent years the nonlinear potential theory associated with $\mathscr{A}$-harmonic functions has been studied quite intensively [GLM1, 2], [HK1, 2], [HKM1, 2], [K], [LM]. The problem of determining harmonic morphisms is central to the theory which is partly motivated by its applications to quasiregular mappings. Let us recall that a continuous mapping f from an open set Ω in $\mathbf{R}^n$ into $\mathbf{R}^n$, $n \geq 2$, is *K-quasiregular* if it belongs to the Sobolev space $W^{1;n}_{\text{loc}}(\Omega)$ and satisfies the functional inequality

$$|f'(x)|^n \leq KJ_f(x) \tag{1.4}$$

almost everywhere in Ω. Here $f'(x)$ designates the formal derivative matrix of f at a point x with $|f'(x)|$ its supremum norm, and $J_f(x)$ is the Jacobian determinant.

When $n = 2$ and $K = 1$ in (1.4) we recover complex analytic functions which are harmonic morphisms for the Laplace equation. In fact, any continuous mapping f which is a harmonic morphism for the Laplacian (that is, an $(\mathscr{A}, \mathscr{A})$-harmonic morphism with $\mathscr{A}(x, h) = h$) is necessarily conformal in that it is

(i) analytic or antianalytic when $n = 2$,

(ii) of the form $f(x) = \lambda Ox + a$ for some $\lambda \in \mathbf{R}$, $a \in \mathbf{R}^n$, and an orthogonal $n \times n$ matrix O when $n > 2$.

For this result, see [GH], [F1], [I] where references to earlier works can also be found. The systematic study of harmonic morphisms in potential theory apparently began in the article [CC].

When $p = n$, the dimension of the underlying euclidean space, it is a fundamental property of quasiregular mappings that $u \circ f$ is $\mathscr{A}$-harmonic for some $\mathscr{A}$ whenever u is n-harmonic, i.e. u satisfies equation (1.2) for $p = n$. Even more is true: given any mapping $\mathscr{A} \in \mathbf{A}_n$, a new mapping $\mathscr{A}^*$ can be defined such that $\mathscr{A}^* \in \mathbf{A}_n$ and that f is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism. The explicit expression for $\mathscr{A}^*$ is given by the formula

$$\mathscr{A}^*(x, h) = J_f(x) f'(x)^{-1} \mathscr{A}(f(x), f'(x)^{-1T} h)\,,$$

if $f'(x)$ exists and is invertible, and

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$$\mathscr{A}^*(x, h) = |h|^{n-2}h ,$$

otherwise. Above, T denotes transpose. For these results as well as more information on quasiregular mappings, see [GLM1], [MRV1, 2], [Re], [Ri2].

Thus quasiregular mappings provide examples of $\mathbf{A}_n$-harmonic morphisms. Many fundamental properties of quasiregular mappings can be proved by using this morphism property only. For example, the recent proof of the Picard theorem given by Eremenko and Lewis [EL] only hinges on the fact that $\log|f(x) - b|$ is $\mathscr{A}$-harmonic in $\mathbf{R}^n$ if f is entire quasiregular and omits the point b. The natural question therefore is: are there other mappings with that property? The main theorem of this paper answers to this question negatively by asserting that *every sense-preserving* $\mathbf{A}_n$*-harmonic morphism is a quasiregular mapping.* Moreover, when $n = 2$ we can show that every $\mathbf{A}_2$-harmonic morphism is either sense-preserving or sense-reversing in each component of Ω. This characterization parallels the classical situation explained above and essentially solves the problem in the Sobolev borderline case $p = n$.

For p different from n the situation seems to be more complicated. There is a subclass of quasiregular mappings, called *mappings of bounded length distortion*, which are $\mathbf{A}_p$-harmonic morphisms for all $p > 1$ [MV]. Roughly, these are maps that are uniformly locally bilipschitz outside the branch set. We have been able to prove that if $1 < p < n$ and if f: $\Omega \to \mathbf{R}^n$ is an $\mathbf{A}_p$-harmonic morphism subject to additional topological restrictions, then f is a mapping of bounded length distortion in every compact subset of Ω. We can display a fairly trivial example showing that one cannot expect f to be a mapping of bounded length distortion in all of Ω although we do not know if that is true in a single component of Ω; in our example Ω has infinitely many components.

In the situation when $p > n$ very little is known to us. We are able to prove that in any case harmonic morphisms satisfy a maximum principle and that they map each component of Ω either onto an open set or to a point.

The paper is organized such that first we study properties of morphisms in general. Then we establish estimates for singular $\mathscr{A}$-harmonic functions (some of these estimates may have independent interest). In Section 4 we characterize $\mathbf{A}_p$-harmonic morphisms in the borderline case $p = n$ and in Section 5 we examine the case $1 < p < n$.

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TERMINOLOGY. Throughout this paper Ω will denote an open subset of $\mathbf{R}^n$, $n \geq 2$. For $1 < p < \infty$ the set $\mathbf{A}_p$ is the collection of all mappings $\mathscr{A}$ for which there exist numbers $0 < \alpha \leq \beta < \infty$ such that the following conditions hold:

(1.5) the mapping $x \mapsto \mathscr{A}(x, h)$ is measurable for all $h \in \mathbf{R}^n$ and the mapping $h \mapsto \mathscr{A}(x, h)$ is continuous for a.e. $x \in \mathbf{R}^n$;

for all $h \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$

$$\mathscr{A}(x, h) \cdot h \geq \alpha |h|^p ; \tag{1.6}$$

$$|\mathscr{A}(x, h)| \leq \beta |h|^{p-1} ; \tag{1.7}$$

$$(\mathscr{A}(x, h_1) - \mathscr{A}(x, h_2)) \cdot (h_1 - h_2) > 0 \tag{1.8}$$

whenever $h_1 \neq h_2$; and

$$\mathscr{A}(x, \lambda h) = |\lambda|^{p-2} \lambda \mathscr{A}(x, h) \tag{1.9}$$

for $\lambda \in \mathbf{R}$, $\lambda \neq 0$.

We call the parameters n, p, α, β the *structure* attached both to the mapping $\mathscr{A}$ and to equation (1.1).

If C is a compact subset of Ω, then the *p-capacity* of the *condenser* (C, Ω) is the number

$$\operatorname{cap}_p(C, \Omega) = \inf \int_\Omega |\nabla \varphi|^p dx ,$$

where the infimum is taken over all $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on C. The definition is extended in a natural way first to open subsets of Ω by taking the supremum over their compact subsets, and then to arbitrary subsets of Ω by taking the infimum over their open neighborhoods. A set E is said to have zero p-capacity if $\operatorname{cap}_p(E \cap \Omega, \Omega) = 0$ for all bounded Ω. Then a point has zero p-capacity if and only if $1 < p \leq n$. In general, if $1 < p < n$ and $0 < r \leq 1/2$, then $\operatorname{cap}_p(\overline{B}(x_0, r), B(x_0, 1)) \approx r^{n-p}$, whereas $\operatorname{cap}_n(\overline{B}(x_0, r), B(x_0, R)) = \omega_{n-1}(\log R/r)^{1-n}$.

§ 2. Basic properties of $\mathbf{A}_p$-harmonic morphisms

In this section we consider some basic properties of an $\mathbf{A}_p$-harmonic morphism $f\colon \Omega \to \mathbf{R}^n$. We first establish that under weak additional assumptions f is either an open mapping or constant in each component of Ω. This, in turn, leads to a maximum principle for $\mathbf{A}_p$-harmonic mor-

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phisms. We record a Picard type result concerning the size of the omitted set and show that the preimage of a polar set (equivalently, a set of p-capacity zero) is again a polar set. This latter result, when applied to the particular case when the polar set is one point, is crucial when we prove the discreteness of $\mathbf{A}_p$-harmonic morphisms for values $n-1<p\leq n$.

