Canad. J. Math. Vol. **51** (2), 1999 pp. 347–371

Exceptional Moufang Quadrangles of Type F_4

Bernhard Mühlherr and Hendrik Van Maldeghem

Abstract. In this paper, we present a geometric construction of the Moufang quadrangles discovered by Richard Weiss (see Tits & Weiss [18] or Van Maldeghem [19]). The construction uses fixed point free involutions in certain mixed quadrangles, which are then extended to involutions of certain buildings of type F_4. The fixed flags of each such involution constitute a generalized quadrangle. This way, not only the new exceptional quadrangles can be constructed, but also some special type of mixed quadrangles.

1 Introduction

1.1 Generalized Polygons and the Moufang Condition

Generalized polygons were introduced by Tits in [9]. They can be defined as the rank 2 geometries whose incidence graph has girth $2n$ and diameter n, for some natural number $n \geq 2$ (in which case the generalized polygon is also called a *generalized n-gon*). In fact, generalized polygons can be seen as the rank 2 (weak) spherical buildings. All (thick) spherical buildings of rank ≥ 3 are classified by Tits [10]. In the rank 2 case, it seems reasonable to accept that no classification is possible, since there are (many variations of) free contructions (well-known for projective planes, which are essentially the generalized 3-gons), see Tits [13]. However, it was observed by Tits in the addenda of [10] that under the hypothesis of the so-called *Moufang condition*, there is reasonable hope to classify all (Moufang) generalized n-gons (with $n \geq 3$, excluding the more or less trivial case $n = 2$). For projective planes, this classification follows from the combined results of Moufang [5], Kleinfeld [4] and Bruck & Kleinfeld [1]. For generalized hexagons, the classification was carried out by Jacques Tits already in the 1960s, although it was never published. Furthermore, Tits [12], [14] and Weiss [20] show that for a Moufang generalized n-gon, one has $n \in \{2, 3, 4, 6, 8\}$ (assuming thickness, *i.e.*, all points and lines are incident with at least 3 elements). For generalized octagons, the classification is contained in Tits [15]. That leaves the case $n = 4$. It was conjectured by Tits [10], [11] that all Moufang generalized quadrangles arise from classical groups, algebraic groups or mixed groups of type B_2, and an explicit enumeration is contained in Tits [11]. Recently, the classification of Moufang quadrangles was completed, and the classification of Moufang polygons revised by Tits & Weiss [18]. It turned out that the list of Moufang generalized quadrangles as given by Tits [11] was incomplete, and a new class of examples (discovered in February 1997 by Richard Weiss, to appear in [18]) had to be added. This class of examples was defined by Weiss in terms of commutation relations. In this paper, we show that it is related to mixed groups of type F_4, and so Tits' conjecture stated above remains true if one rephrases it as

Received by the editors June 5, 1998; revised October 12, 1998.
The second author is a Research Director of the Fund for Scientific Research FWO – Flanders (Belgium).
AMS subject classification: 51E12, 51E24.

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

all Moufang generalized quadrangles arise from classical groups, algebraic groups or mixed groups. Hence every Moufang quadrangle is of "algebraic origin".

For convenience, we shall call the recently discovered Moufang quadrangles *the new Moufang quadrangles*. Only after our construction, one can call them *exceptional Moufang quadrangles of type* F_4.

1.2 Main Result

Let M be a spherical Coxeter diagram over a set I, let Δ be a building of type M and let typ: $\Delta \to \mathcal{P}(I)$ by a type function (where $\mathcal{P}(I)$ denotes the set of all subsets of I). We are interested in type preserving involutions $\tau \in \operatorname{Aut}\Delta$ with the property that $\tilde{\Delta} := \operatorname{Fix}_\Delta(\tau)$, the fixed point structure of τ in Δ, is a thick building. We call such involutions *homogeneous*.

Given a homogeneous involution τ in $\operatorname{Aut}(\Delta)$, then $\operatorname{typ}(A) = \operatorname{typ}(B)$ for any two maximal simplices of $\tilde{\Delta}$; the Tits diagram of τ is the pair $\big(M, \operatorname{typ}(A)\big)$ where A is a maximal simplex in $\tilde{\Delta}$. It is a fact that the Tits diagram of τ determines the type (i.e. the Coxeter diagram) of $\tilde{\Delta}$. In the particular case where $M = \mathsf{F}_4$ (labelled by $I := \{1,2,3,4\}$ in a linear order) and the Tits diagram of τ is $(\mathsf{F}_4, \{1,4\})$ it turns out that the diagram of $\tilde{\Delta}$ is equal to C_2; in other words: $\tilde{\Delta}$ is a generalized quadrangle. Moreover, if the building Δ is thick, then $\tilde{\Delta}$ is a Moufang quadrangle because Δ is a Moufang building.

There is a very special sort of buildings of type F_4 — namely those which are associated to 'mixed algebraic groups' of type F_4. The main goal of the present paper is to give the commutation relations for all Moufang quadrangles which arise as sets of fixed points of a homogeneous involution which acts on a building arising from mixed algebraic groups of type F_4 and which has Tits diagram $(\mathsf{F}_4, \{1,4\})$. The motivation for our investigations is that we obtain the following result as a consequence:

Theorem *The new Moufang quadrangles arise as sets of fixed points of homogeneous involutions acting on buildings of mixed type* F_4.

Our guess that the theorem above might be true was based on the observation that there are subquadrangles of the new quadrangles which are related to buildings of type D_4, mixed type C_4 and mixed type B_4, and that all these buildings can be embedded into a building of mixed type F_4. The appearence of these subquadrangles had been already mentioned by Tits [17].

Our original proof of the theorem above was based on these observations: we used the Extension Theorem (Theorem 4.1.2 in Tits [10]) in order to construct the desired homogeneous involutions on certain mixed F_4-buildings; the fact that the fixed point quadrangles are the 'right' ones was verified by showing the existence of certain subquadrangles related to certain D_4- and mixed C_4-buildings.

In our present approach we establish the desired homogeneous involutions as automorphisms of the corresponding group; this approach provides the commutation relations of the fixed point quadrangle directly and it is therefore more efficient for our purposes. However, the geometric evidence is not transparent in our new proof. In the last section of this paper we will describe these geometric aspects of our main result by an *a posteriori* explanation.

Contents

As already mentioned above, a generalized Moufang quadrangle is uniquely determined if one knows the commutation relations for the positive root groups. In Section 2, we mention several examples of Moufang quadrangles which will be of interest in the sequel by giving their commutation relations. In Section 3, we consider type preserving automorphisms of spherical Moufang buildings whose set of fixed points is again a thick building; these automorphisms are called homogeneous. We discuss the Moufang structure of the fixed point building and we draw special attention to homogeneous involutions acting on Moufang buildings of characteristic 2. In Section 4, we introduce a set of involutions $\mathcal{J}$ for a given building of mixed type (namely, those with diagram $\mathsf{C}_2, \mathsf{B}_4, \mathsf{C}_4$ and F_4). The elements of $\mathcal{J}$ are given by certain parameters and it will turn out that each homogeneous involution which is of interest for our purposes is conjugate to an element of $\mathcal{J}$. In Section 5 we restrict to buildings of mixed type F_4: assuming that an element of $\mathcal{J}$ is indeed homogeneous we calculate the commutation relations of the fixed point quadrangle in terms of its parameters. In Section 6 we characterize those parameters which are actually the parameters of the homogeneous involutions in $\mathcal{J}$. In order to do this we have to consider fixed point free involutions acting on certain buildings of mixed type C_2; we obtain two conditions which turn out to be equivalent in the situation in which we are mainly interested. Sections 4 and 6 provide a 'classification' of the homogeneous involutions of type $(\mathsf{F}_4, \{1, 4\})$ and Section 5 gives the fixed point quadrangles. Therefore it remains to put things together and give a precise statement of our results. This will be done in Section 7 whereas in Section 8 we give some geometric background of the whole procedure.

2 Moufang Quadrangles and Commutation Relations

2.1 Definitions and Notation

We give some definitions in order to fix notation and terminology.

A *generalized quadrangle* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a rank 2 geometry with point set $\mathcal{P}$, line set $\mathcal{L}$ and (symmetric) incidence relation $\mathbf{I}$, such that the following axioms are satisfied:

(GQ1) every point lies on at least two, but not on all lines; dually, every line carries at least two, but not all points;

(GQ2) for any given pair $(p, L) \in \mathcal{P} \times \mathcal{L}$, with p not incident with L, there exists a unique pair $(q, M) \in \mathcal{P} \times \mathcal{L}$ such that $p \,\mathbf{I}\, M \,\mathbf{I}\, q \,\mathbf{I}\, L$.

A *subquadrangle* of Γ is a subgeometry $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$, $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, where $\mathbf{I}'$ is the restriction of $\mathbf{I}$ to $\mathcal{P}' \times \mathcal{L}' \cup \mathcal{L}' \times \mathcal{P}'$, which is a generalized quadrangle. A subquadrangle Γ' is called *full*, respectively *ideal*, if every element of Γ incident with any line, respectively point, of Γ', is an element of Γ'. A generalized quadrangle is called *thick* if every element is incident with at least 3 elements. An *apartment* is a subquadrangle such that every element is incident with exactly 2 elements, hence an apartment is an *ordinary quadrangle*. For a given element x of Γ, we denote by $\Gamma(x)$ the set of elements of Γ incident with x. A *collineation* of Γ is a permutation of $\mathcal{P}$ inducing via the incidence relation a permutation of $\mathcal{L}$.

Now let Σ be an apartment of some generalized quadrangle Γ, and name the elements of Σ as x_i, $i = 0, 1, 2, \ldots, 6, 7 \mod 8$, with $x_i \,\mathbf{I}\, x_{i+1}$. The *Moufang condition* states that, for all

i, the group U_i of collineations fixing $\Gamma(x_i)\cup\Gamma(x_{i+1})\cup\Gamma(x_{i+2})$ acts transitively (and hence regulary) on the set $\Gamma(x_{i+3})\setminus\{x_{i+2}\}$ (or, equivalently, on the set $\Gamma(x_{i-1})\setminus\{x_i\}$). Put $U_+=\langle U_1,U_2,U_3,U_4\rangle$. Then Tits, Section 4 of [11], shows that U_+ determines Γ completely and that $U_+=U_1U_2U_3U_4$ (and this decomposition is unique, *i.e.*, every element u of U_+ can be written in a unique way as $u_1u_2u_3u_4$, with $u_i\in U_i$, $i=1,2,3,4$). Hence the Moufang quadrangle Γ is determined by the groups U_1,U_2,U_3,U_4 and the commutators $[u_i,u_j]$, $1\le i<j\le 4$.

2.2 Examples

Our first example is taken from Tits [11]. Let K be a field of characteristic 2, and let K' be a subfield of K containing all squares of K. Let L be a subspace of the vector space K over K', and let L' be a subspace of the vector space K' over K^2 (the subfield of all squares). Suppose that $1\in L\cap L'$ and that L and L' generate K and K', respectively, as rings (these conditions are just needed for a canonical description). Then we define the groups U_1 and U_3 as copies of the additive group of L', and U_2 and U_4 are copies of the additive group of L. We denote the embeddings by $U_i\to\mathrm{K}\colon x_i\mapsto x$. The commutation relations now are:

$$[U_1,U_2]=[U_2,U_3]=[U_3,U_4]=[U_1,U_3]=[U_2,U_4]=\{0\},$$

where we denote the trivial element of each U_i by 0, and

$$[x_1,y_4]=(xy)_2(xy^2)_3.$$

The Moufang quadrangles thus defined are exactly the *mixed quadrangles* $\mathsf{Q}(\mathrm{K},\mathrm{K}';L,L')$.

As we remarked, the conditions on L and L' generating K and K' respectively are there for a canonical notation. In fact, it is enough to ask that L and L' are additive subgroups of some field F of characteristic 2, that $1\in L\cap L'$, that L generates some subfield K of F (as a ring), and that $LL'\subseteq L$, $L^2L'\subseteq L'$. It is then clear that L' generates some subfield K' (as a ring) containing K^2, that $L'L'\subseteq L$ and hence $\mathrm{K}'\subseteq L$. Consequently L can be viewed as a vector space over K', and similarly, L' is a vector space over K^2. So we obtain the mixed quadrangle $\mathsf{Q}(\mathrm{K},\mathrm{K}';L,L')$.

