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SOME SATURATED VARIETIES OF SEMIGROUPS

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We show that a semigroup satisfying a heterotypical identity of which at least one side has no repeated variable is saturated and find sufficient conditions on a homotypical identity which is not a permutation identity and of which at least one side has no repeated variable, to ensure that any semigroup satisfying the identity is saturated.

1. Introduction and summary

The general question studied in the papers [*3*], [*4*], [*5*], [*9*], [*10*] is as follows: which varieties of semigroups are saturated? The author [*9*, Theorem 3.4] has answered this question for commutative varieties jointly with Higgins [*4*, Theorem 4] and more generally, for permutative varieties [*10*, Theorem 5.4].

A necessary condition for a semigroup variety to be saturated is that it admits an identity, not a permutation identity, of which at least one side has no repeated variable [*3*, Theorem 6]. The author has established that this condition is also sufficient for commutative [*9*, Theorem 3.1] and permutative varieties [*10*, Theorem 5.1]. In this paper we are able to show that a semigroup satisfying a heterotypical identity of which at least one side has no repeated variable is saturated. Further we find sufficient conditions on a homotypical identity, not a permutation identity and of which at least one side has no repeated variable, to ensure that any semigroup satisfying the identity is saturated. To find, however, a complete

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determination of all saturated varieties of semigroups remains still an open problem.

2. Preliminaries

Let U and S be any semigroups with U a subsemigroup of S. Following Howie and Isbell [7], we say that U *dominates* an element $d \in S$ if for every semigroup T and for all homomorphisms $\beta, \gamma : S \to T$, $\beta|U = \gamma|U$ implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the *dominion of* U in S, and we denote it by $\mathrm{Dom}_S(U)$. One can easily verify that $\mathrm{Dom}_S(U)$ is a subsemigroup of S containing U. A semigroup U is called *saturated* if $\mathrm{Dom}_S(U) \neq S$ for every properly containing semigroup S. A variety V of semigroups will be called *saturated* if every member of V is saturated. A subsemigroup U of a semigroup S is *closed in* S if $\mathrm{Dom}_S(U) = U$. The *content of* w, for any word w, is the (necessarily finite) set of variables appearing in w, and we denote it by $C(w)$.

Semigroup dominions are characterized by the following result.

RESULT 1 (Isbell's Zigzag Theorem [8, Theorem 2.3] or [6, Theorem VII.2.13]). Let U be any subsemigroup of any semigroup S, and let d be any element of S. Then $d \in \mathrm{Dom}_S(U)$ if and only if either $d \in U$ or there are elements $a_0, a_1, \ldots, a_{2m} \in U$, $y_1, y_2, \ldots, y_m$, $t_1, t_2, \ldots, t_m \in S$ such that

$$d = a_0t_1, \qquad a_0 = y_1a_1,$$

$$(1) \qquad y_ia_{2i} = y_{i+1}a_{2i+1}, \quad a_{2i-1}t_i = a_{2i}t_{i+1} \quad (i = 1, 2, \ldots, m-1),$$

$$a_{2m-1}t_m = a_{2m}, \qquad y_ma_{2m} = d.$$

These equations are called a *zigzag of length* m *over* U *with value* d and with *spine* $a_0, a_1, \ldots, a_{2m}$.

A semigroup U is called *permutative* if it satisfies some nontrivial permutation identity.

RESULT 2 [10, Theorem 5.1]. A permutative semigroup U is

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saturated if it satisfies an identity I such that

(i) I is not a permutation identity, and

(ii) at least one side of I has no repeated variable.

RESULT 3 [*10*, Result 4]. Let U and S be any semigroups with U a subsemigroup of S and $\mathrm{Dom}_S(U) = S$. Then for any $d \in S \backslash U$ and for any positive integer k, there exist $a_1, a_2, \ldots, a_k \in U$ and $d_k \in S \backslash U$ such that $d = a_1 a_2 \ldots a_k d_k$.

In general we shall use the notations and conventions of Clifford and Preston [*1*, *2*] or Howie [*6*].

