THE PRODUCT OF TWO ULTRASPHERICAL POLYNOMIALS

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1. Let

$$\sum_{n=0}^{\infty} C_n^{\nu}(x)t^n = (1-2xt+t^2)^{-\nu}.$$

It is familiar that

$$\sum_{n=0}^{\infty} C_n^{\nu}(x)\frac{t^n}{(2\nu)_n} = \Gamma(\nu+\tfrac{1}{2})e^{xt}\{\tfrac{1}{2}t(1-x^2)^{\frac{1}{2}}\}\ J_{\nu-\frac{1}{2}}\{t(1-x^2)^{\frac{1}{2}}\}. \tag{1}$$

In the addition theorem [3, p. 363]

$$\frac{J_\nu(w)}{(\frac{1}{2}w)^\nu} = \Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)\frac{J_{\nu+n}(t)}{(\frac{1}{2}t)^\nu}\frac{J_{\nu+n}(z)}{(\frac{1}{2}z)}C_n^{\nu}(\cos\theta), \tag{2}$$

where

$$w = (t^2+z^2-2tz\cos\theta)^{\frac{1}{2}},$$

take $\theta=\pi$ and replace ν by $\nu-\frac{1}{2}$. Since

$$C_n^{\nu-\frac{1}{2}}(-1) = (-1)^n\frac{(2\nu-1)_n}{n!},$$

we get

$$\frac{J_{\nu-\frac{1}{2}}(t+z)}{\{\frac{1}{2}(t+z)\}^{\nu-\frac{1}{2}}} = \Gamma(\nu-\tfrac{1}{2})\sum_{n=0}^{\infty}(-1)^n(\nu-\tfrac{1}{2}+n)\frac{(2\nu-1)_n}{n!}\frac{J_{\nu-\frac{1}{2}+n}(t)}{(\frac{1}{2}t)^{\nu-\frac{1}{2}}}\frac{J_{\nu-\frac{1}{2}+n}(z)}{(\frac{1}{2}z)^{\nu-\frac{1}{2}}}.$$

We now replace t and z by $t(1-x^2)^{\frac{1}{2}}$ and $z(1-x^2)^{\frac{1}{2}}$, respectively, and use (1). The result is

$$\sum_{k=0}^{\infty} C_k^{\nu}(x)\frac{(t+z)_\nu^k}{(2\nu)_k} = \sum_{r=0}^{\infty}(-1)^r\frac{\nu-\frac{1}{2}+r}{\nu-\frac{1}{2}}\frac{(2\nu-1)_r(1-x^2)^r}{r!\{(\nu+\frac{1}{2})_r\}^2}$$
$$\times(\tfrac{1}{4}tz)^r\sum_{m=0}^{\infty}C_m^{\nu+r}(x)\frac{t^m}{(2\nu+2r)_m}\sum_{n=0}^{\infty}C_n^{\nu+r}(x)\frac{z^n}{(2\nu+2r)_n}.$$

Comparing coefficients of t^mz^n on both sides, we get, for $\nu\neq\frac{1}{2}$,

$$\binom{m+n}{m}\frac{C_{m+n}^{\nu}(x)}{(2\nu)_{m+n}} = \sum_{r=0}^{\min(m,n)}(-\tfrac{1}{4})^r\frac{\nu-\frac{1}{2}+r}{\nu-\frac{1}{2}}\frac{(2\nu-1)_r}{r!}\frac{(1-x^2)^r}{\{(\nu+\frac{1}{2})_r\}^2}\frac{C_{m-r}^{\nu+r}(x)C_{n-r}^{\nu+r}(x)}{(2\nu+2r)_{m-r}(2\nu+2r)_{n-r}}. \tag{3}$$

For $\nu=\frac{1}{2}$ we have, however,

$$P_{m+n}(x) = P_m(x)P_n(x)+2\,.\,m!n!\sum_{r=1}^{\min(m,n)}(-\tfrac{1}{4})^r\frac{(1-x^2)^r}{r!r!}\frac{C_{m-r}^{\frac{1}{2}+r}(x)}{(2r+1)_{m-r}}\frac{C_{n-r}^{\frac{1}{2}+r}(x)}{(2r+1)_{n-r}}, \tag{4}$$

where $P_n(x)$ is the Legendre polynomial. Since [4, p. 329]

$$C_{n-r}^{\frac{1}{2}+r}(x) = \frac{(x^2-1)^{-\frac{1}{2}r}}{2^r(\frac{1}{2})_r}P_n^r(x),$$

