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INVERSE CONNECTION FORMULAE FOR GENERALISED BESSEL POLYNOMIALS

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Abstract

We solve the problem of finding the inverse connection formulae for the generalised Bessel polynomials and their reciprocals, the reverse generalised Bessel polynomials. The connection formulae express monomials in terms of the generalised Bessel polynomials. They enable formulae for the elements of change of basis matrices for both kinds of generalised Bessel polynomials to be derived and proved correct directly.

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1. Introduction

The properties of generalised Bessel polynomials have been studied extensively [6, 14] and they have applications in solutions to the wave equation [2, 8] and signal processing filters.

The contribution of this work is to give inverse connection formulae. They provide coefficients for the mappings from the monomials to either the generalised Bessel polynomials or the reverse generalised Bessel polynomials.

The connection formulae enable us to derive change of basis matrices directly from any sequence of polynomials that spans $(1, x, \ldots, x^n)$ to a basis that spans the same vector space and includes polynomials from either family of generalised Bessel polynomials.

Suppose that $(f_0(x), f_1(x), \ldots, f_n(x))$ is a sequence of polynomials over $\mathbb{R}[x]$ that spans the same vector space as $(1, x, \ldots, x^n)$ and where $f_i(x)$ has degree i in x for $0 \le i \le n$. The inverse connection formula we consider is to find a function $\alpha_1 : \{0, 1, \ldots, n\}^2 \to \mathbb{R}$ such that

$$x^k = \sum_{v=0}^{k} \alpha_1(k, i) f_{k-i}(x),$$

where $0 \le k \le n$.

Suppose we have a sequence of polynomials $(h_0(x), h_1(x), \ldots, h_n(x))$ that spans $(1, x, \ldots, x^n)$ where

$$h_i(x) = \sum_{v=0}^{i} \alpha_2(i, v) x^{i-v}$$

and $\alpha_2 : \{0, 1, \ldots, n\}^2 \to \mathbb{R}$. The basis $(h_0(x), h_1(x), \ldots, h_n(x))$ could include classical orthogonal polynomials such as Chebyshev polynomials of the first kind or nonorthogonal polynomials such as Bernstein polynomials. We select the Laguerre polynomials $(L_0(x), L_1(x), \ldots, L_n(x))$ as a running example, so that

$$\alpha_2(n, k) = \frac{(-1)^{n-k}}{(n-k)!}\binom{n}{k}. \tag{1.1}$$

This equation follows from [7, (18.18.18)]. We have

$$\alpha(n, k) = \sum_{v=0}^{k} \alpha_1(n-v, k-v)\, \alpha_2(n, v). \tag{1.2}$$

The element $m_{i,j}$ of the change of basis matrix from $(h_0(x), h_1(x), \ldots, h_n(x))$ to $(f_0(x), f_1(x), \ldots, f_n(x))$ is given by

$$m_{i,j} = \begin{cases} \alpha(j, j-i) & \text{if } j \geq i, \\ 0 & \text{if } j < i, \end{cases}$$

where $0 \leq i, j \leq n$.

The right-hand side of (1.2) is an expression for the element in the row with index $n-k$ of the last column of the product of two upper triangular change of basis matrices: that from $(1, x, \ldots, x^n)$ to $(f_0(x), f_1(x), \ldots, f_n(x))$ with that from $(h_0(x), h_1(x), \ldots, h_n(x))$ to $(1, x, \ldots, x^n)$.

The elements can be found more directly because (1.2) separates finding α into two sub-problems both of which can be solved independently, that is, finding solutions for α_1 and α_2. We now solve the sub-problem for α_1 for both kinds of generalised Bessel polynomials.

2. Generalised Bessel polynomials

The generalised Bessel polynomials can be defined by

$$y_n(x; \alpha, \beta) = \sum_{l=0}^{n} \binom{n}{l} (n+\alpha-1)_l \left(\frac{x}{\beta}\right)^l, \tag{2.1}$$

where $(x)_l$ is the Pochhammer symbol or rising factorial. This is derived from Krall and Frink [8, (34)].