We give a second example. Let K be a field of characteristic 2 and let L be a separable quadratic extension of K. Denote by $x\mapsto\overline{x}$ the non-trivial (involutory) field automorphism of L fixing K pointwise. Let K' be a subfield of K containing the field K^2 of all squares of K and let L' be the subfield of L generated by L^2 and K'. We then have $\mathrm{L}^2\subseteq\mathrm{L}'\subseteq\mathrm{L}$ and L' is a separable quadratic extension of K' (because the map $x\mapsto\overline{x}$ restricts to an automorphism of L' and the fixed subfield is exactly K'). Now let there be given two elements $\alpha\in\mathrm{K}'$ and $\beta\in\mathrm{K}$ such that, for all $u,v\in\mathrm{L}$, and all $a\in\mathrm{K}'$,

$$u\overline{u}+\alpha v\overline{v}+\beta a=0 \tag{1}$$

implies that $u=v=a=0$, and, for all $x,y\in\mathrm{L}'$, and all $b\in\mathrm{K}$,

$$x\overline{x}+\beta^2y\overline{y}+\alpha b^2=0 \tag{2}$$

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

implies that $x = y = b = 0$. We identify U_1 and U_3 with the direct product $\mathrm{L}' \times \mathrm{L}' \times \mathrm{K}$ (additively), and U_2 and U_4 with $\mathrm{L} \times \mathrm{L} \times \mathrm{K}'$. We define the quadrangle $\mathsf{Q}(\mathrm{K}, \mathrm{L}, \mathrm{K}', \alpha, \beta)$ as the Moufang quadrangle with commutation relations (see Tits & Weiss [18]):

$$[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = \{0\}$$

and

$$\begin{aligned}
\big[(x, y, b)_1, (x', y', b')_3\big] &= \Big(0, 0, \alpha\big(x\overline{x}' + x'\overline{x} + \beta^2(y\bar{y}' + y'\bar{y})\big)\Big)_2 \\
\big[(u, v, a)_2, (u', v', a')_4\big] &= \Big(0, 0, \beta^{-1}\big(u\overline{u}' + u'\overline{u} + \alpha(v\overline{v}' + v'\overline{v})\big)\Big)_3 \\
\big[(x, y, b)_1, (u, v, a)_4\big] &= \Big(bu + \alpha(\overline{x}v + \beta y\overline{v}), bv + xu + \beta y\overline{u}, b^2a + a\alpha(x\overline{x} + \beta^2 y\bar{y}) \\
&\qquad + \alpha\big(u^2x\bar{y} + \overline{u}^2\overline{x}y + \alpha(\overline{v}^2xy + v^2\overline{x}\bar{y})\big)\Big)_2 \\
&\qquad \cdot \Big(ax + \overline{u}^2y + \alpha v^2\bar{y}, ay + \beta^{-2}(u^2x + \alpha v^2\overline{x}), ab \\
&\qquad\quad + b\beta^{-1}(u\overline{u} + \alpha v\overline{v}) \\
&\qquad\quad + \alpha\big(\beta^{-1}(xu\overline{v} + \overline{x}\overline{u}v) + y\overline{u}\overline{v} + \bar{y}uv\big)\Big)_3.
\end{aligned}$$

These are the new Moufang quadrangles as discovered by Richard Weiss, and which we would like to give the name: *exceptional Moufang quadrangles of type* F_4, as opposed to the exceptional Moufang quadrangles of type E_i, $i = 6, 7, 8$, see Tits [11], [16].

Now we restrict in the above formulae from L and L' down to K and K', respectively. We define

$$\begin{aligned}
L &= \{\beta^{-1}(u^2 + \alpha v^2) + a : u, v \in \mathrm{K}, a \in \mathrm{K}'\} \subseteq \mathrm{K}; \\
L' &= \{\alpha(x^2 + \beta^2y^2) + b^2 : x, y \in \mathrm{K}', b \in \mathrm{K}\}.
\end{aligned}$$

It is clear that both L and L' are additive subgroups of K and that $1 \in L \cap L'$. We now show that $LL' \subseteq L$. It is easily calculated that a generic element of LL' has the form

$$\beta^{-1}\Big(\big(bu + \alpha(vx + \beta vy)\big)^2 + \alpha(bv + ux + \beta uy)^2\Big) + \big(ab^2 + a\alpha(x^2 + \beta^2y^2)\big), \tag{3}$$

which clearly belongs to L. Similarly, $L^2L' \subseteq L'$. Since L generates $\mathrm{K}'(\beta)$ as a ring, and L' generates $\mathrm{K}^2(\alpha)$ as a ring, there exists a mixed quadrangle $\mathsf{Q}\big(\mathrm{K}'(\beta), \mathrm{K}^2(\alpha); L, L'\big)$. Moreover, restricting the commutation relations of $\mathsf{Q}(\mathrm{K}, \mathrm{L}, \mathrm{K}', \alpha, \beta)$ according to the restrictions from L and L' down to K and K', we see that we obtain the commutation relations of $\mathsf{Q}\big(\mathrm{K}'(\beta), \mathrm{K}^2(\alpha); L, L'\big)$ with respect to the additive isomorphisms

$$\begin{aligned}
\mathrm{K} \times \mathrm{K} \times \mathrm{K}' \to L &: (u, v, a) \mapsto \beta^{-1}(u^2 + \alpha v^2) + a, \\
\mathrm{K}' \times \mathrm{K}' \times \mathrm{K} \to L' &: (x, y, b) \mapsto \alpha(x^2 + \beta^2y^2) + b^2.
\end{aligned}$$

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

We will show that both quadrangles $\mathsf{Q}(\mathrm{K},\mathrm{L},\mathrm{K}';\alpha,\beta)$ and $\mathsf{Q}\big(\mathrm{K}'(\beta),\mathrm{K}^2(\alpha);L,L'\big)$ arise as fixed point structure of an involution in a certain building of mixed type F_4. Therefore, we have to classify fixed point free involutions in mixed quadrangles $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ ("fixed point free" means here: without any fixed elements, either points or lines).

We give a third example. In the commutation relations of the exceptional Moufang quadrangle $\mathsf{Q}(\mathrm{K},\mathrm{L},\mathrm{K}';\alpha,\beta)$ above, we restrict U_1 and U_3 from $\mathrm{L}'\times\mathrm{L}'\times\mathrm{K}$ down to $\{0\}\times\{0\}\times\mathrm{K}$. Writing $(0,0,b)$ in this set as b, we obtain the commutation relations

$$[U_1,U_2]=[U_1,U_3]=[U_2,U_3]=[U_3,U_4]=\{0\}$$

and

$$\big[(u,v,a)_2,(u',v',a')_4\big]=\Big(\beta^{-1}\big(u\overline{u}'+u'\overline{u}+\alpha(v\overline{v}'+v'\overline{v})\big)\Big)_3$$
$$\big[b_1,(u,v,a)_4\big]=(bu,bv,b^2a)_2\cdot\big(b\beta^{-1}(u\overline{u}+\alpha v\overline{v}+\beta a)\big)_3$$

In fact, this is a classical Moufang quadrangle (see Tits [16]). In Van Maldeghem [19], it is called a *Moufang quadrangle of type* $(C-CB)_2$. Below, we will show that it arises from a mixed building of type C_4. We will denote it by $\mathsf{Q}(\mathrm{K},\mathrm{L}\times\mathrm{L}\times\mathrm{K}';\alpha,\beta)$.

3 Homogeneous Involutions and Moufang Buildings

Throughout these notes we will consider buildings as simplicial chamber complexes endowed with an apartment system and we adopt the notation of Tits [10]. All buildings considered in this paper are assumed to be spherical but we do not assume in general that a building is thick (note however that our definition of Moufang buildings will require thickness!). Moreover, an automorphism of a building is always meant to be type preserving. We remark however that everything included in this section remains true (or has a true analogue) without this restriction.

3.1 Homogeneous Automorphisms and Tits Diagrams

Throughout this subsection let M be a spherical Coxeter diagram over a set I, let Δ be a (not necessarily thick) building of type M and let $\mathrm{typ}\colon\Delta\to\mathcal{P}(I)$ be a type function (where $\mathcal{P}(I)$ denotes the set of all subsets of I).

We let $\big(W,(s_i)_{i\in I}\big)$ be the Coxeter system of type M, for $J\subseteq I$ we set $W_J=\langle s_j\mid j\in J\rangle$ and r_J denotes the longest element in W_J.

Let τ be an automorphism of Δ and let $\tilde{\Delta}:=\mathrm{Fix}_\Delta(\tau)$ be the set of simplices fixed under the action of τ. The automorphism τ will be called *homogeneous* if $\tilde{\Delta}$ is a thick building (with the partial ordering induced from Δ).

Lemma 3.1 *Given a homogeneous automorphism τ of Δ and two maximal simplices A, B in $\tilde{\Delta}$, then* $\mathrm{typ}(A)=\mathrm{typ}(B)$. *Moreover* $\mathrm{typ}\,|_{\tilde{\Delta}}\colon\tilde{\Delta}\to\mathcal{P}(\tilde{I})$ *is a type function on $\tilde{\Delta}$ where* $\tilde{I}:=\mathrm{typ}(A)$.

Let τ be a homogeneous involution of the building Δ and let $\tilde{I}$ be as in the previous lemma. The *Tits diagram* of τ is defined to be the pair $(M,\tilde{I})$. It is visualized by drawing circles around the nodes of the Coxeter diagram M which are contained in $\tilde{I}$.

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

It is a fact that the Tits diagram of a homogeneous involution determines the type (*i.e.*, the Coxeter diagram $\tilde{M} = (\tilde{m}_{\tilde{\imath}\tilde{\jmath}})_{\tilde{\imath},\tilde{\jmath}\in\tilde{I}})$ completely: if we set $\tilde{I}_0 : I \setminus \tilde{I}$ then $\tilde{m}_{\tilde{\imath}\tilde{\jmath}}$ is precisely the order of the product $r_{\tilde{I}_0\cup\{\tilde{\imath}\}} r_{\tilde{I}_0\cup\{\tilde{\jmath}\}}$.

3.2 Spherical Moufang Buildings

In this subsection we explain the Moufang condition in order to fix some additional notation which is needed in the sequel. Let Δ be an irreducible building of rank at least 2 and let $\Sigma \subseteq \Delta$ be an apartment. We denote the set of all half apartments of Σ by $\Theta(\Sigma)$; sometimes half apartments are also called 'roots', but we don't do this here because we want to reserve the word 'root' for an element of an abstract root system.

Let θ be any half apartment of Δ. A chamber C is said to be in the interior of θ if $|\operatorname{Cham}\theta \cap \operatorname{Cham}\operatorname{St} A| = 2$ for each face A of C of codimension 1. The set of chambers which are in the interior of θ will be denoted by θ°. The root group U_θ associated to θ is the subgroup of $\operatorname{Aut}(\Delta)$ which stabilizes any chamber adjacent to some chamber in θ°.

The building Δ is called *Moufang* if it is thick and if for each half apartment θ the root group U_θ acts transitively on the set of apartments containing θ. For any Moufang building we denote the group generated by the set of root groups by $G(\Delta)$. It is readily verified that it suffices to take the root groups associated to the half apartments of a single apartment in order to generate $G(\Delta)$. Moreover, $G(\Delta)$ is a normal subgroup of $\operatorname{Aut}(\Delta)$ and hence we can interpret each automorphism of Δ as an automorphism of $G(\Delta)$ as well.

A Moufang building will be said to be of *characteristic* 2 if all root groups are 2-groups; *i.e.*, if the order of any element in a root group has order 2^n for some natural number n.