We say that a mapping $f\colon \Omega\to\mathbf{R}^n$ has the *Radó property* if $f^{-1}(y)$ has no interior points whenever $y\in\mathbf{R}^n$. Clearly every open mapping has the Radó property whereas the converse is not true; for example, let $f\colon \mathbf{R}^n\to\mathbf{R}^n$ be a projection onto any affine proper subspace.

2.1. THEOREM. *Suppose that $f\colon \Omega\to\mathbf{R}^n$ is an $\mathbf{A}_p$-harmonic morphism having the Radó property. Then f is open.*

Proof. Fix $x_0\in\Omega$ and assume, on the contrary, that there are arbitrarily small balls B centered at x_0 whose images are not neighborhoods of $f(x_0)$. Choose a ball $B=B(x_0,r)$, compactly contained in Ω, such that $f(x_0)\in\partial f(B)$ and that $f(B)\Subset B'=B(f(x_0),1)$. Because f has the Radó property, we can find a point $z_0\in B$ with $f(z_0)\neq f(x_0)$.

Suppose first that $1<p\leq n$ so that all singletons have zero p-capacity. Let $y_i\in B'\setminus f(B)$, $i=1,2,\cdots$, be a sequence of points converging to $f(x_0)$ and let u_i be a singular $\mathscr{A}$-harmonic function in $B'\setminus\{y_i\}$. More precisely, let u_i be $\mathscr{A}$-harmonic and nonnegative in $B'\setminus\{y_i\}$ such that $\lim_{z\to y_i}u_i(z)=\infty$ and that $\lim_{z\to y}u_i(z)=0$ for each $y\in\partial B'$; the existence of such a function is well known (cf. [S], [HKM2], [H]). Since $f(z_0)\neq y_i$, we may assume that $u_i(f(z_0))=1$ for each i. It then follows from Harnack's inequality that for each compact set $K\subset B'\setminus\{f(x_0)\}$ there is i_0 such that the sequence u_i, $i\geq i_0$, is a uniformly bounded family of $\mathscr{A}$-harmonic functions, hence equicontinuous on K, and we may select a subsequence, call it still u_i, which converges locally uniformly to an $\mathscr{A}$-harmonic function u_0 in $B'\setminus\{f(x_0)\}$ (see [S], [HK1], [HKM2, Chapter 6]). Because $u_i(f(z_0))=1$ for all i, we easily infer from the uniform Wiener type boundary estimate [Maz], [HKM2, Chapter 6] that $\lim_{z\to y}u_0(z)=0$ for all $y\in\partial B'$, and because $u_0(f(z_0))=1$, it follows that u_0 is not constant. Next, u_0 is nonnegative, and it is an easy consequence of the dilation invariant Harnack inequality that the limit $\lim_{z\to f(x_0)}u_0(z)$ exists; moreover, since sets of zero p-capacity are removable for bounded $\mathscr{A}$-harmonic functions [S], [HK1], the maximum principle forces this limit to be ∞.

From the definition of an $\mathbf{A}_p$-harmonic morphism we see that, for some $\mathscr{A}^* \in \mathbf{A}_p$, the functions $v_i = u_i \circ f$ are $\mathscr{A}^*$-harmonic in B; observe that no preimage of y_i meets B. Now v_i are nonnegative with $v_i(z_0) = 1$, and arguing as before we obtain a subsequence of v_i which converges locally uniformly in B to an $\mathscr{A}^*$-harmonic function v_0. But then we obviously have that $v_0 = u_0 \circ f$ in B, which implies the impossible equality $v_0(x_0) = u_0(f(x_0)) = \infty$. Thus f is open at x_0 and the proof is complete when $1 < p \leq n$.

Suppose next that $p > n$. As above we choose a sequence $y_i \in B' \setminus f(B)$ converging to $f(x_0)$. Now all points have positive p-capacity (in particular they are regular for the Dirichlet problem [Maz], [HKM2, Chapter 6]) and we let u_i be the unique $\mathscr{A}$-harmonic function in $B' \setminus \{y_i, f(z_0)\}$ with boundary values 0 on $\partial B' \cup \{f(z_0)\}$ and 1 on $\{y_i\}$. It is easily seen that u_i converges uniformly in B' to the $\mathscr{A}$-harmonic function u_0 in $B' \setminus \{f(x_0), f(z_0)\}$ which has boundary values 0 on $\partial B' \cup \{f(z_0)\}$ and 1 on $f(x_0)$; this is, in fact, again a consequence of the uniform Wiener type boundary estimate applied to singletons with positive p-capacity. Moreover, the functions $u_i \circ f$ converge to an $\mathscr{A}^*$-harmonic function v_0 in $B \setminus \{f^{-1}(f(z_0))\}$. Since $v_0 \leq 1$ and $v_0(x_0) = 1$, the maximum principle implies $v_0 \equiv 1$ in the x_0-component of $B \setminus \{f^{-1}(f(z_0))\}$. Because $\lim_{x \to \{f^{-1}(f(z_0))\}} v_0(x) = 0$, we have the desired contradiction, and the theorem follows.

An inspection of the above proof readily implies that if Ω is a domain, $x \in \Omega$, and if $f\colon \Omega \to \mathbf{R}^n$ is a nonconstant $\mathbf{A}_p$-harmonic morphism, then $f(x)$ belongs to the interior of $f(U)$ whenever U is a neighborhood of a point from the nonempty set $\partial\{f^{-1}(f(x))\} \cap \Omega$. This is more succinctly expressed in the following theorem.

2.2. THEOREM. *If Ω is a domain and $f\colon \Omega \to \mathbf{R}^n$ is a nonconstant $\mathbf{A}_p$-harmonic morphism, then for each subdomain U of Ω we have that $f(U)$ is either a point or an open set.*

Theorem 2.2 leads to the openness of $\mathbf{A}_p$-harmonic morphisms for $1 \leq p \leq n$. This result in the linear axiomatic setting is due to Fuglede [F2].

2.3. THEOREM. *Suppose that $f\colon \Omega \to \mathbf{R}^n$ is a nonconstant $\mathbf{A}_p$-harmonic morphism and Ω is connected. If $1 < p \leq n$, then f is an open mapping.*

Proof. Suppose that f is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism and that U is a nonempty open subset of Ω. We want to show that $f(U)$ is open and for that we may assume U is connected. By Theorem 2.2, $f(U)$ is either open or a point; assume $f(U) = \{y_0\}$. As in the proof of Theorem 2.1 we can construct a singular $\mathscr{A}$-harmonic function u in $B\backslash\{y_0\}$ for some ball B centered at y_0. Since f is not constant, $f(\Omega)$ is open by Theorem 2.2, and we may assume that B is compactly contained in $f(\Omega)$. Let $B' = f^{-1}(B)$. Then the pull back $v = u \circ f$ is an $\mathscr{A}^*$-harmonic function in the open set $B'\backslash f^{-1}(y_0)$, and it can be extended continuously to have the value ∞ on $f^{-1}(y_0) \cap B'$. The extended function is $\mathscr{A}^*$-superharmonic in B' which means that $f^{-1}(y_0) \cap B'$ is $\mathscr{A}^*$-polar, hence of zero p-capacity (see the discussion after Theorem 2.7). This is a contradiction since $U \subset f^{-1}(y_0)$ and since no set of zero p-capacity can have interior points. The theorem follows.