The connection formula for the mapping to the monomials is therefore

$$\alpha(n, k, \alpha, \beta) = \binom{n}{k} \frac{(n+\alpha-1)_{n-k}}{\beta^{n-k}}. \tag{2.2}$$

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For the inverse connection formula for the mapping from monomials, we use the following equation from Doha and Ahmed [5, (8)] as the starting point for an Ansatz. It is equivalent to that of Sánchez-Ruiz and Dehesa [13, (2.32)]:

$$\pi_{ni} = \binom{n}{i}\frac{(-1)^{n-i}2^n(2i+\alpha+1)\Gamma(i+\alpha+1)}{\Gamma(n+i+\alpha+2)}.$$

This is the inverse connection formula for the term $y_i(x;\alpha+2,2)$, where $0 \le i \le n$. To find the Ansatz, we replace i by $n-k$ to be consistent with our notation, α by $\alpha-2$, the ratio of gamma functions by a Pochhammer symbol and the constant 2 by β. This gives the equation

$$\alpha(n,k,\alpha,\beta) = \binom{n}{k}\frac{(-1)^k\beta^n(2(n-k)+\alpha-1)}{(n-k+\alpha-1)_{n+1}}. \tag{2.3}$$

THEOREM 2.1. *Equation (2.3) is the connection formula for the mapping from monomials to generalised Bessel polynomials.*

PROOF. The proof is by mathematical induction. It uses back substitution to find the connection formula for the last column of the inverse matrix of the change of basis matrix from generalised Bessel polynomials to the monomials. For clarity, we call the function in (2.2) α_0.

In the base case,

$$\alpha_1(n,0,\alpha,\beta) = \frac{1}{\binom{n}{0}\frac{(n+\alpha-1)_n}{\beta^n}},$$

from (2.2). This equals the $\alpha(n,0,\alpha,\beta)$ from (2.3) when $k=0$, as required.

The induction hypothesis is that (2.3) holds for all $k : 0 \le k < m \le n$. From this induction hypothesis and back substitution,

$$\alpha_1(n,m,\alpha,\beta) = \frac{\sum_{k=1}^{m}\alpha_0(n-m+k,k,\alpha,\beta)\alpha(n,m-k,\alpha,\beta)}{-\alpha_0(n-m,0,\alpha,\beta)}. \tag{2.4}$$

We find that $\alpha_1(n,m,\alpha,\beta) = \alpha(n,m,\alpha,\beta)$ by applying Gosper's algorithm from the 'fastZeil' package [12] to the right-hand side of (2.4) and simplifying. The result holds for the base case and from the induction hypothesis for $k = m$. By the principle of mathematical induction, it holds for all $k : 0 \le k \le n$. □

As an example,

$$\begin{aligned} x^2 &= \sum_{k=0}^{2}\alpha(2,k,\alpha,\beta)y_{2-k}(x;\alpha,\beta) \\ &= \frac{\beta^2}{\alpha(1+\alpha)} - \frac{2\beta^2(1+\frac{\alpha x}{\beta})}{\alpha(2+\alpha)} + \frac{\beta^2(1+\frac{2(1+\alpha)x}{\beta}+\frac{(1+\alpha)(2+\alpha)x^2}{\beta^2})}{(1+\alpha)(2+\alpha)}. \end{aligned}$$

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3. Reverse generalised Bessel polynomials

The reverse generalised polynomials $\theta_n(x;\alpha,\beta)$ are the reciprocal polynomials of the generalised Bessel polynomials, that is,

$$\theta_n(x;\alpha,\beta) = x^n y_n(x^{-1};\alpha,\beta), \tag{3.1}$$

where $n \geq 0$. Burchnall [2, Section 5] discussed $\theta_n(x;\alpha,\beta)$ (see also Grosswald [6, Ch. 1] for historical details and the overview of Srivastava [14, (15)]). It follows from (2.1) and (3.1) that the reverse polynomials can be defined by

$$\theta_n(x;\alpha,\beta) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{\beta^{n-k}} (n+\alpha-1)_{n-k},$$

so that the connection formula to the monomials is

$$\alpha(n,k,\alpha,\beta) = \binom{n}{k} \frac{(n+\alpha-1)_k}{\beta^k}. \tag{3.2}$$

We want to find the connection formula for the inverse mapping, that is, from the monomials to reverse generalised Bessel polynomials. To do this, we use back substitution to find the connection formula for the last column of the inverse matrix of the change of basis matrix from reverse generalised Bessel polynomials to the monomials. We call the function in (3.2) α_0, to distinguish it from the inverse connection formula, α_1.