3.3 Homogeneous Involutions Acting on Moufang Buildings

Throughout this subsection we assume that Δ is a spherical Moufang building. We denote the centralizer of the element g in the group G by $\mathcal{C}_G(g)$. We start with an observation concerning homogeneous involutions:

Lemma 3.2 *Let τ be a homogeneous involution acting on Δ and let $\tilde{\Delta} = \operatorname{Fix}_\Delta(\tau)$. Given an apartment $\tilde{\Sigma}$ of $\tilde{\Delta}$, then there exists an apartment Σ in Δ stabilized by τ and such that $\tilde{\Sigma} = \operatorname{Fix}_\Sigma(\tau)$. In particular, each chamber of $\tilde{\Delta}$ is contained in an apartment which is stable under τ.*

Proof Let $\tilde{C}$ be a chamber of $\tilde{\Sigma}$ and let C be a chamber of Δ which contains $\tilde{C}$. Then $\tau(C)$ is opposite to C in $\operatorname{St}\tilde{C}$ because $\tau|_{\operatorname{St}\tilde{C}}$ is a fixed point free involution. It follows that the convex closure of $\tilde{\Sigma}, C$ and $\tau(C)$ is an apartment of Δ which is stable under τ. The second assertion follows from the fact that each chamber of $\tilde{\Delta}$ is contained in an apartment of $\tilde{\Delta}$. ■

The following proposition can be seen as a combinatorial version of some basic facts which are well known from the theory of Galois descents in semi-simple algebraic groups. A detailed proof of this geometric version can be easily extracted from Mühlherr [6].

Proposition 3.3 *Let τ be a homogeneous involution acting on Δ and suppose that $\tilde{\Delta}$ is irreducible and of rank at least 2. Then $\tilde{\Delta} := \operatorname{Fix}_\Delta(\tau)$ is a Moufang building. Let $\tilde{\Sigma}$ be an*

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

apartment of $\tilde{\Delta}$, *let* $\tilde{\theta}_1$, $\tilde{\theta}_2$ *be non-opposite half apartments contained in* $\tilde{\Sigma}$ *and let* $\tilde{U}_1, \tilde{U}_2$ *be the corresponding root groups. Then there exist subgroups* $\tilde{U}_{\tilde{\theta}_1}, \tilde{U}_{\tilde{\theta}_2}$ *of* $\mathcal{C}_{G(\Delta)}(\tau)$ *such that* $\tilde{U}_{\tilde{\theta}_1}$ *(respectively* $\tilde{U}_{\tilde{\theta}_2}, \langle \tilde{U}_{\tilde{\theta}_1}, \tilde{U}_{\tilde{\theta}_2} \rangle$*) is mapped isomorphically onto* $\tilde{U}_1$ *(respectively* $\tilde{U}_2, \langle \tilde{U}_1, \tilde{U}_2 \rangle$*) under the canonical homomorphism* $\mathcal{C}_{G(\Delta)}(\tau) \to \mathrm{Aut}(\tilde{\Delta})$.

Since it will be of interest in Section 5, we describe in some detail how the group $\tilde{U}_{\tilde{\theta}}$ for a given half apartment of $\tilde{\Delta}$ can be obtained. Let Σ be an apartment of Δ which contains $\tilde{\theta}$ and which is stabilized by τ (*cf.* Lemma 3.2) and set $\tilde{\Sigma} = \mathrm{Fix}_{\Sigma}(\tau)$; let H_0 denote the set of half apartments of Σ which contain $\tilde{\Sigma}$, let $H'_{\tilde{\theta}}$ denote the set of half apartments of Σ which contain $\tilde{\theta}$ and put $H_{\tilde{\theta}} := H'_{\tilde{\theta}} \setminus H_0$ and $U_{\tilde{\theta}} := \langle U_\theta \mid \theta \in H_{\tilde{\theta}} \rangle$. Now $\tilde{U}_{\tilde{\theta}}$ is precisely the centralizer of τ in $U_{\tilde{\theta}}$.

We close this subsection with an observation in the characteristic 2 case:

Lemma 3.4 *Suppose that* Δ *is of characteristic* 2 *and that* τ *is an involution acting on* Δ. *Then the following are equivalent:*

(i) τ *is homogeneous.*
(ii) There exists a pair $(\tilde{C}, \Sigma)$ *consisting of a maximal simplex* $\tilde{C}$ *in* $\mathrm{Fix}_{\Delta}(\tau)$ *and an apartment* Σ *of* Δ *such that* $\tilde{C}$ *is contained in* Σ *and such that* Σ *is stabilized by* τ.

Proof That (i) implies (ii) follows from Lemma 3.2.

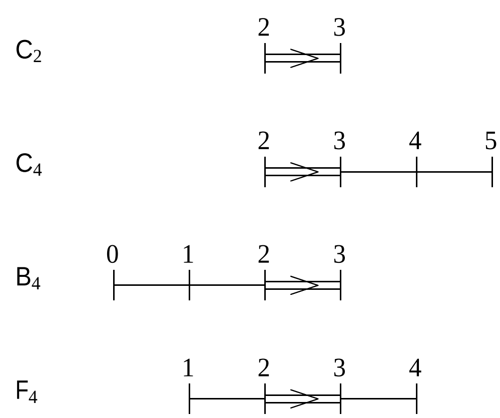
In order to show the converse implication we remark that it is sufficient to show that the chamber complex $\tilde{\Delta} := \mathrm{Fix}_{\Delta}(\tau)$ is thick. Thus we can reduce the problem to the case where the rank of $\tilde{\Delta}$ is one. But in this case the assertion is trivial because an involution acting on a 2-group U has a non-trivial centralizer. ■

4 Coordinates of Homogeneous Involutions Acting on Certain Buildings of Mixed Type

4.1 Conventions

From now on, until the end of this section, we will use the following convention:

We assume that X is a Dynkin diagram in the set $\mathbf{X} := \{\mathsf{C}_2, \mathsf{B}_4, \mathsf{C}_4, \mathsf{F}_4\}$. These diagrams will be labelled according to the following picture:



For each $X \in \mathbf{X}$ we denote the set of its labels by $I(X)$; for instance $I(\mathsf{B}_4) = \{0, 1, 2, 3\}$.

Let $\mathbf{\Phi}$ denote the root system of type X. The set of short (respectively long) roots in Φ is denoted by Φ_s (respectively Φ_l). Moreover, $\mathcal{B} = \{\eta_i \mid i \in I(X)\}$ will be a basis of the root system $\mathbf{\Phi}$. The Weyl group of $\mathbf{\Phi}$ will be denoted by W and for each $\phi \in \Phi$ we let $s_\phi \in W$ be the reflection associated to ϕ; we put $s_i := s_{\eta_i}$ for $i \in I(X)$.

We assume that L is a field of characteristic 2 and that L' is a subfield of L containing L^2. We denote the set of all automorphisms of L which stabilize L' by $\mathrm{Aut}(\mathrm{L}, \mathrm{L}')$ and for $X \in \mathbf{X}$ we put $T_X(\mathrm{L}, \mathrm{L}') = \{(\lambda_i)_{i \in I(X)} \mid \lambda_i \in \mathrm{L} \setminus \{0\} \text{ for } i \geq 3, \lambda_i \in \mathrm{L}' \text{ for } i \leq 2\}$.

We let Δ be the building of mixed type associated to the group $X(\mathrm{L}, \mathrm{L}')$ (*cf.* Tits [10]), Σ is an apartment of Δ and Θ denotes the set of all half apartments of Δ which are contained in Σ.

We let $\mathrm{Spe}(\Delta)$ be the set of all type preserving automorphisms of Δ and $\hat{N}$ denotes the stabilizer of Σ in $\mathrm{Spe}(\Delta)$.

Remark It follows from Tits [10] that Δ is a Moufang building of characteristic 2, the root groups being isomorphic to the additive groups of L and L'.

4.2 Parametrisations

A *parametrisation* of Δ is a triple $\Pi = \big(\omega, (x_\phi)_{\phi \in \Phi_s}, (y_\phi)_{\phi \in \Phi_l}\big)$ where ω is a bijection from Φ onto Θ and where x_ϕ (respectively $y_{\phi'}$) is an isomorphism from the additive group of L (respectively L') onto $U_{\omega(\phi)}$ (respectively $U_{\omega(\phi')}$) for each $\phi \in \Phi_s$ (respectively $\phi' \in \Phi_l$) such that the following conditions are satisfied:

(i) The mapping $B \to \bigcap_{\eta \in B} \omega(\eta)$ defines a bijection from the set of bases of $\mathbf{\Phi}$ onto the set of chambers in Σ.

(ii) Given $\delta, \delta' \in \Phi_s$ satisfying $\delta' \notin \{+\delta, -\delta\}$ and $t, t' \in \mathrm{L}$, then $[x_\delta(t), x_{\delta'}(t')] = x_{\delta+\delta'}(tt')$ if $\delta + \delta' \in \Phi_s$ and $[x_\delta(t), x_{\delta'}(t')] = \mathrm{id}_\Delta$ in the remaining cases.

(iii) Given $\delta \in \Phi_s$, $\delta' \in \Phi_l$, $t \in \mathrm{L}$ and $s \in \mathrm{L}'$ then $[x_\delta(t), y_{\delta'}(s)] = x_{\delta+\delta'}(ts) y_{2\delta+\delta'}(t^2 s)$ if $\delta + \delta' \in \Phi$ and $[x_\delta(t), y_{\delta'}(s)] = \mathrm{id}_\Delta$ in the remaining cases.

(iv) Given $\delta, \delta' \in \Phi_l$ satisfying $\delta' \notin \{+\delta, -\delta\}$ and $s, s' \in \mathrm{L}'$, then $[y_\delta(s), y_{\delta'}(s')] = y_{\delta+\delta'}(ss')$ if $\delta + \delta' \in \Phi$ and $[y_\delta(s), y_{\delta'}(s')] = \mathrm{id}_\Delta$ in the remaining cases.

The following proposition is immediate from Tits [10, 10.3.2].

Proposition 4.1 *There exists a parametrisation of* Δ.

From now on until the end of this section we fix a parametrisation $\Pi = \big(\omega, (x_\phi)_{\phi \in \Phi_s}, (y_\phi)_{\phi \in \Phi_l}\big)$ of Δ and we put $C = \omega(\mathcal{B})$.

4.3 Coordinatization of $\hat{N}$

Let $\tau \in \hat{N}$. Then there exists a unique element $(w, \sigma, \lambda) \in W \times \mathrm{Aut}(\mathrm{L}, \mathrm{L}') \times T_X(\mathrm{L}, \mathrm{L}')$ such that the following holds:

(1) $\tau\big(x_{\eta_i}(t)\big) = x_{w(\eta_i)}\big(\lambda_i \sigma(t)\big)$ for all $i \geq 3$ and $t \in \mathrm{L}$.

(2) $\tau\big(y_{\eta_i}(s)\big) = y_{w(\eta_i)}\big(\lambda_i \sigma(s)\big)$ for all $i \leq 2$ and $s \in \mathrm{L}'$.

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

Moreover, if $\phi = \Sigma_{i \in I(X)} n_i \eta_i \in \Phi_s$, then $\tau\big(x_\phi(t)\big) = x_{w(\phi)}\big(\lambda_\phi \sigma(t)\big)$ for all $t \in \mathrm{L}$ with $\lambda_\phi := \Pi_{i \in I(X)} \lambda_i^{n_i}$. The analogous statement holds also for $\phi \in \Phi_l$ if we replace x by y and $t \in \mathrm{L}$ by $s \in \mathrm{L}'$.

Given $\tau \in \hat{N}$ then the triple (w, σ, λ) is called the *coordinate* of τ and we denote it by $[\tau]$.

We have also the 'converse' of the previous fact: Given $\psi = (w, \sigma, \lambda) \in W \times \mathrm{Aut}(\mathrm{L}, \mathrm{L}') \times T_X(\mathrm{L}, \mathrm{L}')$ then there exists an element $\tau \in \hat{N}$ such that $[\tau] = \psi$; this element will be denoted by τ_ψ.

Remark The statements made so far follow from Steinberg [8] and Carter [2] and are actually well known if $\mathrm{L} = \mathrm{L}'$ (in other words: if Δ is the building associated to the Chevalley group of type X over the field L). Using Tits [10, 10.3.2 and Theorem 10.4], one easily verifies that they remain valid for mixed groups.

The next lemma follows from an elementary calculation (bearing in mind that $[\mathrm{id}_\Delta] = \big(1_W, \mathrm{id}_\mathrm{L}, (1_\mathrm{L}, \ldots, 1_{\mathrm{L}'})\big)$).