2.4. *Remark.* We do not know whether $\mathbf{A}_p$-harmonic morphisms are open mappings for $p > n$. This does not follow from Theorems 2.1 and 2.2 as there are nonconstant mappings $f: \mathbf{R}^n \to \mathbf{R}^n$ which do not have the Radó property but nevertheless map each open set U in $\mathbf{R}^n$ either onto an open set or to a point. To exhibit a particular example in $\mathbf{R}^2$, set first $g_1: \mathbf{R} \to \mathbf{R}$ by

$$g_1(t) = \begin{cases} 1 + t, & -\infty < t < -1, \\ 0, & -1 \leq t < 0, \\ t, & 0 \leq t < \infty. \end{cases}$$

Then define $g: \mathbf{R}^2 \to \mathbf{R}^2$ by $g(x) = (g_1(x_1), x_2)$ for $x = (x_1, x_2)$ and let $h: \mathbf{R}^2 \to \mathbf{R}^2$, $h(x_1, x_2) = (r, \varphi)$, be the mapping $r = |x_1|$, $\varphi = x_2/x_1$ in the polar coordinates (r, φ) of $\mathbf{R}^2$. The mapping $f = h \circ g$ has the desired properties; it maps the vertical strip $\{x \in \mathbf{R}^2: -1 \leq x_1 \leq 0\}$ to the origin. Note that h is an open mapping of $\mathbf{R}^2$ onto itself which sends the x_2-axis to 0.

Theorem 2.2 implies the following maximum principle for $\mathbf{A}_p$-harmonic morphisms.

2.5. THEOREM. *If $f: \Omega \to \mathbf{R}^n$ is an $\mathbf{A}_p$-harmonic morphism, then for each $x \in \Omega$ it holds that*

$$|f(x)| \leq \limsup_{y \to \partial\Omega} |f(y)|;$$

here ∞ is included in $\partial\Omega$ if Ω is unbounded. Moreover, if Ω is connected and if f is nonconstant, then the above inequality is strict.

The following Picard type theorem is almost an immediate consequence of the definition for $\mathbf{A}_p$-harmonic morphisms. When $p = n$,

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Theorem 4.1 in Section 4 implies that a (sense-preserving) $\mathbf{A}_p$-harmonic morphism f: $\mathbf{R}^n \to \mathbf{R}^n$ is quasiregular and hence, a fortiori, the Picard theorem for quasiregular mappings [EL], [Ri1] tells us that the omitted set is at most finite. A similar remark pertains to discrete $\mathbf{A}_p$-harmonic morphisms when $1 < p < n$ (see Theorem 5.7).

2.6. THEOREM. *Suppose that f: $\mathbf{R}^n \to \mathbf{R}^n$ is a nonconstant $\mathbf{A}_p$-harmonic morphism. If $p = n$, then $\mathbf{R}^n \setminus f(\mathbf{R}^n)$ has zero n-capacity. If $p \neq n$, then f is onto.*

Proof. If f a nonconstant $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism, then $\Omega' = f(\mathbf{R}^n)$ is open by Theorem 2.2.

If $p = n$ and the complement of Ω' has positive n-capacity, then by the Kellogg property $\partial\Omega'$ has at least two regular boundary points, say y_0 and y_1, for the $\mathscr{A}$-harmonic Dirichlet problem [K, 5.6]; see also [HKM2]. Thus the Perron method provides a bounded nonconstant $\mathscr{A}$-harmonic function u in Ω' with limit values 0 at y_0 and 1 at y_1. Consequently, the composition $u \circ f$ would be a bounded nonconstant $\mathscr{A}^*$-harmonic function in $\mathbf{R}^n$, contradicting the Liouville theorem. Thus $\mathbf{R}^n \setminus f(\mathbf{R}^n)$ has zero n-capacity as desired.

Suppose then that $p \neq n$ and that there is a point $x_0 \in \partial\Omega'$. Now we may construct a nonconstant positive $\mathscr{A}$-harmonic function u in $\mathbf{R}^n \setminus \{x_0\}$ (see the proof of [K, Lemma 3.2] or [HKM2, Chapter 7]). Then, by Harnack's inequality, $u \circ f$ is a bounded nonconstant $\mathscr{A}^*$-harmonic function in $\mathbf{R}^n$, contradicting the Liouville theorem.

Next we discuss how $\mathbf{A}_p$-harmonic morphisms pull back $\mathscr{A}$-superharmonic functions. Recall that a lower semicontinuous function u: $\Omega \to (-\infty, \infty]$ is *$\mathscr{A}$-superharmonic* in Ω if it is not identically infinite in any component of Ω and if it satisfies the comparison principle relative to $\mathscr{A}$-harmonic functions: for each domain $D \Subset \Omega$ and each $\mathscr{A}$-harmonic function h in D, $h \in C(\bar{D})$, the inequality $h \leq u$ on ∂D implies $h \leq u$ inside D. It is in fact a recent theorem of Laine [L] that if f: $\Omega \to \mathbf{R}^n$ is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism which is not constant in any of the components of Ω and if u is $\mathscr{A}$-superharmonic in an open set Ω', then, in each component of $f^{-1}(\Omega')$, $u \circ f$ is either $\mathscr{A}^*$-superharmonic or identically infinite. Laine works in an axiomatic set up but it is easy to see that the nonlinear potential theory of $\mathscr{A}$-superharmonic functions embodies his axiom system. See also [HKM2, Chapter 13]. We can record the following theorem.

2.7. THEOREM. *Suppose that* $f: \Omega \to \mathbf{R}^n$ *is an* $(\mathscr{A}^*, \mathscr{A})$*-harmonic morphism which is not constant in any of the components of* Ω. *Then the function* $u \circ f$ *is* $\mathscr{A}^*$*-superharmonic in* $f^{-1}(\Omega')$ *whenever* u *is* $\mathscr{A}$*-superharmonic in an open set* Ω'.

Before the proof we recall that a set $E \subset \mathbf{R}^n$ is $\mathscr{A}$*-polar* if there is a neighborhood U of E and an $\mathscr{A}$-superharmonic function u in U such that $u = \infty$ on E. It is known that $\mathscr{A}$-polar sets admit a characterization as sets of zero p-capacity [HK2]; see also [HKM2]. In particular, for $p > n$ the sole polar set is the empty set.

Proof. In light of the above mentioned result of Laine the only thing that prevents $u \circ f$ from being $\mathscr{A}^*$-superharmonic in $f^{-1}(\Omega')$ is the possibility that $u \circ f = \infty$ in a component U. If this happens, we deduce that u is identically infinite in the image $f(U)$ which immediately forces $p \leq n$. But $f(U)$ is open by assumption and by Theorem 2.3 violating the characterization of polar sets as sets of zero capacity. The theorem follows.

Theorem 2.7 leads to the following corollary on polar sets.

2.8. COROLLARY. *Suppose that* $f: \Omega \to \mathbf{R}^n$ *is an* $(\mathscr{A}^*, \mathscr{A})$*-harmonic morphism which is not constant in any of the components of* Ω. *If* $E \subset \mathbf{R}^n$ *is* $\mathscr{A}$*-polar, then* $f^{-1}(E)$ *is* $\mathscr{A}^*$*-polar. Equivalently, if* $E \subset \mathbf{R}^n$ *is of zero* p*-capacity, then* $f^{-1}(E)$ *is of zero* p*-capacity.*

Proof. Since the empty set is the only set of zero p-capacity for $p > n$, we may assume that $1 < p \leq n$. If E is $\mathscr{A}$-polar, there is an entire $\mathscr{A}$-superharmonic function u in $\mathbf{R}^n$ such that $u = \infty$ on E [K, Theorem 4.1]. Consequently, by Theorem 2.7, $v = u \circ f$ is an $\mathscr{A}^*$-superharmonic function in Ω with $v = \infty$ on $f^{-1}(E)$ which means that $f^{-1}(E)$ is an $\mathscr{A}^*$-polar set as asserted.

A mapping $f: \Omega \to \mathbf{R}^n$ is *light* if for each $y \in \mathbf{R}^n$ the preimage $f^{-1}(y)$ is a totally disconnected set, i.e. its components are singletons, and f is *discrete* if $f^{-1}(y)$ is a discrete set in Ω.