(4) may be written as

$$P_{m+n}(x) = P_m(x)P_n(x)+2\sum_{r=1}^{\min(m,n)}\frac{P_m^r(x)P_n^r(x)}{(m+1)_r(n+1)_r}. \tag{5}$$

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By taking $\nu = q+\frac{1}{2}$ in (3) we get the identity

$$\binom{m+n}{m}\frac{P^q_{m+n+q}(x)}{q!(2q+1)_{m+n}} = (x^2-1)^{\frac{1}{2}q}\sum_{r=0}^{\min(m,n)}\frac{(2q+r)!}{r!}\frac{P_m^{q+r}(x)P_n^{q+r}(x)}{(m+2q+r)!(n+2q+r)!} \quad (q \geqq 1). \quad (6)$$

2. To invert (3) we require the formula

$$\sum_{r=0}^{\infty}\frac{1}{\Gamma(\nu+r+1)}\frac{(2\nu+r+1)_r}{r!}\frac{J_{\nu+r}(t+z)}{\{\frac{1}{2}(t+z)\}^{\nu+r}}(\tfrac{1}{4}tz)^r = \frac{J_\nu(t)}{(\frac{1}{2}t)^\nu}\frac{J_\nu(z)}{(\frac{1}{2}z)^\nu}. \quad (7)$$

Indeed, on making use of (2), it is clear that the left member of (7) is equal to

$$\sum_{r=0}^{\infty}\frac{\nu}{\nu+r}\frac{(2\nu+r+1)_r}{r!}\sum_{s=0}^{\infty}(-1)^s(\nu+r+s)\frac{(2\nu+2r)_s J_{\nu+r+s}(t)}{s!(\frac{1}{2}t)^\nu}\frac{J_{\nu+r+s}(z)}{(\frac{1}{2}z)^\nu}$$

$$= \sum_{n=0}^{\infty}(\nu+n)\frac{J_{\nu+n}(t)}{(\frac{1}{2}t)^\nu}\frac{J_{\nu+n}(z)}{(\frac{1}{2}z)^\nu}\cdot\sum_{r+s=n}\frac{(-1)^s}{\nu+r}\frac{(2\nu+r+1)_r(2\nu+2r)_s}{r!s!}.$$

Now, for $n \geqq 1$ the inner sum is equal to

$$\sum_{r=0}^{n}\frac{(-1)^{n-r}}{-r}\frac{(2\nu+r+1)_r(2\nu+2r+1)_{s-1}(2\nu+2r)}{r!s!} = \frac{2}{n!}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}(2\nu+r+1)_{n-1} = 0,$$

since the nth difference of a polynomial of degree $n-1$ vanishes. For $n = 0$, on the other hand, the inner sum is $1/\nu$. This evidently proves (7). Note that the formula holds for $\nu = 0$.

We now replace ν by $\nu-\frac{1}{2}$, t and z by $t(1-x^2)^{\frac{1}{2}}$ and $z(1-x^2)^{\frac{1}{2}}$, respectively, and again use (1). Then

$$\sum_{m=0}^{\infty}C_m^\nu(x)\frac{t^m}{(2\nu)_m}\sum_{n=0}^{\infty}C_n^\nu(x)\frac{z^n}{(2\nu)_n} = \sum_{r=0}^{\infty}\frac{(2\nu+r)_r}{r!(\nu+\frac{1}{2})_r(\nu+\frac{1}{2})_r}(\tfrac{1}{4}tz)^r(1-x^2)^r\sum_{n=0}^{\infty}C_n^{\nu+r}(x)\frac{(t-z)^n}{(2\nu+2r)_n}.$$

Equating coefficients we get

$$\frac{C_m^\nu(x)}{(2\nu)_m}\frac{C_n^\nu(x)}{(2\nu)_n} = \sum_{r=0}^{\min(m,n)}\frac{(2\nu+r)_r}{r!(\nu+\frac{1}{2})_r(\nu+\frac{1}{2})_r}\binom{m+n-2r}{m-r}\frac{(1-x^2)^r}{4^r}\frac{C_{m+n-2r}^{\nu+r}(x)}{(2\nu+2r)_{m+n-2r}}$$

or, if we prefer,

$$C_m^\nu(x)C_n^\nu(x) = \frac{(2\nu)_m(2\nu)_n}{(2\nu)_{m+n}}\sum_{r=0}^{\min(m,n)}\binom{m+n-2r}{m-r}\frac{(2\nu+r)_r(\nu)_r}{r!(\nu+\frac{1}{2})_r}(1-x^2)^r C_{m+n-2r}^{\nu+r}(x)$$

$$= \frac{(2\nu)_m(2\nu)_n}{(2\nu)_{m+n}}\sum_{r=0}^{\min(m,n)}\binom{m+n-2r}{m-r}\frac{(\nu)_r(\nu)_r}{r!(2\nu)_r}4^r(1-x^2)\,C_{m+n-2r}^{\nu+r}(x). \quad (8)$$