Lemma 4.2 *Let $\tau \in \hat{N}$ be such that $\tau^2 = \mathrm{id}_\Delta$ and let (w, σ, λ) be its coordinate. Then:*

(i) $w^2 = 1_W$
(ii) $\sigma^2 = \mathrm{id}_L$
(iii) If λ_ϕ is defined as above, then $\lambda_\phi \sigma(\lambda_{w(\phi)}) = 1_\mathrm{L}$ for all $\phi \in \Phi$.

The converse holds as well: Given a triple $\psi = (w, \sigma, \lambda) \in W \times \mathrm{Aut}(\mathrm{L}, \mathrm{L}') \times T_X(\mathrm{L}, \mathrm{L}')$ satisfying (i)–(iii), then $\tau_\psi^2 = \mathrm{id}_\Delta$.

4.4 Homogeneous Involutions Having Tits Diagram $\big(X, I(X) \setminus \{2, 3\}\big)$

Throughout this section we assume that $\sigma \in \mathrm{Aut}(\mathrm{L}, \mathrm{L}')$ satisfies $\sigma^2 = \mathrm{id}_\mathrm{L}$ and we write $\bar{t}$ instead of $\sigma(t)$ for $t \in \mathrm{L}$. We put $\mathrm{K} = \{t \in \mathrm{L} \mid t = \bar{t}\}$ and $\mathrm{K}' = \mathrm{L}' \cap \mathrm{K}$. We set $w_1 = s_2 s_3 s_2 s_3$ and define (writing $\mathrm{F}^\times := \mathrm{F} \setminus \{0\}$ for an arbitrary field F)

$$P(\mathsf{C}_2) = \mathrm{K}'^\times \times \mathrm{K}^\times, \qquad P(\mathsf{B}_4) = \mathrm{K}'^\times \times \mathrm{K}'^\times,$$
$$P(\mathsf{C}_4) = (\mathrm{K}^2)^\times \times \mathrm{K}^\times, \qquad P(\mathsf{F}_4) = (\mathrm{K}^2)^\times \times \mathrm{K}'^\times.$$

For $(b, a) \in P(X)$ we define $\lambda_{(b,a)} = \lambda \in T_X(\mathrm{L}, \mathrm{L}')$ as follows: $\lambda_1 := (ab)^{-1}$, $\lambda_2 := b$, $\lambda_3 := a$, $\lambda_4 := (b^{1/2} a)^{-1}$ and $\lambda_0 = \lambda_5 := 1_\mathrm{L}$. Finally we define for $X \in \mathbf{X}$ the set $\mathcal{I}(X) = \{\tau_{w_1, \sigma, \lambda_{(b,a)}} \mid (b, a) \in P(X)\}$.

Lemma 4.3 *Let $\tau \in \hat{N}$ be a homogeneous involution of Δ having Tits diagram $\big(X, I(X) \setminus \{2, 3\}\big)$, $X \in \mathbf{X}$, and stabilizing the simplex of cotype $\{2, 3\}$ contained in C. Let $[\tau] = (w, \sigma, \lambda)$ be its coordinate. Then*

(i) $w = w_1$
(ii) There exists $\mu \in T_X(\mathrm{L}, \mathrm{L}')$ such that $\tau_{(1_W, \mathrm{id}_L, \mu)} \tau \tau^{-1}_{(1_W, \mathrm{id}_L, \mu)} = \tau_{(w_1, \sigma, \lambda_{(b,a)})}$ for some $(b, a) \in P(X)$.

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

Proof Assertion (i) is obvious.

We now show (ii). We have

$$w_1(\eta_0) = \eta_0, \qquad w_1(\eta_1) = \eta_1 + 2\eta_2 + 2\eta_3, \qquad w_1(\eta_2) = -\eta_2,$$
$$w_1(\eta_3) = -\eta_3, \qquad w_1(\eta_4) = \eta_2 + 2\eta_3 + \eta_4, \qquad w_1(\eta_5) = \eta_5$$

and by Lemma 4.2(iii) it follows that $\overline{\lambda}_0\lambda_0 = \overline{\lambda}_1\lambda_1\lambda_2^2\lambda_3^2 = \overline{\lambda}_2\lambda_2^{-1} = \overline{\lambda}_3\lambda_3^{-1} = \overline{\lambda}_4\lambda_4\lambda_2\lambda_3^2 = \overline{\lambda}_5\lambda_5 = 1_{\mathrm{L}}$ and in particular that $\lambda_2 \in \mathrm{K}'$, $\lambda_3 \in \mathrm{K}$.

We put

$$\lambda^* = \begin{cases} 1_{\mathrm{L}} + \lambda_1\lambda_2\lambda_3 & \text{if } \overline{\lambda}_1 \neq \lambda_1, \\ 1_{\mathrm{L}} & \text{else,} \end{cases}$$

$$\mu^* = \begin{cases} 1_{\mathrm{L}} + \lambda_2\lambda_3^2\lambda_4^2 & \text{if } \overline{\lambda}_4 \neq \lambda_4, \\ 1_{\mathrm{L}} & \text{else,} \end{cases}$$

$$\mu_0 = \begin{cases} 1_{\mathrm{L}} + \overline{\lambda}_0 & \text{if } \overline{\lambda}_0 \neq \lambda_0, \\ 1_{\mathrm{L}} & \text{else,} \end{cases}$$

$$\mu_5 = \begin{cases} 1_{\mathrm{L}} + \overline{\lambda}_5 & \text{if } \overline{\lambda}_5 \neq \lambda_5, \\ 1_{\mathrm{L}} & \text{else,} \end{cases}$$

and $\mu_1 = 1_{\mathrm{L}}$, $\mu_2 = \mu^{*-1}$, $\mu_3 = \mu^*\lambda^{*-1}$, $\mu_4 = \mu^{*-1}\lambda^*$.

Now we define $\lambda' \in T_X(\mathrm{L}, \mathrm{L}')$ via the identity

$$\tau_{(w_1,\sigma,\lambda')} = \tau_{(1_W,\mathrm{id}_{\mathrm{L}},\mu)}\tau\tau^{-1}_{(1_W,\mathrm{id}_{\mathrm{L}},\mu)},$$

where $\mu \in T_X(\mathrm{L}, \mathrm{L}')$ is defined by $\mu = (\mu_i)_{i\in I(X)}$.

Now one calculates that $\lambda_0' = \lambda_5' = 1$, $\lambda_1' = (\lambda_1^{-1} + \overline{\lambda}_1^{-1})^{-1}$ (if $\lambda_1 \neq \overline{\lambda}_1$; otherwise $\lambda_1' = \lambda_1$) and $\lambda_4' = (\lambda_4 + \overline{\lambda}_4)(\lambda^*\overline{\lambda}^*)^{-1}$ (if $\lambda_4 \neq \overline{\lambda}_4$; otherwise $\lambda_4' = \lambda_4$). Hence $\overline{\lambda}_i' = \lambda_i'$ for all $i \in I(X)$ and as $\tau_{(w_1,\sigma,\lambda')}$ is a homogeneous involution the claim follows from Lemma 4.2(iii) if we put $b := \lambda_2' = (\mu_2\bar{\mu}_2)^{-1}\lambda_2$ and $a := \lambda_3' = (\mu_3\bar{\mu}_3)^{-1}\lambda_3$. ■

Corollary 4.4 *Given any homogenous involution τ of Δ having Tits-diagram $\big(X, I(X) \setminus \{2,3\}\big)$, then τ is conjugate to an element of $\mathcal{I}(X)$.*

Proof By Lemma 3.4 there exists a pair $(\tilde{C}, \Sigma')$ consisting of a fixed simplex $\tilde{C}$ of type $I(X)\setminus\{2,3\}$ and an apartment Σ' containing $\tilde{C}$ which is stable under τ. Thus τ is conjugate (in Spe(Δ)) to an element of $\hat{N}$ which stabilizes the $\big(I(X)\setminus\{2,3\}\big)$-face of C. The assertion follows from the previous lemma. ■

The following Lemma is an easy consequence of Lemma 3.4:

Lemma 4.5 *Let τ be an involution in $\mathcal{I}(X)$ and let B denote the face of cotype $\{2,3\}$ of C. Then τ is a homogenous involution of Δ with Tits-diagram $\big(X, I(X) \setminus \{2,3\}\big)$ if and only if $\tau|_{St(B)}$ acts fixed point freely.*

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

5 The Fixed Point Structure of Homogeneous Involutions Acting on Mixed F_4-Buildings and Having Tits Diagram $(\mathsf{F}_4, \{1, 4\})$

In this section we are going to calculate the commutation relations of the fixed point quadrangle of a homogeneous involution acting on a mixed F_4-building and having Tits diagram $(\mathsf{F}_4, \{1, 4\})$. It follows from Corollary 4.4, that we only have to consider homogeneous involutions which are contained in $\mathcal{I}(\mathsf{F}_4)$. We adopt the conventions of the previous sections and we assume that $X = \mathsf{F}_4$. According to Lemma 4.3, we let $(b, a) \in K^2 \times K'$ and we put $\tau = \tau_{(w_1,\sigma,((ab)^{-1},b,a,(ba^2)^{-1/2}))}$. Note that in Section 7, we will identify the commutation relations of the fixed point quadrangle of τ with the commutation relations of the new Moufang quadrangles, or of certain mixed quadrangles.

We set $c_1 = (ab)^{-1}$, $c_2 = b$, $c_3 = b$, $c_4 = (ba^2)^{-1/2}$, $\tilde{\Sigma} = \mathrm{Fix}_{\Sigma}(\tau)$ and $\tilde{C}$ denotes the face of type $\{1, 4\}$ of the chamber C.

5.1 The 'Positive' Half Apartments Contained in $\tilde{\Sigma}$

We introduce some more notation. From now on, until the end of this section, we will represent the root $\phi = n_1\eta_1 + n_2\eta_2 + n_3\eta_3 + n_4\eta_4$ as $n_1n_2n_3n_4$. Moreover, we will write U_ϕ for $U_{\omega(\phi)}$.

With the notation of Subsection 3.3 we are going to describe the groups $\tilde{U}_{\tilde{\theta}}$. Clearly, $\tilde{\Sigma}$ is the flag complex of an ordinary quadrangle. Let $p_0, \dots, p_3$ (respectively $l_0, \dots, l_3$) denote the simplices of type 1 (respectively 4) contained in $\tilde{\Sigma}$. Without loss, we may assume that $\tilde{C} = p_0 \cup l_0$ and that p_i is incident to l_j whenever $j = i$ or $j \equiv i + 1 \mod 4$.

We define the half apartments $\tilde{\theta}_i$ of $\tilde{\Sigma}$ as follows:

$$\tilde{\theta}_1 = l_3 \cup p_4 \cup l_4 \cup p_0 \cup l_0, \qquad \tilde{\theta}_2 = p_4 \cup l_4 \cup p_0 \cup l_0 \cup p_1,$$
$$\tilde{\theta}_3 = l_4 \cup p_0 \cup l_0 \cup p_1 \cup l_1, \qquad \tilde{\theta}_4 = p_0 \cup l_0 \cup p_1 \cup l_1 \cup p_2.$$

With the notation of Subsection 3.3 we define for $i = 1, \dots, 4$ the subset R_i of Φ by $R_i = \omega^{-1}(H_{\tilde{\theta}_i})$. One verifies that

$$R_1 = \{1000, 1100, 1110, 1120, 1220\}$$
$$R_2 = \{1111, 1121, 2342, 1221, 1231\}$$
$$R_3 = \{1122, 1222, 1232, 1242, 1342\}$$
$$R_4 = \{0001, 0011, 0122, 0111, 0121\}$$

Note that we can already see that the groups $U_{\tilde{\theta}_i} = \langle U_\phi \mid \phi \in R_i \rangle$ (with the notation of Subsection 3.3) are abelian. Indeed, for instance $U_{\tilde{\theta}_1} = \langle U_{1000}, U_{1100}, U_{1110}, U_{1120}, U_{1220} \rangle$ and the latter is abelian by Subsection 4.2(*ii*).