Since, for $1 < p \leq n$, a point always has zero p-capacity, the preimage $f^{-1}(y)$ under a nonconstant (in any component) $\mathbf{A}_p$-harmonic morphism f likewise has zero p-capacity by Corollary 2.8. In particular, if $n - 1 < p \leq n$, then $f^{-1}(y)$ has Hausdorff dimension strictly less than one (see

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Remark 2.12 (b) below) and hence cannot contain a continuum. Consequently, this and Theorem 2.3 imply

2.9. THEOREM. *Suppose that* $f\colon \Omega \to \mathbf{R}^n$ *is an* $\mathbf{A}_p$*-harmonic morphism such that* f *is not constant in any of the components of* Ω. *If* $n-1<p \leq n$, *then* f *is light and open.*

For $n=2$, light, open maps are necessarily discrete [LV, p. 244], and in any dimension open, discrete maps are either sense-preserving or sense-reversing [Ch1, 2], [V1]. Here recall that a continuous mapping $f\colon \Omega \to \mathbf{R}^n$ is *sense-preserving* (resp. *sense-reversing*) if the topological degree of f satisfies $\mu(y,f,D)>0$ (resp. $\mu(y,f,D)<0$) for every $y \in f(D)\setminus f(\partial D)$ and every domain D with compact closure in Ω. For the definition and the properties of the topological degree, see [RR] (see also [MRV1], [Re], [Ri2]). We can state the following theorem.

2.10. THEOREM. *Suppose that* $f\colon \Omega \to \mathbf{R}^2$ *is a nonconstant* $\mathbf{A}_p$*-harmonic morphism in a plane domain* Ω *and* $1<p\leq 2$. *Then* f *is discrete and open. In particular*, f *is either sense-preserving or sense-reversing.*

Theorem 2.10 means, in particular, that for $1<p\leq 2$ a plane $\mathbf{A}_p$-harmonic morphism is topologically equivalent to an analytic function, since by Stoïlow's theorem every discrete and open mapping f in the plane is of the form $f=g\circ h$ where h is a homeomorphism and g is an analytic function.

The following theorem is yet another result in this vein.

2.11. THEOREM. *Suppose that* $f\colon \Omega \to \mathbf{R}^n$ *is a nonconstant* $\mathbf{A}_p$*-harmonic morphism in a domain* Ω *and suppose that* $n-1<p\leq n$. *If* f *is sense-preserving or sense-reversing, then* f *is discrete. If* f *is discrete, then* f *is either sense-preserving or sense-reversing.*

Proof. By Theorem 2.9, f is light and open. Now a result of Titus and Young [TY, Theorem 4] tells us that sense-preserving light mappings are discrete. Since the result applies to sense-reversing mappings as well, the first assertion follows. The second assertion follows from the above mentioned results of Chernavskii and Väisälä [Ch1, 2], [V1].

2.12. *Remarks.* (a) It is clear that the conclusions of Theorem 2.7 and Corollary 2.8 are not necessarily true if $1<p\leq n$ and if f maps a component of Ω onto a point. However, for $p>n$ the assertions are retained for any $\mathbf{A}_p$-harmonic morphism f, because then $\mathscr{A}$-superharmonic functions are real-valued and $\mathscr{A}$-polar sets empty.

(b) Corollary 2.8 admits a formulation in terms of Hausdorff measures: If $f: \Omega \to \mathbf{R}^n$ is an $\mathbf{A}_p$-harmonic morphism and $E \subset \mathbf{R}^n$ has finite Hausdorff $(n-p)$-measure, then $f^{-1}(E)$ has Hausdorff dimension at most $n-p$. This follows from the well known relations between Hausdorff measures and capacities; see e.g. [HKM2], [Re, p. 120], [V2].

(c) We do not know whether sense-preserving $\mathbf{A}_p$-harmonic morphisms are discrete in general. This does not follow from Corollary 2.8 combined with the Titus-Young theorem because for $1 < p \leq n-1$ sets of zero p-capacity may contain nondegenerate continua. Nor do we know whether $\mathbf{A}_p$-harmonic morphisms are light in general.

(d) Because quasiregular mappings ($p = n$) and mappings of bounded length distortion (all p's) are $\mathbf{A}_p$-harmonic morphism, the above results can be applied to those mappings. In this case, however, such results are well known; see [MRV1, 2], [Re], [Ri2], [MV].

§ 3. Estimates for singular $\mathscr{A}$-harmonic functions

Our basic strategy in investigating the local distortion of $\mathbf{A}_p$-harmonic morphisms is to analyze the behavior of singular solutions. In this section we prove some auxiliary results concerning the asymptotic behavior of $\mathscr{A}$-harmonic functions near an essential isolated singularity. Results of this sort appear e.g. in [H], [Re], [S], where we also refer to for the construction of singular solutions.

Throughout this section we assume that $1 < p \leq n$ and that $\mathscr{A} \in \mathbf{A}_p$ with structure constants $0 < \alpha \leq \beta < \infty$. The letter c will stand for various constants depending only on (n, p, β, α), and the expression $a \approx b$ means $c^{-1}a \leq b \leq ca$.

By a *singular* $\mathscr{A}$-harmonic function u in a ball B will *singularity* $\xi \in B$ we mean a positive $\mathscr{A}$-harmonic function u in a punctured ball $B \setminus \{\xi\}$ with $\lim_{x \to \xi} u(x) = \infty$. We set $u(\xi) = \infty$; then u becomes $\mathscr{A}$-superharmonic in B. For simplicity we also assume that u is bounded near ∂B so that for all large $a > 0$ the set $\{u > a\}$ is compactly contained in B. We define the *flux* of u at ξ by the formula

$$\operatorname{flux}(u) = \int_B \mathscr{A}(x, \nabla u) \cdot \nabla \varphi \, dx$$

where $\varphi \in C_0^\infty(B)$ is such that $\varphi = 1$ in a neighborhood of ξ. Then $\operatorname{flux}(u)$

does not depend on the particular choice of φ, and by approximating we may allow φ to be a compactly supported function in $W_0^{1,p}(B)$ with $\varphi = 1$ in a neighborhood of ξ.

For the next two lemmas we assume that u is a singular $\mathscr{A}$-harmonic function in a ball B with singularity ξ.

3.1. LEMMA. *There is a constant $c = c(n, p, \beta, \alpha) > 0$ such that for all numbers $0 < b < a$ for which the set $\{u > b\}$ is compactly contained in B we have*

$$\frac{1}{c}\operatorname{flux}(u)(a-b)^{1-p} \le \operatorname{cap}_p(\{u \ge a\}, \{u > b\}) \le c\operatorname{flux}(u)(a-b)^{1-p}.$$

Proof. Consider the function

$$v(x) = \max\left\{\min\left\{\frac{u(x)-b}{a-b}, 1\right\}, 0\right\}.$$

Combining the quasiminimizing property of $\mathscr{A}$-harmonic functions with the obvious fact that v is an admissible test function for the condenser $(\{u \ge a\}, \{u > b\})$, we find that

$$\operatorname{cap}_p(\{u \ge a\}, \{u > b\}) \le \int_{\{b<u<a\}} |\nabla v|^p dx \le c\operatorname{cap}_p(\{u \ge a\}, \{u > b\}).$$

The quasiminimizing property in this case means

$$\int_{\{b<u<a\}} |\nabla v|^p dx \le c\int_{\{b<u<a\}} |\nabla \psi|^p dx$$

for any function ψ with $v - \psi \in W_0^{1,p}(\{b < u < a\})$; this is an easy consequence of Hölder's inequality. Now because $\mathscr{A}(x, \nabla v)\cdot\nabla v \approx |\nabla v|^p$ and because

$$\int_B \mathscr{A}(x, \nabla v)\cdot\nabla v\, dx = (a-b)^{1-p}\int_B \mathscr{A}(x, \nabla u)\cdot\nabla v\, dx = (a-b)^{1-p}\operatorname{flux}(u),$$

the lemma follows.