First, we find an Ansatz for the inverse connection formula by generalising successive connection formulae found through back substitution. In the base case, we have $\alpha_1(n,0,\alpha,\beta) = 1/\alpha(n,0,\alpha,\beta)$, from (3.2), so that $\alpha_1(n,0,\alpha,\beta) = 1$. Equation (3.3) follows from back substitution:

$$\begin{aligned} \alpha_1(n,m,\alpha,\beta) &= \frac{\sum_{k=1}^{m} \alpha_0(n-m+k,k,\alpha,\beta)\alpha(n,m-k,\alpha,\beta)}{-\alpha_0(n-m,0,\alpha,\beta)} \\ &= -\sum_{k=1}^{m} \alpha_0(n-m+k,k,\alpha,\beta)\alpha(n,m-k,\alpha,\beta). \end{aligned} \tag{3.3}$$

By successively applying (3.3), we find the following sequence of specific connection formulae:

$$\begin{aligned} \alpha_1(n,1,\alpha,\beta) &= -\frac{\binom{n}{1}(n+\alpha-1)}{\beta}, \\ \alpha_1(n,2,\alpha,\beta) &= \frac{\binom{n}{2}(n+\alpha-1)(n+\alpha-4)}{\beta^2}, \\ \alpha_1(n,3,\alpha,\beta) &= -\frac{\binom{n}{3}(n+\alpha-1)(n+\alpha-6)(n+\alpha-5)}{\beta^3}, \\ \alpha_1(n,4,\alpha,\beta) &= \frac{\binom{n}{4}(n+\alpha-1)(n+\alpha-8)(n+\alpha-7)(n+\alpha-6)}{\beta^4}. \end{aligned}$$

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We use the following generalisation of the sequence from $\alpha_1(n,0,\alpha,\beta)$ to $\alpha_1(n,4,\alpha,\beta)$ to give the Ansatz:

$$\alpha_1(n,k,\alpha,\beta) = (-1)^k \frac{\binom{n}{k}}{\beta^k}(n+\alpha-1)(n+\alpha-2k)_{k-1}. \tag{3.4}$$

Equation (3.3) cannot be simplified by applying Gosper's algorithm to its right-hand side. It is an mth-order recurrence and it is not convenient to use in a proof by mathematical induction. We can apply Zeilberger's algorithm from the 'fastZeil' package [12] to (3.3) to obtain a second-order recurrence for the inverse connection formula. The 'fastZeil' package generates the following recurrence with the condition that $k > 1$:

$$\begin{aligned} k(k+1)(n-k)\mathrm{SUM}[k] + \beta(k+1)(n-k+\alpha-2)\mathrm{SUM}[k+1] \\ = (k-n)(n-3k+\alpha-2)(n-k+\alpha-1)\alpha_1(n,k,\alpha,\beta). \end{aligned} \tag{3.5}$$

THEOREM 3.1. *Equation (3.4) is the connection formula for the mapping from monomials to the reverse generalised Bessel polynomials.*

PROOF. The proof is by mathematical induction on k. The case for $k = 0$ was verified above. The induction hypothesis is that the result holds for all $k : 0 \le k < m \le n$.

The condition that $k > 1$ for the recurrence is too strict. We can check that (3.5) also holds when $k = 0$ and $k = 1$. Suppose that $k > 0$, apply (3.5), substitute $\alpha_1(n,k,\alpha,\beta)$ for SUM[k] and solve it for SUM[$k+1$]. After simplifications, we find that SUM[$k+1$] $= \alpha_1(n,k+1,\alpha,\beta)$, as required. The result holds for the base cases, and from the induction hypothesis for $k = m$. By the principle of mathematical induction, it holds for all $k : 0 \le k \le n$. □

As an example,

$$\begin{aligned} x^2 &= \sum_{k=0}^{2} \alpha_1(2,k,\alpha,\beta)\theta_{2-k}(x;\alpha,\beta) \\ &= \left(\frac{(1+\alpha)(2+\alpha)}{\beta^2} + \frac{2(1+\alpha)x}{\beta} + x^2\right) - \frac{2(1+\alpha)(\frac{\alpha}{\beta}+x)}{\beta} + \frac{(-2+\alpha)(1+\alpha)}{\beta^2}. \end{aligned}$$

4. Related work

The Bessel polynomials, $y_n(x) = y_n(x;2,2)$, were named in 1949 by Krall and Frink [8], and had previously appeared (see, for example, Grosswald [6]). They are a sequence of polynomials that are orthogonal on the unit circle of the complex plane. The reverse Bessel polynomials were identified by Burchnall and Chaundry [3, page 478] (see also Burchnall [2, (8)]).