5.2 The Root Groups $\tilde{U}_{\tilde{\theta}_i}$

In order to determine the root groups $\tilde{U}_{\tilde{\theta}_i}$ we have to determine $\mathcal{C}_{U_{\tilde{\theta}_i}}(\tau)$. In order to do so we introduce the following definitions:

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

For $p \in \mathrm{K}'$ we put

$$v_1(p) = y_{0122}(p), \qquad v_2(p) = y_{2342}(p),$$

and for $i = 1, 2$ we set $V_i = \{v_i(p) \mid p \in \mathrm{K}'\}$.

For $q \in \mathrm{K}$ we put

$$z_1(q) = x_{1110}(q), \qquad z_2(q) = x_{1232}(q),$$

and for $i = 1, 2$ we set $Z_i = \{z_i(q) \mid q \in \mathrm{K}\}$.

For $s \in \mathrm{L}'$ we put

$$y_1(s) = y_{1000}(s) y_{1220}(c_1 \bar{s}), \qquad y_2(s) = y_{1100}(s) y_{1120}(c_1 c_2 \bar{s}),$$
$$y_3(s) = y_{1122}(s) y_{1342}(c_1 \bar{s}), \qquad y_4(s) = y_{1222}(s) y_{1242}(c_1 c_2 \bar{s}),$$

and for $i = 1, \dots, 4$ we set $Y_i = \{y_i(s) \mid s \in \mathrm{L}'\}$.

For $t \in \mathrm{L}$ we put

$$x_1(t) = x_{0001}(t) x_{0121}(c_4 \bar{t}), \qquad x_2(t) = x_{0011}(t) x_{0111}(c_3 c_4 \bar{t}),$$
$$x_3(t) = x_{1111}(t) x_{1231}(c_4 \bar{t}), \qquad x_4(t) = x_{1121}(t) x_{1221}(c_3 c_4 \bar{t}),$$

and for $i = 1, \dots, 4$ we put $X_i = \{x_i(t) \mid t \in \mathrm{L}\}$.

It is easily verified, that the V_i, Z_i, Y_i, X_i are subgroups of $G(\Delta)$ centralizing τ. Moreover, one has the following:

$$\tilde{U}_{\tilde{\theta}_1} = Y_1 Y_2 Z_1, \qquad \tilde{U}_{\tilde{\theta}_2} = X_3 X_4 V_2,$$
$$\tilde{U}_{\tilde{\theta}_3} = Y_3 Y_4 Z_2, \qquad \tilde{U}_{\tilde{\theta}_4} = X_1 X_2 V_1.$$

5.3 Commutation Relations for the $\tilde{U}_{\tilde{\theta}_i}$

We will consider the $\tilde{U}_{\tilde{\theta}_i}$ as products of the V_i, Z_i, Y_i and X_i and give the commutation relations in terms of these factors at the end of this subsection. As the computations are a bit involved we give only the results of the two most important intermediate steps. The reader should have no difficulties to reconstruct the details, provided he has enough patience.

First Step

$$\big[v_1(p), z_1(q)\big] = z_2(pq) v_2(pq^2),$$
$$\big[v_1(p), y_1(s)\big] = v_2(c_1 p s \bar{s}) y_3(ps),$$
$$\big[v_1(p), y_2(s)\big] = v_2(c_1 c_2 p s \bar{s}) y_4(ps),$$
$$\big[z_1(q), x_1(t)\big] = z_2(c_4 q t \bar{t}) x_3(qt),$$
$$\big[z_1(q), x_2(t)\big] = z_2(c_3 c_4 q t \bar{t}) x_4(qt),$$

$$
\begin{aligned}
\big[y_1(s), y_3(s')\big] &= v_2\big(c_1(s\bar{s}' + \bar{s}s')\big),\\
\big[y_2(s), y_4(s')\big] &= v_2\big(c_1c_2(s\bar{s}' + \bar{s}s')\big),\\
\big[x_1(t), x_3(t')\big] &= z_2\big(c_4(t\bar{t}' + \bar{t}t')\big),\\
\big[x_2(t), x_4(t')\big] &= z_2\big(c_3c_4(t\bar{t}' + \bar{t}t')\big),\\
\big[y_1(s), x_1(t)\big] &= x_4(c_4\bar{t}s)y_4(c_1t^2\bar{s}),\\
\big[y_1(s), x_2(t)\big] &= x_3(c_3c_4\bar{t}s)y_4\big((c_3c_4)^2\bar{t}^2 s\big),\\
\big[y_2(s), x_1(t)\big] &= x_4(c_1c_2t\bar{s})y_3(c_1c_2t^2\bar{s}),\\
\big[y_2(s), x_2(t)\big] &= x_3(ts)y_3(t^2 s).
\end{aligned}
$$

All other commutation relations between two groups of the V_i, Z_i, Y_i and X_i are trivial.

Second Step

$$
\begin{aligned}
\big[y_1(s)y_2(s'), x_1(t)\big] &= x_4(c_1c_2t\bar{s}' + c_4\bar{t}s)v_2\big(c_1^2c_2(s'\bar{t}^2 s + \bar{s}'t^2\bar{s})\big)y_3(c_1c_2t^2\bar{s}')y_4(c_1t^2\bar{s}),\\
\big[y_1(s)y_2(s'), x_2(t')\big] &= x_3(t's' + c_3c_4\bar{t}'s)v_2\big(c_1(s't'^2\bar{s} + \bar{s}'\bar{t}'^2 s)\big)y_3(t'^2 s')y_4(c_4^2c_3^2\bar{t}'^2 s),\\
\big[y_1(s)y_2(s'), x_1(t)x_2(t')\big] &= \big[y_1(s)y_2(s'), x_1(t)\big]\big[y_1(s)y_2(s'), x_2(t')\big]\\
&\quad \cdot z_2\big(c_3c_4^2(t't\bar{s} + \bar{t}\bar{t}'s) + c_4(t'\bar{t}s' + \bar{t}'t\bar{s}')\big).
\end{aligned}
$$

The Final Result

$$
\begin{aligned}
\big[y_1(s_1)y_2(s_2)z_1(q_1), y_3(s_3)y_4(s_4)z_2(q_2)\big] &= v_2\Big(c_1\big(s_1\bar{s}_3 + \bar{s}_1s_3 + c_2(s_2\bar{s}_4 + \bar{s}_2s_4)\big)\Big),\\
\big[x_1(t_1)x_2(t_2)v_1(p_1), x_3(t_3)x_4(t_4)v_2(p_2)\big] &= z_2\Big(c_4\big(t_1\bar{t}_3 + \bar{t}_1t_3 + c_3(t_2\bar{t}_4 + \bar{t}_2t_4)\big)\Big),\\
\big[y_1(s_1)y_2(s_2)z_1(q_1), x_1(t_1)x_2(t_2)v_1(p_1)\big] &= x_3(t_3)x_4(t_4)v_2(p_2)y_3(s_3)y_4(s_4)z_2(q_2),
\end{aligned}
$$

where

$$
\begin{aligned}
t_3 &= t_2s_2 + c_3c_4\bar{t}_2s_1 + q_1t_1,\\
t_4 &= c_1c_2t_1\bar{s}_2 + c_4\bar{t}_1s_1 + q_1t_2,\\
p_2 &= c_1^2c_2(\bar{t}_1^2s_1s_2 + t_1^2\bar{s}_1\bar{s}_2) + c_1(t_2^2\bar{s}_1s_2 + \bar{t}_2^2s_1\bar{s}_2) + c_1p_1(s_1\bar{s}_1 + c_2s_2\bar{s}_2) + p_1q_1^2,\\
s_3 &= c_1c_2t_1^2\bar{s}_2 + t_2^2s_2 + p_1s_1,\\
s_4 &= c_1t_1^2\bar{s}_1 + (c_4c_3)^2\bar{t}_2^2s_1 + p_1s_2,\\
q_2 &= c_3c_4^2(t_1t_2\bar{s}_1 + \bar{t}_1\bar{t}_2s_1) + c_4(\bar{t}_1t_2s_2 + t_1\bar{t}_2\bar{s}_2) + c_4q_1(t_1\bar{t}_1 + c_3t_2\bar{t}_2) + p_1q_1.
\end{aligned}
$$

6 Fixed Point Free Involutions of the Mixed Quadrangle $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$

In this section, we investigate the conditions on (b,a) for $\tau = \tau_{(w_1,\sigma,\lambda_{(b,a)})}$ to be a homogeneous involution having Tits diagram $(\mathsf{F}_4,\{1,4\})$. To that end, it suffices to look at the action of τ on any residue of type C_2 it stabilizes. We have to express that τ does not fix any element in such a residue. This requires some explicit calculations, and therefore we have to consider a model of such a residue. We choose to work geometrically. Everything could also be done on the group-theoretical level. For clarity, we will denote the restriction of τ to a residue of type C_2 by θ in this section.

Since we are interested in involutions which act fixed point free on both the set of points and the set of lines of a mixed quadrangle, we first find a (geometric) representation of the dual of such a quadrangle, and the action of a collineation on that dual. We could use the Klein quadric, but we have chosen to use coordinates in the sense of Hanssens & Van Maldeghem [3].

6.1 The Quadrangles $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ and $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$

The symplectic quadrangle $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$ can be defined as follows: The points are the points of the projective space $\mathbf{PG}(3,\mathrm{L})$, and we relabel them according to the following table:

POINTS	
Coordinates in $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$	Coordinates in $\mathbf{PG}(3,\mathrm{L})$
(∞)	$(1,0,0,0)$
(a)	$(a,0,1,0)$
(k,b)	$(b,0,k,1)$
(a,l,a')	$(l+aa',1,a',a)$

The lines are the lines of $\mathbf{PG}(3,\mathrm{L})$ which are totally isotropic with respect to an alternating bilinear form, *e.g.*, $X_0Y_1+X_1Y_0+X_2Y_3+X_3Y_2$. If we denote by $\langle a,b\rangle$ the line generated by the points a and b, then we can represent the lines of $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$ as follows:

LINES	
Coordinates in $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$	Coordinates in $\mathbf{PG}(3,\mathrm{L})$
$[\infty]$	$\langle(1,0,0,0),(0,0,1,0)\rangle$
$[k]$	$\langle(1,0,0,0),(0,0,k,1)\rangle$
$[a,l]$	$\langle(a,0,1,0),(l,1,0,a)\rangle$
$[k,b,k']$	$\langle(b,0,k,1),(k',1,b,0)\rangle$

It is readily checked that the incidence relation can be written as (see *e.g.* Van Maldeghem [19])

$$[k,b,k']\,\mathbf{I}(k,b)\,\mathbf{I}[k]\,\mathbf{I}(\infty)\,\mathbf{I}[\infty]\,\mathbf{I}(a)\,\mathbf{I}[a,l]\,\mathbf{I}(a,l,a')$$

and

$$\begin{cases} k' = a^2k+l \\ a' = ak+b \end{cases}$$

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

If we want to assign to the lines coordinates of points of $\mathbf{PG}(3,\mathrm{L})$, and to the points coordinates of totally isotropic lines of $\mathbf{PG}(3,\mathrm{L})$ (with respect to the above alternating form), then the dual of the incidence relation must look the same as above. This can be achieved by mapping $[k,b,k']$ onto (k,b^2,k') and (a,l,a') onto $[a^2,l,a'^2]$. This way, we get a representation of $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$ where the lines are certain points of $\mathbf{PG}(3,\mathrm{L})$, as follows:

POINTS	
Coordinates in $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$	Coordinates in $\mathbf{PG}(3,\mathrm{L})$
$[\infty]$	$(1,0,0,0)$
$[k]$	$(k,0,1,0)$
$[a,l]$	$(l,0,a^2,1)$
$[k,b,k']$	$(b^2+kk'',1,k',k)$

Notice that a point (x_0,x_1,x_2,x_3) of $\mathbf{PG}(3,\mathrm{L})$ represents a line of $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$ if and only if $x_0x_1+x_2x_3\in\mathrm{L}^2$.