For all small enough $r > 0$ write

$$M(r) = \max_{|x-\xi|=r} u(x)$$

and

$$m(r) = \min_{|x-\xi|=r} u(x).$$

3.2. LEMMA. *There is a constant $c = c(n, p, \beta, \alpha) > 0$ such that for* *all sufficiently small* $r > 0$ *we have*

$$M(r) - m(r) \leq c \operatorname{flux}(u)^{1/(n-1)},$$

when $p = n$, *and*

$$\frac{1}{c} r^{(p-n)/(p-1)} \operatorname{flux}(u)^{1/(p-1)} \leq m(r) \leq M(r) \leq c r^{(p-n)/(p-1)} \operatorname{flux}(u)^{1/(p-1)},$$

when $1 < p < n$.

Proof. Assume first that $p = n$. We let $r > 0$ be so small that $\{u \geq m(r)\}$ is compact in B and apply Lemma 3.1 (note that there is nothing to prove if $M(r) = m(r)$). Indeed, by the maximum principle the set $\{u \geq M(r)\}$ is connected and meets both ξ and the sphere $\partial B(\xi, r)$, while the boundary of the set $\{u > m(r)\}$ meets the same sphere. A standard symmetrization argument (see [G] or [Re, Theorem 3.5, p. 121]) yields

$$\operatorname{cap}_n(\{u \geq M(r)\}, \{u > m(r)\}) \geq c(n) > 0$$

and the assertion for $p = n$ follows from Lemma 3.1.

Next consider the case when $1 < p < n$. Let $a > 0$ be a number large enough such that $\{u \geq a\}$ is compact in B. We choose $r > 0$ so small that $m(r) > 2a$ and set

$$v(x) = \min\left\{\frac{u(x) - a}{M(r) - a}, 1\right\}.$$

Then v is the unique $\mathscr{A}$-harmonic function in the open set $\{u > a\} \setminus \{u \geq M(r)\}$ having boundary values 1 and 0 on the inner and outer boundary components, respectively. The capacity estimate [HK3, Lemma 3.2] implies that

$$\operatorname{cap}_p(\{u \geq M(r)\}, \{u > a\}) \geq c\left(\frac{m(r) - a}{M(r) - a}\right)^{p-1} \operatorname{cap}_p\left(\left\{v \geq \frac{m(r) - a}{M(r) - a}\right\}, \{u > a\}\right),$$

and the dilation invariant Harnack inequality guarantees that for all small $r > 0$ we have

$$M(r) \leq c m(r).$$

Hence

$$\operatorname{cap}_p(\{u \geq \{M(r)\}, \{u > a\}) \geq c \operatorname{cap}_p(\{u \geq m(r)\}, \{u > a\}).$$

Next, the maximum principle implies the inclusions

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$$\{u \geq M(r)\} \subset \bar{B}(\xi, r) \subset \{u \geq m(r)\}$$

which means that the above capacities can be estimated from above and below by a constant times the capacity $\mathrm{cap}_p(\bar{B}(\xi, r), \{u > a\})$, which behaves asymptotically like r^{n-p} as $r \to 0$. Putting these estimates together and again invoking the quasiminimizing property of $\mathscr{A}$-harmonic functions, we obtain

$$\int_{\{u>a\}} \mathscr{A}(x, \nabla v) \cdot \nabla v \, dx \approx \mathrm{cap}_p(\{u \geq M(r)\}, \{u > a\}) \approx r^{n-p}.$$

Since

$$\int_{\{u>a\}} \mathscr{A}(x, \nabla v) \cdot \nabla v \, dx = (M(r) - a)^{1-p} \int_{\{u>a\}} \mathscr{A}(x, \nabla u) \cdot \nabla v \, dx \approx M(r)^{1-p} \,\mathrm{flux}\,(u)$$

and since

$$M(r) \approx m(r),$$

the lemma follows.

3.3. LEMMA. *Suppose that u is a singular $\mathscr{A}$-harmonic function in $B(x_0, R)$ with singularity at $\xi \in B(x_0, R/4)$. Suppose further that $u(x) \to 0$ as $x \to \partial B(x_0, R)$ and that $u(z_0) = 1$ for some fixed point z_0 with $|z_0 - x_0| = R/2$. Then there is a constant $c = c(n, p, \beta, \alpha) > 0$ such that*

$$\frac{1}{c} R^{n-p} \leq \mathrm{flux}\,(u) \leq cR^{n-p}.$$

Proof. If M denotes the maximum value of u on the sphere $\partial B(x_0, R/2)$ and m the minimum value there, we know from Harnack's inequality that $m \leq 1 \leq M \leq cm$. Hence Lemma 3.1 implies

$$\mathrm{flux}\,(u) \approx \mathrm{cap}_p(\{u \geq M\}, B(x_0, R)) \approx \mathrm{cap}_p(\{u \geq m\}, B(x_0, R)).$$

On the other hand, by the maximum principle the set $\{u \geq M\}$ is contained in the closed ball $\bar{B}(x_0, R/2)$ and the set $\{u \geq m\}$ contains the same ball, so that

$$\mathrm{cap}_p(\{u \geq M\}, B(x_0, R)) \leq \mathrm{cap}_p(\bar{B}(x_0, R/2), B(x_0, R)) \leq \mathrm{cap}_p(\{u \geq m\}, B(x_0, R)).$$

Because

$$\mathrm{cap}_p(\bar{B}(x_0, R/2), B(x_0, R)) \approx R^{n-p},$$

the assertion follows.

3.4. LEMMA. *Suppose that $p = n$ and that u is a singular $\mathscr{A}$-harmonic function in $B = B(x_0, R)$ as in Lemma 3.3. For all sufficiently large positive numbers $a > 0$ the following holds: if $B(\xi, \lambda)$ is the largest open ball centered at ξ and contained in the set $\{u > a\}$, and if $\bar{B}(\xi, \Lambda)$ is the smallest closed ball centered at ξ and containing the set $\{u \geq a\}$, then*

$$\frac{\Lambda}{\lambda} \leq c < \infty .$$

Proof. From the preceding lemma we obtain

$$0 < \frac{1}{c} \leq \operatorname{flux}(u) \leq c , \tag{3.5}$$

where $c = c(n, \alpha, \beta) > 0$. Next, fix $a > 0$ large and let $0 < \lambda \leq \Lambda < \infty$ be the radii as described in the claim. (It is enough to assume $a > \max_{|x-x_0|=R/4} u(x)$ so that the closed ball $\bar{B}(\xi, \Lambda)$ lies inside $B(x_0, R)$). We may obviously assume that the middle inequality is strict. Let M_1 denote the maximum value of u on the sphere $\partial B(\xi, \lambda)$ and m_1 the minimum value on the sphere $\partial B(\xi, \Lambda)$. Then by the maximum principle $M_1 > a > m_1$, and Lemma 3.1 together with (3.5) implies that

$$\operatorname{cap}_n(\{u \geq M_1\}, \{u > a\}) \approx (M_1 - a)^{1-n}$$

and that

$$\operatorname{cap}_n(\{u \geq a\}, \{u > m_1\}) \approx (a - m_1)^{1-n} .$$

To estimate these capacities, we employ the symmetrization device once more and conclude that they both are bounded from below by a dimensional constant. Hence

$$M_1 - m_1 = M_1 - a + a - m_1 \leq c < \infty .$$

Using this inequality, Lemma 3.1, (3.5), and a trivial capacity estimate, we infer

$$0 < c \leq (M_1 - m_1)^{1-n} \leq c \operatorname{cap}_n(\{u \geq M_1\}, \{u > m_1\}) \leq c\left(\log \frac{\Lambda}{\lambda}\right)^{1-n}$$

which gives the desired result.

In the final lemma in this section we need to consider a situation which is slightly more general than that above. We say that an $\mathscr{A}$-harmonic function u has singularities $\{\xi_1, \cdots, \xi_k\} \subset B$ if u is $\mathscr{A}$-harmonic in $B\setminus\{\xi_1, \cdots, \xi_k\}$ and $\lim_{x\to\xi_i} u(x) = \infty$ for all i. We define the flux of u by the same integral formula

$$\operatorname{flux}(u) = \int_B \mathscr{A}(x, \nabla u)\cdot\nabla\varphi\, dx$$

as before; the test function $\varphi \in C_0^\infty(B)$ now satisfies $\varphi = 1$ in a neighborhood of each singularity ξ_i.