In particular, for $\nu = \frac{1}{2}$, (8) becomes

$$\binom{m+n}{m}P_m(x)P_n(x) = \sum_{r=0}^{\min(m,n)}\binom{m+n-2r}{m-r}\frac{(2r)!}{(r!)^3}\frac{(x^2-1)^{\frac{1}{2}r}}{2^r}P_{m+n-r}^r(x). \quad (9)$$

More generally for $\nu = q+\frac{1}{2}$ we get

$$\frac{(2q+1)_{m-n}}{(2q+1)_m(2q+1)_n}P_{m+q}^q(x)P_{n+q}^q(x) = (2q)!\sum_{r=0}^{\min(m,n)}(-1)^r\binom{m+n-2r}{m-r}$$

$$\times\frac{(2q+2r)!}{r!(q+r)!(2q+r)!}\frac{(x^2-1)^{\frac{1}{2}(q+r)}}{2^{q+r}}P_{m+n+q-r}^{q+r}(x). \quad (10)$$

3. Note that (10) differs from the formula found by Bailey [**2**] for the product of associated Legendre polynomials. Similarly (6) differs from the inverse formula found by Al-Salam [**1**]. However we shall now show that (10) does indeed imply Bailey's identity.

We recall that

$$4\nu(n+\nu-1)(1-x^2)C_{n-2}^{\nu+1}(x) = (n+2\nu-1)(n+2\nu-2)C_{n-2}^{\nu}(x)-n(n-1)C_n^{\nu}(x). \tag{11}$$

We shall show that generally

$$\frac{4^r(\nu)_r(n-2r)!}{(2\nu)_n}(1-x^2)^r C_{n-2r}^{\nu+r}(x) = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{n+\nu-2s}{(n+\nu-r-s)_{r+1}}\frac{(n-2s)!}{(2\nu)_{n-2s}}C_{n-2s}^{\nu}(x), \tag{12}$$

for $2r \leqq n$. For $r=1$, (12) evidently reduces to (11). Now assuming that (12) holds for the value r, we get (replacing n by $n-2$ and ν by $\nu+1$)

$$\begin{aligned}
&\frac{4^{r+1}(\nu)_{r+1}(n-2r-2)!}{(2\nu)_n}(1-x^2)^{r+1}C_{n-2r-2}^{\nu+r+1}(x)\\
&= \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{(n-2s-2)!}{(n+\nu-r-s-1)_{r+1}(2\nu)_{n-2s}}\,4\nu(n+\nu-2s-1)(1-x^2)C_{n-2s-2}^{\nu+1}(x)\\
&= \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{(n-2s-2)!}{(n+\nu-r-s-1)_{r+1}(2\nu)_{n-2s}}\\
&\qquad\times\{(n+2\nu-2s-1)(n+2\nu-2s-2)C_{n-2s-2}^{\nu}(x)-(n-2s)(n-2s-1)C_{n-2s}^{\nu}(x)\}\\
&= \sum_{s=0}^{r+1}(-1)^{r+1-s}\left\{\binom{r}{s-1}\frac{1}{(n+\nu-r-s)_{r+1}}+\binom{r}{s}\frac{1}{(n+\nu-r-s-1)_{r+1}}\right\}\frac{(n-2s)!}{(2\nu)_{n-2s}}C_{n-2s}^{\nu}(x)\\
&= \sum_{s=0}^{r+1}(-1)^{r+1-s}\binom{r+1}{s}\frac{n+\nu-2s}{(n+\nu-r-s-1)_{r+2}}\frac{(n-2s)!}{(2\nu)_{n-2s}}C_{n-2s}^{\nu}(x),
\end{aligned}$$

so that (12) holds for the value $r+1$.