Now the quadrangle $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ is obtained by restricting the coordinates k, k', l above to L'. Hence a point (x_0,x_1,x_2,x_3) of $\mathbf{PG}(3,\mathrm{L})$ is a point of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ if and only if $x_0x_1+x_2x_3\in\mathrm{L}'$. Consequently, the lines of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ are the points (y_0,y_1,y_2,y_3) of $\mathbf{PG}(3,\mathrm{L}')$ (because $\mathrm{L}^2\subseteq\mathrm{L}'$) with $y_0y_1+y_2y_3\in\mathrm{L}^2$.

6.2 Involutions of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$

Now let θ be a (fixed point free) involution of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$. Then θ arises from some involution of $\mathbf{PG}(3,\mathrm{L})$. Since the collineation group of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ acts transitively on the set of all apartments, we may assume that $(\infty)^\theta=(0,0,0)$, $(0,0,0)^\theta=(\infty)$, $(0,0)^\theta=(0)$ and $(0)^\theta=(0,0)$. Let σ be an automorphism of L of order at most 2 (and let K and K$'$ be as before; also denote $\sigma(t)=\bar t$, for all $t\in\mathrm{L}$). Then we may represent θ as

$$\begin{pmatrix}x_0\\x_1\\x_2\\x_3\end{pmatrix}\mapsto\begin{pmatrix}0&\beta&0&0\\1&0&0&0\\0&0&0&\gamma\\0&0&\alpha^{-1}&0\end{pmatrix}\begin{pmatrix}\overline{x}_0\\\overline{x}_1\\\overline{x}_2\\\overline{x}_3\end{pmatrix},$$

with $\alpha,\beta,\gamma\in\mathrm{L}^\times$. If we denote the above 4×4 matrix by M, and if we denote by A the matrix of the alternating bilinear form (*i.e.*, A is the matrix with a 1 on the positions $(1,2)$, $(2,1)$, $(3,4)$ and $(4,3)$, and everywhere else 0), then θ preserves the lines of $\mathsf{Q}(\mathrm{L},\mathrm{L};\mathrm{L},\mathrm{L})$ (a necessary condition) if and only if $M^TAM=mA$, where $m\in\mathrm{L}^\times$. One calculates easily that this means $\alpha\beta=\gamma$. Now the matrix M together with σ respresents a collineation of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ if and only if θ preserves the point set of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$. This is true if and only if $x_0'x_1'+x_2'x_3'\in\mathrm{L}'$, for all $(x_0',x_1',x_2',x_3')=(x_0,x_1,x_2,x_3)^\theta$, with $x_0x_1+x_2x_3\in\mathrm{L}'$. An elementary calculation shows that this is equivalent to $\beta\in\mathrm{L}'$ (by choosing $x_0=x_1=1$ and $x_2=x_3=0$) and $\overline{x}_1\in\mathrm{L}'$ whenever $x_1\in\mathrm{L}'$. So σ preserves L'. Moreover, expressing that θ is an involution, we obtain that $M\overline{M}$ is a multiple of the identity matrix. Hence one calculates that $\beta=\overline{\beta}=\gamma\overline{\alpha}^{-1}$. Consequently $\beta\in\mathrm{K}'$ and $\alpha=\overline{\alpha}$, $\gamma=\overline{\gamma}$, implying both γ

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

and α belong to K. We can now write θ as follows:

$$(4)\qquad \begin{cases} x_0' = \beta\overline{x}_1, \\ x_1' = \overline{x}_0, \\ x_2' = (\alpha\beta)\overline{x}_3, \\ x_3' = \alpha^{-1}\overline{x}_2. \end{cases}$$

6.3 Parametrisation of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ and Coordinates of θ

Let Σ be the apartment of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ which is determined by the points $[0]$, $[0,0]$, $[0,0,0]$ and $[\infty]$, let $\mathbf{\Phi} = (\Phi, E)$ be the root system of type C_2 and let $\{\eta_2, \eta_3\}$ be a base of $\mathbf{\Phi}$.

It is an elementary exercise to verify that there exists a parametrisation

$$\Pi = \big(\omega, (x_\phi)_{\phi\in\Phi_s}, (y_\phi)_{\phi\in\Phi_l}\big)$$

of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$ (with respect to the apartment Σ) such that y_{η_2} maps $s \in \mathrm{L}'$ onto the automorphism

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and such that x_{η_3} maps $t \in \mathrm{L}$ onto the automorphism

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Now, if θ denotes the the involution of the previous subsection, then we obtain after an elementary calculation that $\theta\big(y_{\eta_2}(s)\big) = \theta y_{\eta_2}(s)\theta = y_{-\eta_2}(\beta\overline{s})$, for each $s \in \mathrm{L}'$, and $\theta\big(x_{\eta_3}(t)\big) = \theta x_{\eta_3}(t)\theta = x_{-\eta_3}(\alpha\overline{t})$, for all $t \in \mathrm{L}$.

We now show:

Proposition 6.1 *With the notation of Section 4, suppose that $X = \mathsf{C}_2$ and $(\beta,\alpha) \in \mathrm{K}' \times \mathrm{K}$ $(= \mathfrak{I}(\mathsf{C}_2))$. Set $\tau = \tau_{(w_1,\sigma,(\beta,\alpha))}$. Then $\mathrm{Fix}_\Delta(\tau) = \varnothing$ (with $\Delta = \mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$) if and only if the following two conditions are satisfied:*

(i) If $m \in \mathrm{L}$ is such that $m\overline{m} = \beta$ and if $u, v \in \mathrm{L}$ satisfy $(u,v) \neq (0,0)$, then

$$\overline{m}(u\overline{u} + \alpha v\overline{v}) \notin \mathrm{L}'.$$

(ii) If $p \in \mathrm{L}'$ is such that $p\overline{p} = \alpha^{-2}\beta^2$ and if $x, y \in \mathrm{L}'$ satisfy $(x,y) \neq (0,0)$, then

$$\overline{p}(x\overline{x} + \alpha^2\beta y\overline{y}) \notin \mathrm{L}^2.$$

If β is contained in K^2, then condition (i) is equivalent to the following condition (I), where $\beta = \beta'^2$.

(I) *For all $u, v \in \mathrm{L}$, and all $a \in \mathrm{K}'$, we have*

$$u\overline{u} + \alpha v\overline{v} + \beta' a = 0 \Longleftrightarrow u = v = a = 0.$$

If α is contained in K', then condition (ii) is equivalent to condition (II) below.

(II) *For all $x, y \in \mathrm{L}'$, and all $b \in \mathrm{K}$, we have*

$$x\overline{x} + \beta y\bar{y} + \alpha\beta b^2 = 0 \Longleftrightarrow x = y = b = 0.$$

Proof First, we express that θ does not fix any point of $\mathsf{Q}(\mathrm{L}, \mathrm{L}'; \mathrm{L}, \mathrm{L}')$.

It is readily calculated that a point (x_0, x_1, x_2, x_3) is a fixed point for θ if and only if there exists $m \in \mathrm{L}$ such that $m\overline{m} = \beta$, $x_0 = \overline{m}\overline{x}_1$ and $x_2 = \overline{m}\alpha^{-1}\overline{x}_3$. Hence a generic fixed point in $\mathbf{PG}(3, \mathrm{L})$ has coordinates

$$(\overline{mu}, u, \overline{m}\alpha^{-1}\overline{v}, v),$$

with $m\overline{m} = \beta$. No fixed point may belong to $\mathsf{Q}(\mathrm{L}, \mathrm{L}'; \mathrm{L}, \mathrm{L}')$, in other words,

$$\overline{m}(u\overline{u} + \alpha v\overline{v}) \notin \mathrm{L}', \tag{5}$$

for every m as above, and for all $u, v \in \mathrm{L}$, $(u, v) \neq (0, 0)$. This shows (i). Now suppose that $\beta = \beta'^2 \in \mathrm{K}^2$. Note that from the above follows that, in particular, β cannot be written as $m\overline{m}$ with $m \in \mathrm{L}'$, hence $\beta \notin \mathrm{K}'^2$. If $m \neq \beta'$, then we substitute $u = \overline{(\beta' + \overline{m})u'}$ and $v = (\beta' + \overline{m})v'$ in Equation (5), and we obtain, in view of $(\beta' + \overline{m})\overline{(\beta' + \overline{m})} = \overline{m}^{-1}\beta'(\overline{m}^2 + \beta'^2) \in \overline{m}^{-1}\beta'\mathrm{K}'^2 = \overline{m}^{-1}\beta'^{-1}\mathrm{K}'^2$,

$$\beta'^{-1}(u'\overline{u}' + \alpha v\overline{v}') \notin \mathrm{L}'. \tag{6}$$

Since the left hand side is in K, we deduce that Equation (6) is equivalent to

$$u\overline{u} + \alpha v\overline{v} + \beta' a = 0 \Longleftrightarrow u = v = a = 0, \tag{7}$$

for all $u, v \in \mathrm{L}$, and all $a \in \mathrm{K}'$. If $m = \beta'$, then this follows immediately. We have proved (I).

We now look at the action of θ on the set of lines of $\mathsf{Q}(\mathrm{L}, \mathrm{L}'; \mathrm{L}, \mathrm{L}')$. It is easily seen that θ maps $[\infty]$ to $[0, 0, 0]$ (and vice versa), and $[0]$ to $[0, 0]$ (and vice versa). Hence the matrix of θ in $\mathbf{PG}(3, \mathrm{L}')$, viewed as the extension of the map θ on the dual of $\mathsf{Q}(\mathrm{L}, \mathrm{L}'; \mathrm{L}, \mathrm{L}')$ to $\mathbf{PG}(3, \mathrm{L}')$, has only non-zero entries in the positions $(1, 2)$, $(2, 1)$, $(3, 4)$ and $(4, 3)$. It suffices to look at the image of the point $(1, 1, 1, 1)$ to completely determine this matrix. The point $(1, 1, 1, 1)$ corresponds to the line $[1, 0, 1]$, which is in $\mathbf{PG}(3, \mathrm{L})$ determined by the points $(0, 0, 1, 1)$ and $(1, 1, 0, 0)$. They are mapped under θ to the points $(0, 0, \alpha\beta, \alpha^{-1})$ and $(\beta, 1, 0, 0)$, respectively. These two points determine the line $[\alpha^2\beta, 0, \beta]$, which represents the point with coordinates $\big((\alpha\beta)^2, 1, \beta, \alpha^2\beta\big)$ in $\mathbf{PG}(3, \mathrm{L}')$. Hence we can write θ in $\mathbf{PG}(3, \mathrm{L}')$ as

$$\begin{cases} y_0' = (\alpha\beta)^2\bar{y}_1, \\ y_1' = \bar{y}_0, \\ y_2' = \beta\bar{y}_3, \\ y_3' = \alpha^2\beta\bar{y}_2. \end{cases}$$

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

We verify that θ does not fix any line of $\mathsf{Q}(\mathrm{L},\mathrm{L}';\mathrm{L},\mathrm{L}')$. This is easily done by substituting $\beta^2\alpha^{-2}$ for β; $\beta\alpha^{-2}$ for α; $\mathrm{K}',\mathrm{K}^2,\mathrm{L}'$ and L^2 for $\mathrm{K},\mathrm{K}',\mathrm{L}$ and L', respectively, in the above conditions. Condition (5) becomes (after some rewriting)

$$\overline{p}(x\overline{x} + \alpha^2\beta y\overline{y}) \notin \mathrm{L}^2, \tag{8}$$

for all $p \in \mathrm{L}'$ with $p\overline{p} = \alpha^{-2}\beta^2$, and all $x, y \in \mathrm{L}'$. This shows (ii). In particular, $p \notin \mathrm{L}^2$, hence $\alpha \notin \mathrm{K}^2$. Now suppose that $\alpha \in \mathrm{K}'$. As above, we can reduce the condition (8) to the case $p = \alpha^{-1}\beta$, and so we obtain the condition

$$\alpha^{-1}\beta(x\overline{x} + \beta y\overline{y}) \notin \mathrm{L}^2, \tag{9}$$

for all $x, y \in \mathrm{L}'$. As above, this is equivalent to

$$x\overline{x} + \beta y\overline{y} + \alpha\beta b^2 = 0 \Longleftrightarrow x = y = b = 0, \tag{10}$$

for all $x, y \in \mathrm{L}'$, and all $b \in \mathrm{K}$. We have proved (II).