3.6. LEMMA. *Suppose that K is a compact subset of a ball B and that u and u_j, $j = 1, 2, \cdots$, are nonnegative singular $\mathscr{A}$-harmonic functions whose singularities are contained in K. Then if $u_j \to u$ pointwise in $B\setminus K$, we have that* $\operatorname{flux}(u_j) \to \operatorname{flux}(u)$ *as* $j \to \infty$.

Proof. Fix a test function $\varphi \in C_0^\infty(B)$ such that $\varphi = 1$ in a neighborhood of K. Then the functions u_j are $\mathscr{A}$-harmonic in an open neighborhood of $\operatorname{spt}\nabla\varphi$, the support of $\nabla\varphi$, and since for $\mathscr{A}$-harmonic functions pointwise convergence implies locally uniform convergence, we have that $u_j \to u$ uniformly on $\operatorname{spt}\nabla\varphi$. This implies (see [HK1, 2.32]) that

$$\mathscr{A}(x, \nabla u_j) \longrightarrow \mathscr{A}(x, \nabla u)$$

weakly in $L^{p/(p-1)}(\operatorname{spt}\nabla\varphi)$. In particular, we have that

$$\lim_{j\to\infty}\operatorname{flux}(u_j) = \lim_{j\to\infty}\int_{\operatorname{spt}\nabla\varphi} \mathscr{A}(x, \nabla u_j)\cdot\nabla\varphi\, dx = \int_{\operatorname{spt}\nabla\varphi} \mathscr{A}(x, \nabla u)\cdot\nabla\varphi\, dx = \operatorname{flux}(u)$$

as required.

§ 4. $\mathbf{A}_n$-harmonic morphisms are quasiregular

In this section we establish our main theorem in this paper.

4.1. THEOREM. If $f\colon \Omega \to \mathbf{R}^n$ *is a sense-preserving* $\mathbf{A}_n$*-harmonic morphism, then f is quasiregular.*

Since by Theorem 2.10 every $\mathbf{A}_2$-harmonic morphism in the plane is either sense-preserving or sense-reversing, we obtain

4.2. COROLLARY. *If $f\colon \Omega \to \mathbf{R}^2$ is an $\mathbf{A}_2$-harmonic morphism in a domain Ω, then either f or f composed with a sense-reversing reflection is a quasiregular mapping. In particular, f is of the form $\varphi\circ h$, where h is a quasiconformal homeomorphism and φ is either analytic or anti-analytic.*

To prove Theorem 4.1, we employ the geometric definition for quasiregular mappings given in [MRV1] (see also [Ri2, Section II.6]). A mapping $f\colon \Omega \to \mathbf{R}^n$ is quasiregular if and only if in each component of Ω either f is constant or the following three conditions hold:

(i) f is sense-preserving, discrete, and open;

(ii) the local distortion $H(x, f)$ is locally bounded in Ω;

(iii) there is a real number a such that $H(x, f) \le a$ for a.e. $x \in \Omega \setminus B_f$.

Here

$$H(x, f) = \limsup_{r \to 0} \frac{L(x, r, f)}{l(x, r, f)}$$

with

$$L(x, r, f) = \sup_{|y-x|=r} |f(y) - f(x)|, \qquad l(x, r, f) = \inf_{|y-x|=r} |f(y) - f(x)|,$$

and B_f, the *branch set* of f, is the set where f fails to be a local homeomorphism.

For the rest of this section we assume that $f\colon \Omega \to \mathbf{R}^n$ is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism in an open set Ω for some $\mathscr{A}$ and $\mathscr{A}^*$ from $\mathbf{A}_n$. We let α, β and α^*, β^* denote the structure constants of $\mathscr{A}$ and $\mathscr{A}^*$, respectively, and as usual c stands for various constants depending only on α, β, α^*, β^*, and n.

Proof of (i): This is contained in Theorems 2.3 and 2.11.

Proof of (ii): Fix $x_0 \in \Omega$ and let B' be a ball with radius 1 centered at $y_0 = f(x_0)$. Then choose $r_0 > 0$ so small that $B = B(x_0, r_0)$ is compactly contained in Ω and that $U = f(B) \subset \frac{1}{4}B'$. Fix a normalization point z_0 in B' such that $|z_0 - y_0| = 1/2$ and then for each $y \in U$ choose a singular $\mathscr{A}$-harmonic function u_y with singularity at y such that $u_y(z_0) = 1$; moreover, we demand that $\lim_{z \to \partial B'} u_y(z) = 0$. It follows from Lemma 3.3 that the flux of u_y is bounded from above and from below by constants independent of y. Consequently, we deduce from Lemma 3.1 that

$$\frac{1}{c}(a - b)^{1-n} \le \operatorname{cap}_n(\{u_y \ge a\}, \{u_y > b\}) \le c(a - b)^{1-n} \tag{4.3}$$

for all numbers $0 < b < a$.

Now fix $x_1 \in B$ and $r > 0$ so small that $B(x_1, r) \subset B$ and $\{f^{-1}(f(x_1))\} \cap B(x_1, r) = \{x_1\}$; this is possible because f is discrete. Write $y_1 = f(x_1)$, $u = u_{y_1}$, and $v = u \circ f$. Then v is a singular $\mathscr{A}$-harmonic function in $B(x_1, r)$ with singularity at x_1. Let

$$M = M(r) = \max_{|x-x_1|=r} v(x), \qquad m = m(r) = \min_{|x-x_1|=r} v(x).$$

Since

$$M - m \leq c \operatorname{flux}(v)^{1/(n-1)}$$

by Lemma 3.2, it follows from (4.3) that

$$\text{(4.4)} \qquad \operatorname{cap}_p(\{u \geq M\}, \{u > m\}) \geq \frac{c}{\operatorname{flux}(v)} > 0 .$$

Write

$$L = L(x_1, r, f), \qquad l = l(x_1, r, f).$$

Let λ_1 be the least radius such that the set $\{u \geq M\}$ is contained in the closed ball $\bar{B}(y_1, \lambda_1)$, and let Λ_1 be the radius of the largest open ball $B(y_1, \Lambda_1)$ contained in the set $\{u > m\}$. Since $\{u = M\}$ meets the boundary of the ball $B(y_1, l)$ and $\{u = m\}$ meets the boundary of the ball $B(y_1, L)$, we obtain from Lemma 3.4 that

$$\text{(4.5)} \qquad \lambda_1 \leq cl, \qquad L \leq c\Lambda_1 .$$

Because we may clearly assume that the ratio L/l is so large that $\lambda_1 < \Lambda_1$, we have

$$\operatorname{cap}_n(\{u \geq M\}, \{u > m\}) \leq \operatorname{cap}_n(\bar{B}(y_1, \lambda_1), B(y_1, \Lambda_1)) = \omega_{n-1}\left(\log \frac{\Lambda_1}{\lambda_1}\right)^{1-n},$$

and an appeal to (4.4) yields

$$\frac{\Lambda_1}{\lambda_1} \leq \exp\{c \operatorname{flux}(v)^{1/(n-1)}\}.$$

Combining this with (4.5), we arrive at

$$\frac{L}{l} \leq C \exp\{c \operatorname{flux}(v)^{1/(n-1)}\},$$

and since the right hand side does not depend on r, we have shown that

$$H(x_1, f) \leq C \exp\{c \operatorname{flux}(v)^{1/(n-1)}\} < \infty ,$$

where C and c depend only on α, β, α^*, β^*, and n.