3. Saturated varieties of semigroups

An identity $f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$ is called *heterotypical* if $C(f) \neq C(g)$ and *homotypical* if $C(f) = C(g)$.

LEMMA 3.1. *Let U and S be any semigroups with U a subsemigroup of S. If for all $a, b \in U$, and $s \in S \backslash U$ there exists $w \in U^1$ such that $as = awbs$, then U is closed in S.*

Proof. Suppose to the contrary that U is not closed in S. There exists, therefore, $d \in S \backslash U$ such that $d \in \mathrm{Dom}_S(U)$. We may, by Result 1, let (1) be a zigzag of shortest possible length m over U with value d. Now

$$
\begin{aligned}
d = a_0 t_1 &= a_0 w_0 a_1 t_1 \quad (\text{for some } w_0 \in U^1) \\
&= a_0 w_0 a_2 t_2 \quad (\text{from the equations (1)}) \\
&= a_0 w_0 a_2 w_2 a_3 t_2 \quad (\text{for some } w_2 \in U^1) \\
&\vdots \\
&= a_0 w_0 a_2 w_2 \ldots a_{2m-2} w_{2m-2} a_{2m-1} t_m \quad (\text{for some } w_4, w_6, \ldots, w_{2m-2} \in U^1) \\
&= a_0 w_0 a_2 w_2 \ldots a_{2m-2} w_{2m-2} a_{2m} \in U ,
\end{aligned}
$$

a contradiction. This proves the lemma.

THEOREM 3.2. *If a semigroup U satisfies a heterotypical identity*

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I of which at least one side has no repeated variable, then U is saturated.

Proof. Now I has the form

(2) $$x_1x_2 \ldots x_n = f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_k)$$

where $k \geq 0$, $C(f) \neq \{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_k\} \subseteq C(f)$.

Take any semigroup U satisfying (2) and suppose to the contrary that U is not saturated. There exists, therefore, a semigroup S containing U properly and such that $\mathrm{Dom}_S(U) = S$.

Case (i). $x_1, x_2, \ldots, x_n \subseteq C(f)$. In this case, by considering the last occurrence of y_1 in the word f , we see that I has the form

(3) $$x_1x_2 \ldots x_n = u(\hat{x})y_1g(\tilde{x})$$

for some words u and g and where

$$\hat{x} = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_k) ,$$

and

$$\tilde{x} = (x_1, x_2, \ldots, x_n, y_2, y_3, \ldots, y_k) .$$

Now take any $a, b \in U$, and $s \in S \backslash U$. By Result 3, since $s \in S\backslash U$, we have $s = b_1b_2 \ldots b_ns_n$ for some $b_1, b_2, \ldots, b_n \in U$ and $s_n \in S\backslash U$.

For any fixed $y_1, y_2, \ldots, y_k \in U$ put

$$\tilde{b} = (b_1, b_2, \ldots, b_n, y_2, y_3, \ldots, y_k) ,$$

$$\hat{b} = (b_1, b_2, \ldots, b_n, y_1, y_2, \ldots, y_k) ,$$

and

$$\bar{b} = (b_1, b_2, \ldots, b_n, bu(\hat{b})y_1, y_2, \ldots, y_k) .$$

Now

$$\begin{aligned} as &= ab_1b_2 \ldots b_ns_n \\ &= af(\bar{b})s_n \quad \text{(since } U \text{ satisfies (2), and by replacing } y_1 \text{ with } bu(\hat{b})y_1\text{)} \\ &= au(\bar{b})bu(\hat{b})y_1g(\tilde{b})s_n \\ &= awbf(\hat{b})s_n \quad \text{(from equation (3), where } w = u(\bar{b}) \in U^1\text{)} \\ &= awbb_1b_2 \ldots b_ns_n \quad \text{(since } U \text{ satisfies (2))} \\ &= awbs \quad \text{(since } s = b_1b_2 \ldots b_ns_n\text{)}. \end{aligned}$$

By Lemma 3.1, U is saturated.