It is also clear now that (i) and (ii) imply that θ acts fixed point freely on Δ (as a building). ■

Combining this with Lemma 4.3, we obtain:

Corollary 6.2 *Let $X \in \{\mathsf{B}_4, \mathsf{C}_4, \mathsf{F}_4\}$ and let $\tau = \tau_{(w_1,\sigma,\lambda_{(\beta,\alpha)})} \in \mathcal{I}(\mathsf{F}_4)$ (respectively $\mathcal{I}(\mathsf{B}_4)$, $\mathcal{I}(\mathsf{C}_4)$), then τ is homogeneous if and only if conditions (I) and (II) (respectively (i) and (II), (I) and (ii)) are satisfied.*

6.4 Equivalence of Conditions (I) and (II)

Lemma 6.3 *Let $(\beta,\alpha) \in \mathrm{K}^2 \times \mathrm{K}'$. Then the conditions (I) and (II) are equivalent.*

Proof First note that each of the conditions (I) and (II) implies that $\alpha \notin \mathrm{K}^2$ and $\beta' \notin \mathrm{K}'$ (where again ${\beta'}^2 = \beta$).

We show that (I) implies (II). This follows from writing (II) as

$$(x + \beta' y)\overline{(x + \beta' y)} + \alpha(\beta' b)(\overline{\beta' b}) + \beta'(x\overline{y} + y\overline{x}) = 0,$$

and it follows from (I) that $b = 0$ and $x + \beta' y = 0$. Since $\beta' \notin \mathrm{K}'$, this implies $x = y = 0$. Conversely, it similarly follows from the identity

$$(u\overline{u} + \alpha v\overline{v} + \beta' a)^2 = (u^2 + \alpha v^2)\overline{(u^2 + \alpha v^2)} + \beta a\overline{a} + \alpha\beta\big(\beta'(u\overline{v} + v\overline{u})\big)^2,$$

for all $u, v \in \mathrm{L}$, $a \in \mathrm{K}'$, that (II) implies (I).

Hence (I) and (II) are equivalent. ■

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

6.5 The Moufang Quadrangle $\mathsf{Q}(\mathrm{K}, \mathrm{L} \times \mathrm{L} \times \mathrm{K}'; \alpha, \beta)$

In this subsection, which will only be important for some geometric considerations, we will be more sketchy about proofs. In fact, we will omit most of the calculations. Our purpose is to identify the Moufang quadrangles arising from a homogeneous involution in a building of mixed type C_4 and having Tits diagram $(\mathsf{C}_4, \{4, 5\})$ (with conventions as before). One could actually do a similar calculation as performed in Section 5, but since the building is embedded in a classical one now, we have a more elementary and direct geometric approach along the lines of the previous section.

We embed the quadrangle $\mathsf{Q}(\mathrm{L}, \mathrm{L}'; \mathrm{L}, \mathrm{L}')$ in the mixed polar space $\mathsf{W}_7(\mathrm{L}, \mathrm{L}')$, which is defined as follows. It is the subspace of the usual symplectic polar space of rank 4 over the field L associated to the alternating bilinear form

$$X_0Y_1 + X_1Y_0 + X_2Y_3 + X_3Y_2 + X_4Y_5 + X_5Y_4 + X_6Y_7 + X_7Y_6$$

in $\mathbf{PG}(7, \mathrm{L})$ obtained by restricting the point set to the points with coordinates

$$(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$$

such that

$$x_0x_1 + x_2x_3 + x_4x_5 + x_6x_7 \in \mathrm{L}'.$$

This is what we call a *mixed polar space of type* C_4, or a *mixed building of type* C_4. The embedding of $\mathsf{Q}(\mathrm{L}, \mathrm{L}'; \mathrm{L}, \mathrm{L}')$ is now achieved by putting $x_4 = x_5 = x_6 = x_7 = 0$. Let us denote by e_i, $i = 4, 5, 6, 7$, the point of $\mathbf{PG}(7, \mathrm{L})$ whose i-th coordinate x_i is equal to 1, and all its other coordinates are 0. Then e_i belongs to $\mathsf{W}_7(\mathrm{L}, \mathrm{L}')$ and the quadrangle $\mathsf{Q}(\mathrm{L}, \mathrm{L}'; \mathrm{L}, \mathrm{L}')$ can be seen as the intersection of the *perps of the* e_i, *i.e.*, $e_4^\perp \cap e_5^\perp \cap e_6^\perp \cap e_7^\perp$ (where the perp $p^\perp$ of a point p is the set of all points collinear with p in the polar space).

Now we are interested in a homogeneous involution θ' of $\mathsf{W}_7(\mathrm{L}, \mathrm{L}')$ which fixes at least an ordinary quadrangle, but which acts fixed point freely on the set of 2 and 3-dimensional projective subspaces of $\mathsf{W}_7(\mathrm{L}, \mathrm{L}')$, and which extends θ of the previous subsections (see Equations (4)).

From Lemma 4.3, it follows (after some calculations) that we may assume that θ' is represented by

$$(11) \qquad \begin{cases} x_0' = \beta'^2 \overline{x}_1, \\ x_1' = \overline{x}_0, \\ x_2' = (\alpha\beta'^2)\overline{x}_3, \\ x_3' = \alpha^{-1}\overline{x}_2, \end{cases} \qquad \begin{cases} x_4' = \beta'\overline{x}_4, \\ x_5' = \beta'\overline{x}_5, \\ x_6' = \beta'\overline{x}_6, \\ x_7' = \beta'\overline{x}_7. \end{cases}$$

Let Γ be the generalized quadrangle fixed by θ'. Now for $b \in \mathrm{K}$ and $(u, v, a) \in \mathrm{L} \times \mathrm{L} \times \mathrm{K}'$,

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

we define the following matrices

$$M_1(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_3(b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & 1 \end{pmatrix}$$

$$M_2(u,v,a) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{v} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & {\beta'}^{-1}v \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \overline{u} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & (\alpha\beta')^{-1}u \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ {\beta'}^{-1}v & \overline{v} & (\alpha\beta')^{-1}u & \overline{u} & 0 & 0 & 1 & (\alpha\beta')^{-1}(u\overline{u}+\alpha v\overline{v}+\beta' a) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_4(u,v,a) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \overline{v} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & {\beta'}^{-1}v & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \overline{u} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & (\alpha\beta')^{-1}u & 0 & 0 \\ {\beta'}^{-1}v & \overline{v} & (\alpha\beta')^{-1}u & \overline{u} & 1 & (\alpha\beta')^{-1}(u\overline{u}+\alpha v\overline{v}+\beta' a) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Clearly, $U_i = \{M_i(b) : b \in \mathrm{K}\}$, $i = 1,3$, and $U_j = \{M_j(u,v,a) : u,v \in \mathrm{L}, a \in \mathrm{K}'\}$, $j = 2,4$, form groups, which apparently stabilize the polar space $\mathsf{W}_7(\mathrm{L},\mathrm{L}')$, but which also stabilize Γ. The latter follows from the fact that the above matrices commute with θ' (and this is readily verified). The group U_1 fixes $\Gamma(e_5) \cup \Gamma(e_5e_6) \cup \Gamma(e_6)$ pointwise and acts transitively on $\Gamma(e_4e_6) \setminus \{e_6\}$; the group U_2 fixes $\Gamma(e_5e_6) \cup \Gamma(e_6) \cup \Gamma(e_4e_6)$ and acts transitively on $\Gamma(e_4) \setminus \{e_4e_6\}$; the group U_3 fixes $\Gamma(e_6) \cup \Gamma(e_4e_6) \cup \Gamma(e_4)$ pointwise and acts transitively on $\Gamma(e_4e_7) \setminus \{e_4\}$; the group U_4 fixes $\Gamma(e_4e_6) \cup \Gamma(e_4) \cup \Gamma(e_4e_7)$ and acts transitively on $\Gamma(e_6) \setminus \{e_4e_6\}$. It is now an elementary exercise to check the following identities (inside the group $U_+ = U_1U_2U_3U_4$):

$$[U_1, U_2] = [U_1, U_3] = [U_2, U_3] = [U_3, U_4]$$

and

$$\big[M_2(u, v, a), M_4(u', v', a')\big] = M_3\Big(\beta'^{-1}\big(u\overline{u}' + u'\overline{u} + \alpha(v\overline{v}' + v'\overline{v})\big)\Big),$$

$$\big[M_1(b), M_4(u, v, a)\big] = M_2(bu, bv, b^2a)M_3\big(b\beta'^{-1}(u\overline{u} + \alpha v\overline{v} + \beta' a)\big).$$

Hence, if $\sigma \neq 1$, we obtain the commutation relations of the quadrangle $\mathsf{Q}(\mathrm{K}, \mathrm{L} \times \mathrm{L} \times \mathrm{K}'; \alpha, \beta)$ defined before. Consequently, Γ is isomorphic to that Moufang quadrangle.

Note that the previous calculations also hold for $\sigma = 1$. In that case, the quadrangle Γ is a mixed quadrangle, as can be checked immediately.

7 The Main Result

In this section we summarize the results of the previous sections.

Main Result Let τ be a homogeneous involution of a building Δ of mixed type F_4, and suppose that it has Tits diagram $(\mathsf{F}_4, \{1, 4\})$. Then $\Gamma := \mathrm{Fix}_\Delta(\tau)$ is isomorphic either to a new Moufang quadrangle of type $\mathsf{Q}(\mathrm{K}, \mathrm{L}, \mathrm{K}'; \alpha, \beta)$ (if the field automorphism involved is non-trivial), or to a mixed quadrangle of type $\mathsf{Q}\big(\mathrm{K}'(\beta), \mathrm{K}^2(\alpha); L, L'\big)$ (if the field automorphism involved is trivial), see Subsection 2.2. All new Moufang quadrangles arise in this way (but not all mixed quadrangles).

Moreover, the usual diagram-trick for finding subquadrangles ("the C_2-system associated to a given $\mathsf{B}\,\mathsf{C}_2$-system") applies here as if the quadrangles arose from algebraic groups. More exactly, if we call "points" in Γ elements which have type 1 in Δ, then an ideal subquadrangle Γ' (of type $\mathsf{Q}(\mathrm{K}, \mathrm{L} \times \mathrm{L} \times \mathrm{K}'; \alpha, \beta)$ if the field automorphism involved is non-trivial) arises from the Tits diagram obtained by considering the extended (affine) diagram $\tilde{F}_4$ "at the right", encircling the new node and deleting the old encircled node 1 (we thus obtain $(\mathsf{C}_4, \{4, 5\})$). Dually, a full subquadrangle is obtained. This procedure can be applied once again to the full and the ideal subquadrangle and we obtain mixed quadrangles of type $\mathsf{Q}(\mathrm{K}, \mathrm{K}'; \mathrm{K}, \mathrm{K}')$.

Proof In the final result of Subsection 5.3, we set $c_1 = \alpha\beta^2, c_2 = \beta^{-2}, c_3 = \alpha^{-1}$ and $c_4 = \alpha\beta$. Moreover, we put

$$(x, y, b)_1 := y_1(y)y_2(x)z_1(b),$$

$$(u, v, a)_2 := x_3(\beta^{-1}v)x_4(\beta^{-1}u)v_2(a),$$

$$(x, y, b)_3 := y_3(y)y_4(x)z_2(b),$$

$$(u, v, a)_4 := x_3(\beta^{-1}v)x_4(\beta^{-1}u)v_2(a).$$

We now obtain the commutation relations of the new Moufang quadrangles if σ is non-trivial (and observe that the conditions (I) and (II) are the same as the conditions (1) and (2) in Section 2.2), and we obtain the commutation relations of the mixed quadrangle $\mathsf{Q}\big(\mathrm{L}'(\beta), \mathrm{L}^2(\alpha); L, L'\big)$ (note $\mathrm{K} = \mathrm{L}$ and $\mathrm{K}' = \mathrm{L}'$ here) if σ is trivial, see Subsection 2.2. The other assertions now follow from Section 6. ■

8 Geometric Aspects

We recall that all buildings considered in this paper are assumed to be spherical but not necessarily to be thick. The following lemma is an easy exercise in the elementary theory of spherical buildings:

Lemma 8.1 *Let Δ be a building of type M and let Σ be an apartment of Δ. Given a convex (in the sense of Tits [10]) subset Δ_1 of Δ containing Σ, then Δ_1 is a building of type M as well.*

The following result can be extracted from Scharlau [7] (to which we refer for the notions undefined here):

Lemma 8.2 *Let Δ be a building of type M. Then the* thin-classes *of the chambers constitute the chamber graph of a thick building Δ° with the induced adjacency relation. The set of reflections of the Weyl group of Δ° can be canonically identified with a subset of the reflections of the Weyl group of Δ.*

The two lemmas above can be used to establish the existence of certain subquadrangles of the new quadrangles in a geometric way. In the remainder of this section we will indicate in some detail how this works.