To complete the proof of (ii) we need to verify that the mapping

$$\Phi\colon B \to (0, \infty), \qquad x \mapsto \Phi(x) = \operatorname{flux}(u_{f(x)} \circ f)$$

satisfies

$$\sup_{|x - x_0| \leq r_0/2} \Phi(x) < \infty .$$

Suppose, on the contrary, that there is a sequence $x_j \in B(x_0, r_0/2)$ such that $\Phi(x_j) \to \infty$. We may assume that $x_j \to \xi \in \bar{B}(x_0, r_0/2)$. Due to the normalization $u_{f(x_j)}(z_0) = 1$, we may use Harnack's inequality and equicontinuity to select a subsequence u_i of $u_{f(x_j)}$ such that u_i converges uniformly on compact subsets of $B'\setminus\{f(\xi)\}$ to an $\mathscr{A}$-harmonic function u. As easily seen, $\lim_{y\to f(\xi)} u(y) = \infty$.

Since f is discrete, open, and sense-preserving, there is a neighborhood U of ξ such that U is compactly contained in B, $f(U) = B(f(\xi), s)$ for some $s > 0$, $\{f^{-1}(f(\xi))\} \cap \bar{U} = \{\xi\}$, and every point in $B(f(\xi), s)$ has at most $i(\xi, f) = k$ preimages in U; here $i(\xi, f)$ is the local topological index at ξ (see [MRV1, Lemma 2.12] or [Ri2, Section I.4]). Then for $i \geq i_0$ the functions $v_i = u_i \circ f$ are singular $\mathscr{A}$-harmonic functions in a ball $B(\xi, t) \subset U$ with at most k singularities at points $\{f^{-1}(f(x_i))\}$. Since $f(x_i) \to f(\xi)$,

$$v_i = u_i \circ f \to u \circ f = v$$

pointwise in $B(\xi, t)\setminus\{\xi\}$, and we readily obtain from Lemma 3.6 that

$$\infty = \lim_{i\to\infty} \operatorname{flux}(v_i) = \operatorname{flux}(v) < \infty\,,$$

a contradiction. Thus (ii) is proved.

Proof of (iii). Fix a point x_0 in $\Omega\setminus B_f$ and choose a small ball $B = B(x_0, r)$ contained in Ω such that f is a homeomorphism in $\bar{B}$. Then pick two points x_1 and x_2 from ∂B such that

$$L = L(x_0, r, f) = |f(x_1) - f(x_0)|\,, \qquad l = l(x_0, r, f) = |f(x_2) - f(x_0)|\,.$$

We want to show that the ratio of L and l is bounded from above independently of x_0 and r. This is achieved by a (nonlinear) harmonic measure argument.

Let $\mathscr{C}$ be a spherical cap on $\partial B(x_0, r)$ containing x_1 and x_2 such that one hemisphere of $\partial B(x_0, r)$ does not intersect $\mathscr{C}$. Then $f(\mathscr{C}) = \mathscr{C}'$ is a continuum joining the two spheres $\partial B(y_0, l)$ and $\partial B(y_0, L)$, where $y_0 = f(x_0)$. Let $\omega = \omega(\mathscr{C}', B(y_0, L); \mathscr{A})$ be the $\mathscr{A}$-harmonic measure of $\mathscr{C}'$ in $B(y_0, L)\setminus\mathscr{C}'$. That is, ω is the Perron solution in $B(y_0, L)\setminus\mathscr{C}'$ to the $\mathscr{A}$-harmonic Dirichlet problem with boundary values 1 on $\mathscr{C}'$ and 0 elsewhere. Since $\mathscr{C}'$ is a continuum, the known boundary estimates for $\mathscr{A}$-harmonic measures (see [Maz], [GLM2], and, in particular, [Mar, Theorem 2.18]) imply that

$$\omega(y_0) \geq 1 - c\left(\log \frac{L}{L-l}\right)^{\delta}$$

where c and δ are positive constants depending only on the structure of $\mathscr{A}$. On the other hand, since f is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism, the function $\omega^* = \omega \circ f$ is the $\mathscr{A}^*$-harmonic measure of $\mathscr{C}$ in $D \backslash \mathscr{C}$, where D is the x_0-component of $f^{-1}(B(y_0, L))$ (since x_0 lies outside the branch set, we may assume that r is small enough so that f^{-1} defines a homeomorphism from $B(y_0, L)$ onto D). Now let F be an unbounded continuum in the complement of D such that F contains the point x_1 and let $\hat{\omega}$ be the $\mathscr{A}^*$-harmonic measure of $\mathscr{C}$ in $\mathbf{R}^n \backslash (\mathscr{C} \cup F)$. Then by comparison,

$$\hat{\omega}(x_0) \geq \omega^*(x_0) = \omega(y_0) \geq 1 - c\left(\log \frac{L}{L-l}\right)^{\delta}.$$

To complete the proof we need to show that $\hat{\omega}(x_0)$ is less than some constant $c < 1$ which depends only on the structure of $\mathscr{A}^*$. This is a consequence of a standard iteration of Harnack's inequality. Indeed, by construction $\mathscr{C}$ lies in a half space bounded by an affine hyperplane going through x_0. It follows from elementary geometric considerations that one can join x_0 to a point in F by a chain of 10 balls B_i such that $2B_i$ does not meet $\mathscr{C}$; the last ball in the chain, say B_0, can be assumed to be centered at a point on F. Since $\hat{\omega}$ vanishes continuously on F, invoking the boundary estimates [Maz], [GLM2] once more, we have that $\hat{\omega} \leq c < 1$ in B_0. Then applying Harnack's inequality ten times we arrive at the desired estimate $\hat{\omega}(x_0) \leq c < 1$.

This completes the proof of (iii), hence that of the theorem.

4.6. *Remark.* The proof of (iii) above reveals that outside the branch set the dilatation of an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism is controlled by a constant that depends only on n and the structure constants of $\mathscr{A}$ and $\mathscr{A}^*$. Since the branch set of a nonconstant quasiregular mapping has zero Lebesgue n-measure, we deduce that, if f in the situation of Theorem 4.1 is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism, the dilatation of f depends only on n and the structure constants of $\mathscr{A}$ and $\mathscr{A}^*$.

A homeomorphic quasiregular mapping is usually said to be quasiconformal. Here it is advantageous to call a homeomorphism $f\colon \Omega \to \mathbf{R}^n$ *quasiconformal* if it belongs to $W^{1,n}_{\mathrm{loc}}(\Omega)$ and for some $K < \infty$ the inequality

$$|f'(x)|^n \leq K|J_f(x)|$$

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hold almost everywhere in Ω; this definition does not exclude sense-reversing mappings. From Theorem 4.1 and similarly for sense-reversing mappings (see also [MRV1, p. 11]) we obtain

4.7. COROLLARY. *If $f\colon \Omega \to \mathbf{R}^n$ is a homeomorphic $\mathbf{A}_n$-harmonic morphism, then f is quasiconformal.*

4.8. *Remark.* J. Manfredi has shown that every n-harmonic morphism is necessarily 1-quasiregular, hence a Möbius transformation in dimensions $n \geq 3$. By an n-harmonic morphism we mean an $(\mathscr{A}, \mathscr{A})$-harmonic morphism with $\mathscr{A}(x, h) = |h|^{n-2}h$. His proof uses delicate properties of singular n-harmonic functions.

§ 5. $\mathbf{A}_p$-harmonic morphisms for $1 < p < n$

We recall that a mapping $f\colon \Omega \to \mathbf{R}^n$ is of *L-bounded length distortion* or L-BLD if the coordinate functions belong to the Sobolev space $W^{1,1}_{\text{loc}}(\Omega)$, if $J_f(x) \geq 0$ a.e. in Ω, and if there is an $L \geq 1$ such that the inequality

$$|h|/L \leq |f'(x)h| \leq L|h| \tag{5.1}$$

holds for all h in $\mathbf{R}^n$ and a.e. x in Ω. Here again $f'(x)$ denotes the formal derivative of f and $J_f(x)$ its Jacobian determinant.