The Bessel polynomials gain significance as one of the four families of classical orthogonal polynomials [4, 10]. The other families are the Hermite, Jacobi and Laguerre polynomials (see Koornwinder *et al.* [7, Section 18.3]). Castillo and Petronilho [4, Section 3.2] formalised how the classical orthogonal polynomials

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comprise these four families only. Maroni [10, Sections 2 and 6] discussed characterisations of the classical orthogonal polynomials.

The literature on the topic of change of basis between classical orthogonal polynomials is extensive [11]. There are various other techniques for finding the connection coefficients that are the elements of change of basis matrices, such as matrix methods [1] and recurrences [9, 11].

We showed that the equation from Sánchez-Ruiz and Dehesa [13, (2.32)] is a special case of the inverse connection formula, (2.3), for the generalised Bessel polynomial. We are not aware of other similar equations in the literature for the Bessel polynomials.

5. Two examples

We give an example of the application of (1.2), where α_1 is the inverse connection formula for the generalised Bessel polynomial of (2.3) and α_2 is the formula for the Laguerre polynomial of (1.1).

On substitution into (1.2),

$$\begin{aligned}\alpha(n,k) &= \sum_{v=0}^{k}\binom{n-v}{k-v}\frac{(-1)^{k-v}\beta^{n-v}(2(n-k)+\alpha-1)}{(n-k+\alpha-1)_{n-v+1}}\frac{(-1)^{n-v}}{(n-v)!}\binom{n}{v}\\ &= (-1)^{n+k}\frac{(2(n-k)+\alpha-1)}{(n-k)!}\sum_{v=0}^{k}\binom{n}{v}\frac{\beta^{n-v}}{(k-v)!\,(n-k+\alpha-1)_{n-v+1}}.\end{aligned}$$

When $n=4$, $\alpha=3$ and $\beta=-\frac{3}{2}$, we obtain the change of basis matrix:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{32} & -\frac{73}{320} & -\frac{2429}{5120}\\ 0 & \frac{1}{2} & \frac{17}{20} & \frac{171}{160} & \frac{2629}{2240}\\ 0 & 0 & \frac{9}{160} & \frac{351}{2240} & \frac{20817}{71680}\\ 0 & 0 & 0 & \frac{3}{1120} & \frac{23}{2240}\\ 0 & 0 & 0 & 0 & \frac{1}{14336}\end{bmatrix}.$$

From the fourth column,

$$\begin{aligned}L_3(x) &= \sum_{v=0}^{3}\alpha(3,v)y_{3-v}\Big(x;3,-\frac{3}{2}\Big)\\ &= \frac{3}{1120}y_3\Big(x;3,-\frac{3}{2}\Big)+\frac{351}{2240}y_2\Big(x;3,-\frac{3}{2}\Big)+\frac{171}{160}y_1\Big(x;3,-\frac{3}{2}\Big)-\frac{73}{320}y_0\Big(x;3,-\frac{3}{2}\Big)\\ &= \frac{3(-\frac{560x^3}{9}+40x^2-10x+1)}{1120}+\frac{351(\frac{80x^2}{9}-\frac{16x}{3}+1)}{2240}+\frac{171}{160}(-2x+1)-\frac{73}{320}\\ &= \frac{1}{6}(-x^3+9x^2-18x+6),\ \text{as required.}\end{aligned}$$

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Similarly, when α_1 is the inverse connection formula for the reverse generalised Bessel polynomial of (3.4) and α_2 is the formula for the Laguerre polynomial of (1.1),

$$\alpha(n,k) = \frac{(-1)^{n+k}}{(n-k)!}\sum_{v=0}^{k}\binom{n}{v}\frac{(n-v+\alpha-1)(n+v+\alpha-2k)_{k-v-1}}{\beta^{k-v}(k-v)!}$$

and

$$\begin{aligned} L_3(x) &= \sum_{v=0}^{3}\alpha(3,v)\theta_{3-v}\Big(x;3,-\frac{3}{2}\Big) \\ &= -\frac{1}{6}\theta_3\Big(x;3,-\frac{3}{2}\Big) + -\frac{1}{6}\theta_2\Big(x;3,-\frac{3}{2}\Big) + \frac{25}{9}\theta_1\Big(x;3,-\frac{3}{2}\Big) - \frac{7}{3}\theta_0\Big(x;3,-\frac{3}{2}\Big) \\ &= \frac{(x^3-10x^2+40x-\frac{560}{9})}{6} - \frac{(x^2-\frac{16x}{3}+\frac{80}{9})}{6} + \frac{25}{9}(x-2) - \frac{7}{3}. \end{aligned}$$

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