Let $\mathrm{L}, \mathrm{L}', \mathrm{K}, \mathrm{K}'$ be as in Section 4, let Δ be the building associated to the group $\mathsf{F}_4(\mathrm{L}, \mathrm{L}')$, let Σ be an apartment of Δ and let C be a chamber contained in Σ. We will make freely use of the definitions and notations introduced in Sections 4 and 5. We put $\Phi_s' = \{0010, 0110, 1110, 1232\}$ and $\Phi_l' = \{0100, 0120, 0122, 2342\}$. We label the Dynkin diagram D_4 by $\{0, 1, 2, 3\}$ by assigning 0 to the unique node of valency 3.

We define some subbuildings of Δ:

- Δ_2: Let $P_{12} = \Big\{A \in \Sigma \mid \mathrm{typ}(A) \in \{\{2,3,4\}, \{1,3,4\}\}\Big\}$. We define Δ_2 as the full convex hull of Σ and $\bigcup_{A \in P_{12}} \mathrm{St}\, A$. By Lemma 8.1 Δ_2 is a building of type F_4 and one verifies that Δ_2° is the building associated to the group $\mathsf{D}_4(\mathrm{L}')$. The set of positive roots associated to the reflections which are in the Weyl-group of Δ_2° is the set Φ_l.
- Δ_1: Let v_4 be the vertex of type $\{4\}$ contained in C and let Δ_1 be the full convex hull of Δ_2 and $\mathrm{St}\, v_4$. Then Δ_1° is the building associated to the (mixed) group $\mathsf{B}_4(\mathrm{L}, \mathrm{L}')$. The set of positive roots associated to the reflections which are in the Weyl-group of Δ_1° is $\Phi_l \cup \Phi_s'$.
- ${}_2\Delta$: The definitions of P_{34} and ${}_2\Delta$ are obtained by 'dualizing' the definitions of P_{12} and Δ_2. Now ${}_2\Delta^\circ$ is the building associated to the group $\mathsf{D}_4(\mathrm{L})$ and the set of positive roots associated to the reflections which are in the Weyl-group of ${}_2\Delta^\circ$ is Φ_s.
- ${}_1\Delta$: Define v_1 and ${}_1\Delta$ 'dually' to v_4 and Δ_1. Then ${}_1\Delta$ is the building associated to the (mixed) group $\mathsf{C}_4(\mathrm{L}, \mathrm{L}')$ and the set of positive roots associated to the reflections which are in the Weyl-group of ${}_1\Delta^\circ$ is the set $\Phi_s \cup \Phi_l'$.
- ${}_1\Delta_1$: We put ${}_1\Delta_1 = {}_1\Delta \cap \Delta_1$. The building ${}_1\Delta_1^\circ$ is associated to the (mixed) group $\mathsf{C}_2(\mathrm{L}, \mathrm{L}') = \mathsf{B}_2(\mathrm{L}, \mathrm{L}')$ and the set of positive roots associated to the reflections which are in the Weyl-group of ${}_1\Delta_1^\circ$ is the set $\Phi_s' \cup \Phi_l'$.

Remark The analogous statements for Δ_1, Δ_2 remain true if we start with a building associated to the group $\mathsf{F}_4(k)$ where k is any field. The validity of the statements about

https://doi.org/10.4153/CJM-1999-018-2 Published online by Cambridge University Press

${}_1\Delta$ and ${}_2\Delta$ is a characteristic 2 phenomenon. Indeed, if k has not characteristic 2 then ${}_1\Delta = \Delta = {}_2\Delta$.

Now let τ be a homogenous involution of Δ having Tits diagram $(\mathsf{F}_4, \{1,4\})$, normalizing Σ and fixing the $\{1,4\}$-face of C. Then τ stabilizes Δ_1, Δ_2, ${}_1\Delta$, ${}_2\Delta$ and ${}_1\Delta_1$ and induces homogenous involutions τ_1, τ_2, ${}_1\tau$, ${}_2\tau$ and ${}_1\tau_1$ on Δ_1°, Δ_2°, ${}_1\Delta^\circ$, ${}_2\Delta^\circ$ and ${}_1\Delta_1^\circ$, respectively. The Tits diagram of τ_1 (respectively τ_2, ${}_1\tau$, ${}_2\tau$, ${}_1\tau_1$) is $(\mathsf{B}_4, \{0,1\})$ (respectively $(\mathsf{D}_4, \{0,1\})$, $(\mathsf{C}_4, \{4,5\})$, $(\mathsf{D}_4, \{0,1\})$, $(\mathsf{C}_2, \{2,3\})$). The fixed point sets of all these involutions arise as subquadrangles Γ_1, Γ_2, ${}_1\Gamma$, ${}_2\Gamma$ and ${}_1\Gamma_1$, respectively, of the fixed point quadrangle Γ of τ in Δ. For instance, the quadrangle $\mathsf{Q}(\mathrm{K}, \mathrm{L}\times\mathrm{L}\times\mathrm{K}'; \alpha, \beta)$ of Section 6.5 is isomorphic to Γ_1.

If $\tau = \tau_{(w_1,\sigma,((ab)^{-1},b,a,(ba^2)^{-1/2}))}$, then we can extract the commutation relations for the subquadrangles Γ_1, Γ_2, ${}_1\Gamma$, ${}_2\Gamma$ and ${}_1\Gamma_1$ by restricing them to the corresponding root groups. For instance in order to obtain the commutation relations of Γ_1 one sets $U_1 = Y_1Y_2Z_1$, $U_2 = V_2$, $U_3 = Y_3Y_4Z_2$, $U_4 = V_1$ in Subsection 5.2.

Suppose that $\sigma \neq \mathrm{id}_\mathrm{L}$. Let H' (respectively H) be the quaternion algebra over K' (respectively K) defined by the separable quadratic extension L' (respectively L) and the element $\alpha \in \mathrm{K}'$ (respectively $\beta' \in \mathrm{K}$). Let $n'\colon \mathrm{H}' \to \mathrm{K}'$ and $n\colon \mathrm{H} \to \mathrm{K}$ be the respective norm forms.

The quadrangle Γ_2 is the quadrangle associated to the quadratic form $q_2\colon \mathrm{H}'\times\mathrm{K}'\times\mathrm{K}'\times\mathrm{K}'\times\mathrm{K}' \to \mathrm{K}'$ defined via $q_2(x_0,x_1,x_2,x_3,x_4) := n'(x_0) + x_1x_2 + x_3x_4$.

The quadrangle ${}_2\Gamma$ is the quadrangle associated to the quadratic form ${}_2q\colon \mathrm{H}\times\mathrm{K}\times\mathrm{K}\times\mathrm{K}\times\mathrm{K} \to \mathrm{K}$ defined via ${}_2q(x_0,x_1,x_2,x_3,x_4) := n(x_0) + x_1x_2 + x_3x_4$.

The quadrangle Γ_1 is the quadrangle $\mathsf{Q}(\mathrm{K}', \mathrm{L}'\times\mathrm{L}'\times\mathrm{K}^2; \beta^2, \alpha)$ associated to the quadratic form $q_1\colon \mathrm{K}\times\mathrm{H}' \to \mathrm{K}'$ defined via $q_1(y,x) = y^2 + \beta^2 n'(x)$.

The quadrangle ${}_1\Gamma$ is the quadrangle $\mathsf{Q}(\mathrm{K}, \mathrm{L}\times\mathrm{L}\times\mathrm{K}'; \alpha, \beta)$ associated to the quadratic form ${}_1q\colon \mathrm{K}'\times\mathrm{H} \to \mathrm{K}$ (with K' a vector space over K by defining the scalar product $k\cdot k' = k^2k'$, $k \in \mathrm{K}$, $k' \in \mathrm{K}'$) defined via ${}_1q(y,x) = y + \alpha n(x)$. Note again that this is the quadrangle of Section 6.5.

Finally the quadrangle ${}_1\Gamma_1$ is the quadrangle $\mathsf{Q}(\mathrm{K}, \mathrm{K}'; \mathrm{K}, \mathrm{K}')$.

Acknowledgement We are very grateful to Richard Weiss for communicating to us the commutation relations of the new quadrangles, and for the permission to reproduce them here. Also, we would like to thank Jacques Tits for some stimulating discussions about the subject of the paper. In particular, we learned from Tits that he (unpublished) showed that every "semi-linear" involution of a building is homogeneous (here, semi-linear basically means that there is a non-trivial field automorphism involved).

References

[1] R. H. Bruck and E. Kleinfeld, *The structure of alternative division rings.* Proc. Amer. Math. Soc. **2**(1951), 878–890.
[2] R. W. Carter, *Simple Groups of Lie Type.* John Wiley and Sons, London, 1972.
[3] G. Hanssens and H. Van Maldeghem, *Coordinatization of generalized quadrangles.* Ann. Discrete Math. **37**(1988), 195–208.
[4] E. Kleinfeld, *Alternative division rings of characteristic 2.* Proc. Nat. Acad. Sci. USA **37**(1951), 818–820.
[5] R. Moufang, *Alternativkörper und der Satz vom vollständigen Vierseit.* Abh. Math. Sem. Univ. Hamburg **9**(1933), 207–222.
[6] B. Mühlherr, *Some Contributions to the Theory of Buildings Based on the Gate Property.* PhD-thesis, Tübingen, 1994.

[7] R. Scharlau, *A structure theorem for weak buildings of spherical type.* Geom. Dedicata **24**(1987), 77–84.
[8] R. Steinberg, *Lectures on Chevalley-Groups.* Yale University, 1967.
[9] J. Tits, *Sur la trialité et certains groupes qui s'en déduisent.* Inst. Hautes Études Sci. Publ. Math. **2**(1959), 13–60.
[10] ———, *Buildings of Spherical Type and Finite BN-Pairs.* Lecture Notes in Math. **386**, Springer, Berlin, 1974.
[11] ———, *Classification of buildings of spherical type and Moufang polygons: a survey.* In: Coll. Intern. Teorie Combin., Acc. Naz. Lincei, Proceedings Roma 1973, Atti dei Convegni Lincei **17**(1976), 229–246.
[12] ———, *Non-existence de certains polygones généralisés, I.* Invent. Math. **36**(1976), 275–284.
[13] ———, *Endliche Spiegelungsgruppen, die als Weylgruppen auftreten.* Invent. Math. **43**(1977), 283–295.
[14] ———, *Non-existence de certains polygones généralisés, II.* Invent. Math. **51**(1979), 267–269.
[15] ———, *Moufang octagons and the Ree groups of type* $^2\mathsf{F}_4$. Amer. J. Math. **105**(1983), 539–594.
[16] ———, *Résumé de cours.* Annuaire du Collège de France, 95e année, 1994-1995, 79–95.
[17] ———, *Quadrangles de Moufang* (suite). Séminaire, Lecture given at the Collège de France, February 1997, Paris.
[18] J. Tits and R. Weiss, *The Classification of Moufang Polygons.* In preparation.
[19] H. Van Maldeghem, *Generalized Polygons.* Monographs Math. **93**, Birkhäuser-Verlag, Basel-Boston-Berlin, 1998.
[20] R. Weiss, *The nonexistence of certain Moufang polygons.* Invent. Math. **51**(1979), 261–266.

Fachbereich Mathematik, Lehrstuhl II
Universität Dortmund
D-44221 Dortmund
Germany
email: bernhard.muehlherr@mathematik.uni-dortmund.de

Vakgroep voor Zuivere Wiskunde en Computeralgebra
Universiteit Gent
Galglaan 2
B-9000 Gent
Belgium
email: hvm@cage.rug.ac.be