We remark that

$$\frac{(\nu+\frac{1}{2})_r(n+1)_{2r}}{(2\nu)_{n+2r}}C_{n+2r}^{\nu}(x) = \sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{(n+\nu+r)^s}{(2\nu+2s)_n}(1-x^2)^s C_n^{\nu+s}(x), \tag{13}$$

which is the inverse of (12), can also be proved by induction with respect to r.

4. Returning to (8) and making use of (12), we get

$$\begin{aligned}
C_m^{\nu}(x)C_n^{\nu}(x) &= (2\nu)_m(2\nu)_n\sum_{r=0}^{\min(m,n)}\binom{m+n-2r}{m-r}\frac{(2\nu+r)_r}{r!(\nu+\frac{1}{2})_r}\\
&\qquad\times\frac{4^{-r}}{(m+n-2r)!}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{m+n+\nu-2s}{(m+r+\nu-r-s)_{r+1}}\frac{(m+n-2s)!}{(2\nu)_{m+n-2s}}C_{m+n-2s}^{\nu}(x)\\
&= (2\nu)_m(2\nu)_n\sum_{s=0}^{\min(m,n)}\frac{m+n+\nu-2s}{m+n+\nu-s}\frac{(m+n-2s)!}{(2\nu)_{m+n-2s}}C_{m+n-2s}^{\nu}(x)\\
&\qquad\times\sum_{r=s}^{\min(m,n)}(-1)^{r-s}\binom{r}{s}\frac{(\nu)_r}{r!(m-r)!(n-r)!(2\nu)_r(m+n+\nu-r-s)_r}.
\end{aligned}$$

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The inner sum is equal to

$$\frac{(\nu)_s}{s!(m-s)!(n-s)!(2\nu)_s(m+n+\nu-2s)_s}\sum_{r=0}^{\min(m-s,\,n-s)}\frac{(-m+s)_r(-n+s)_r(\nu+s)_r}{r!(2\nu+s)_r(1+2s-m-n-\nu)_r}$$

$$=\frac{(\nu)_s}{s!(m-s)!(n-s)!(2\nu)_s(m+n+\nu-2s)_s}\,{}_3F_2\left[\begin{matrix}-m+s,\ -n+s,\ \nu+s\\ 2\nu+s,\ 1+2s-m-n-\nu\end{matrix}\right]$$

$$=\frac{(\nu)_s}{s!(m-s)!(n-s)!(2\nu)_s(m+n+\nu-2s)_s}\,\frac{(\nu)_{m-s}(\nu)_{n-s}(2\nu)_s(2\nu)_{m+n-s}}{(\nu)_{m+n-2s}(2\nu)_m(2\nu)_n},$$

by Saalschütz's theorem. We therefore get

$$C_m^\nu(x)C_n^\nu(x)=\sum_{s=0}^{\min(m,\,n)}\frac{m+n+\nu-2s}{m+n+\nu-s}\,\frac{(\nu)_s(\nu)_{m-s}(\nu)_{n-s}}{s!(m-s)!(n-s)!}\,\frac{(2\nu)_{m+n-s}}{(\nu)_{m+n-s}}\,\frac{(m+n-2s)!}{(2\nu)_{m+n-2s}}C_{m+n-2s}^\nu(x). \quad (14)$$

For $\nu=\frac{1}{2}$, (14) reduces to

$$P_m(x)P_n(x)=\sum_{s=0}^{\min(m,\,n)}\frac{m+n+\frac{1}{2}-2s}{m+n+\frac{1}{2}-s}\,\frac{A_sA_{m-s}A_{n-s}}{A_{m+n-s}}P_{m+n-2s}(x), \quad (15)$$

where

$$A_r=\frac{(\frac{1}{2})_r}{r!};$$

(15) is the familiar formula of Adams and Neumann. For $\nu=q+\frac{1}{2}$, (14) becomes

$$(x^2-1)^{\frac{1}{2}q}P_m^q(x)P_n^q(x)=2^q\sum_{s=0}^{\min(m-q,\,n-q)}\frac{m+n-q-2s+\frac{1}{2}}{m+n-q-s+\frac{1}{2}}\,\frac{(q+\frac{1}{2})_s(\frac{1}{2})_{m-s}(\frac{1}{2})_{n-s}}{s!(m-q-s)!(n-q-s)!}$$

$$\times\frac{(m+n-s)!}{(\frac{1}{2})_{m+n-q-s}}\,\frac{(m+n-2q-2s)!}{(m+n-2s)!}P_{m+n-q-2s}^q(x), \quad (16)$$

which is in agreement with Bailey's formula.

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