Case (ii). $\{x_1, x_2, \ldots, x_n\} \nsubseteq C(f)$. Take any variable x_j, say, which appears only in the left hand side of (2), any $a, b \in U$, and any $s \in S\backslash U$. By Result 3, since $s \in S\backslash U$, we have again that $s = b_1b_2 \ldots b_ns_n$ for some $b_1, b_2, \ldots, b_n \in U$ and $s_n \in S\backslash U$. Now

$$\begin{aligned} as &= ab_1b_2 \ldots b_ns_n \\ &= ab_1b_2 \ldots b_{j-1}(bb_1b_2 \ldots b_j)b_{j+1} \ldots b_ns_n \quad \text{(since the right hand side of (2) is independent of the choice of the variable } x_j\text{; if } j = 1\text{, the product } b_1b_2 \ldots b_{j-1} = 1\text{)} \\ &= awbb_1b_2 \ldots b_ns_n \quad \text{(where } w = b_1b_2 \ldots b_{j-1}\text{)} \\ &= awbs \quad \text{(since } s = b_1b_2 \ldots b_ns_n\text{)} \end{aligned}$$

and therefore, by Lemma 3.1, U is saturated. This completes the proof of Theorem 3.2.

Restating Theorem 3.2 in terms of varieties we get

COROLLARY 3.3. *If a semigroup variety V admits a heterotypical identity of which at least one side has no repeated variable, then V is saturated.*

REMARK 1. Higgins [3, Theorem 15] has strengthened Corollary 3.3 by a different technique by showing that a heterotypical variety (one which admits a heterotypical identity) is saturated if and only if it admits an identity, not a permutation identity, of which at least one side has no repeated variable.

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THEOREM 3.4. *Let U be any semigroup satisfying an identity of the form*

$$(4) \qquad x_1x_2 \ldots x_n = g(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)x_j$$

with $j \neq n$ (or dually, of the form

$$x_1x_2 \ldots x_n = x_jg(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \text{ with } j \neq 1).$$

Then U satisfies the permutation identity

$$x_1x_2 \ldots x_jxyx_{j+1} \ldots x_n = x_1x_2 \ldots x_jyxx_{j+1} \ldots x_n .$$

Proof. Take any $x, y, x_1, x_2, \ldots, x_n \in U$. Now

$$\begin{aligned}
x_1x_2 \ldots x_jxyx_{j+1} \ldots x_n &= x_1x_2 \ldots (x_jx)(yx_{j+1}) \ldots x_n \\
&= g(x_1, x_2, \ldots, x_{j-1}, yx_{j+1}, \ldots, x_n)(x_jx) \\
&\qquad \text{(since } U \text{ satisfies (4))} \\
&= x_1x_2 \ldots x_j(yx_{j+1}) \ldots x_nx \\
&\qquad \text{(since } U \text{ satisfies (4))} \\
&= x_1x_2 \ldots (x_jy)x_{j+1} \ldots x_nx \\
&= g(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)(x_jy)x \\
&\qquad \text{(since } U \text{ satisfies (4))} \\
&= g(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)(x_jyx) \\
&= x_1x_2 \ldots x_jyxx_{j+1} \ldots x_n \\
&\qquad \text{(since } U \text{ satisfies (4))}
\end{aligned}$$

as required.

COROLLARY 3.5. *Let U be any semigroup satisfying an identity of the form $x_1x_2 \ldots x_n = g(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)x_j$ with $j \neq n$, and which is not a permutation identity. Then U is saturated.*

Proof. That U is saturated follows from Result 2 and Theorem 3.4.

REMARK 2. If $j = n$, then the semigroup U in the Corollary 3.5 is not necessarily saturated. For example, not all bands are saturated [5, Corollary 4]. Higgins [3, Theorem 16] has shown a related result, namely that if a variety V admits a homotypical identity of the form $x_1x_2 \ldots x_n = f(x_1, x_2, \ldots, x_n)$ which is not a permutation identity and

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is such that f neither begins with x_1 nor ends with x_n, then V is saturated.

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