It is obvious that condition (5.1) is more restrictive than the inequality required for quasiregularity, and we see that BLD mappings form a strict subclass of quasiregular mappings. In fact, a mapping $f\colon \Omega \to \mathbf{R}^n$ is L-BLD if and only if it is quasiregular and satisfies

$$|f'(x)| \leq L\,, \qquad l(f'(x)) \geq 1/L \tag{5.2}$$

a.e. in Ω. Here $l(f'(x)) = \inf_{|h|=1}|f'(x)h|$. Note in particular that a BLD mapping is never constant. These mappings were introduced in [MV] and the name reflects the path length preserving property they have. For these results and more information on BLD mappings, see [MV].

It was proved in [MV] that BLD mappings are $\mathbf{A}_p$-harmonic morphisms for *all* $p > 1$. Namely, given any p and any mapping $\mathscr{A}$ in $\mathbf{A}_p$, there is a mapping $\mathscr{A}^*$ in $\mathbf{A}_p$ such that f is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism. It seems reasonable to expect that $\mathbf{A}_p$-harmonic morphisms are BLD mappings when $p \neq n$, but the following example shows that this hope is futile.

5.3. EXAMPLE. Let Ω be a disjoint collection of open balls B_i in $\mathbf{R}^n$,

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$i = 1, 2, \cdots$. Define a mapping $f\colon \Omega \to \mathbf{R}^n$ by setting $f(x) = ix$ for x in B_i. Then it is easily seen that for all $1 < p < \infty$ the mapping f is an $(\mathscr{A}, \mathscr{A})$-harmonic morphism for $\mathscr{A} = |h|^{p-2}h$ (in other words, f preserves p-harmonic functions) but f is not L-BLD for any fixed L in Ω.

In this example, however, f is BLD in each component of Ω. We do not know whether, for $p \neq n$, there is a nonconstant $\mathbf{A}_p$-harnomic morphism in a connected open set which is not BLD.

In contrast to the borderline case $p = n$, our knowledge of $\mathbf{A}_p$-harmonic morphisms for $1 < p < n$ is incomplete. However, we have the following partial result.

5.4. THEOREM. *Suppose that $1 < p < n$ and that $f\colon \Omega \to \mathbf{R}^n$ is a nonconstant sense-preserving $\mathbf{A}_p$-harmonic morphism in a domain Ω. If either $n - 1 < p < n$ or f is discrete, then f is BLD on every compact subset of Ω. In particular, any homeomorphic $\mathbf{A}_p$-harmonic morphism for $1 < p < n$ is locally bilipschitz.*

A mapping f is locally bilipschitz in our terminology if every point x has a neighborhood U such that for some constant $L = L(U)$ the double inequality

$$|z - y|/L \leq |f(z) - f(y)| \leq L|z - y|$$

holds for all pairs of points z, y in U; similarly, f is locally Lipschitz, if only the right inequality holds. Then it is clear that the last assertion in the theorem follows from the first. We do not know if for every homeomorphic $\mathbf{A}_p$-harmonic morphism f with $1 < p < n$ the above double inequality is true for L independent of the neighborhood U.

To prove Theorem 5.4, we employ a geometric characterization of BLD mappings akin to the quasiregular case. For $x \in \Omega$ we define

$$L(x, f) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}, \qquad l(x, f) = \liminf_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.$$

Then f is L-BLD if and only if it is in $W^{1,1}_{\mathrm{loc}}(\Omega)$, $J_f(x) \geq 0$ a.e. in Ω and

$$L(x, f) \leq L, \qquad l(x, f) \geq 1/L$$

for all x in Ω [MV, Theorem 2.16].

Proof of Theorem 5.4. Suppose that $f\colon \Omega \to \mathbf{R}^n$ is an $(\mathscr{A}^*, \mathscr{A})$-harmonic morphism for some $\mathscr{A}$ and $\mathscr{A}^*$ in $\mathbf{A}_p$. Denote the structure constants of

$\mathscr{A}$ and $\mathscr{A}^*$ by α, β and α^*, β^*, respectively, and recall the convention $c = c(n, p, \alpha, \beta, \alpha^*, \beta^*)$ with $a \approx b$ meaning $c^{-1}a \le b \le ca$.

First observe that if $n - 1 < p < n$, then f is discrete by Theorem 2.11.

Next, pick a point $x_0 \in \Omega$ and write $y_0 = f(x_0)$. Let B' be a ball of radius 1 centered at y_0 and let $r_0 > 0$ be so small that $f(B(x_0, r_0)) = U$ is contained in $\frac{1}{4}B'$. For each y in U choose a singular $\mathscr{A}$-harmonic function u_y as in the proof of Theorem 4.1. That is, u_y has singularity at y, there is a point z_0 such that $|z_0 - y_0| = 1/2$ and $u_y(z_0) = 1$, and $\lim_{z \to \partial B'} u_y(z) = 0$. Then Lemma 3.3 implies

$$0 < \frac{1}{c} \le \operatorname{flux}(u_y) \le c < \infty . \tag{5.5}$$

Now fix a point $x_1 \in B(x_0, r_0)$ and choose $r > 0$ so small that the ball $B(x_1, r)$ is compactly contained in $B(x_0, r_0)$ and that $\{f^{-1}(f(x_1))\} \cap B(x_1, r) = \{x_1\}$. Choose points w_1 and w_2 on the sphere $\partial B(x_1, r)$ such that

$$L = L(x_1, r, f) = \max_{|x_1 - x| = r} |f(x_1) - f(x)| = |f(x_1) - f(w_1)| ,$$

and that

$$l = l(x_1, r, f) = \min_{|x_1 - x| = r} |f(x_1) - f(x)| = |f(x_1) - f(w_2)| .$$

Write $y_1 = f(x_1)$, $u = u_{y_1}$, and $v = u \circ f$. Then v is a positive singular $\mathscr{A}^*$-harmonic function in $B(x_1, r)$ with singularity at x_1. Moreover, provided r is small enough, we conclude from Lemma 3.2 that

$$v(w_1) \approx v(w_2) \approx r^{(p-n)/(p-1)} \operatorname{flux}(v)^{1/(p-1)} . \tag{5.6}$$

On the other hand, using Lemma 3.2 this time for u and heeding (5.5) we find that

$$u(f(w_1)) \approx L^{(p-n)/(p-1)}$$

and

$$u(f(w_2)) \approx l^{(p-n)/(p-1)} .$$

Combining this with (5.6) yields

$$\frac{L}{r} \approx \frac{l}{r} \approx \operatorname{flux}(v)^{1/(p-n)} .$$

In conclusion, there is a constant c such that

$$\frac{1}{c}\operatorname{flux}(v)^{1/(p-n)} \leq l(x,f) \leq L(x_1,f) \leq c\operatorname{flux}(v)^{1/(p-n)}$$

where v is the singular $\mathscr{A}$-harmonic function $u_{f(x_1)} \circ f$.

Finally, since f is discrete, open, and sense-preserving, we can employ Lemma 3.6 exactly as we did in the proof of Theorem 4.1 to conclude that the function

$$x \mapsto \Phi(x) = \operatorname{flux}(u_{f(x)} \circ f)$$

is bounded both from above and below locally in $B(x_0, r_0)$. This shows that f is locally a Lipschitz mapping in Ω, hence differentiable a.e. in Ω by Rademacher's theorem, and in particular f is a member of the Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega)$. Now since f is sense-preserving, it follows from [MRV1, 2.14] that $J_f(x) \geq 0$ a.e. in Ω. Since also $L(x,f)$ and $l(x,f)$ are locally bounded from above and below, the theorem follows.

The proof of Theorem 5.4 gives

5.7. THEOREM. *If $1 < p < n$, then every discrete sense-preserving $\mathbf{A}_p$-harmonic morphism is a locally Lipschitz quasiregular mapping. Moreover, if $1 < p < n$, every homeomorphic $\mathbf{A}_p$-harmonic morphism is a locally bilipschitz quasiconformal mapping.*

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