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FIXED POINT THEORY AND COMPLEMENTARITY PROBLEMS IN HILBERT SPACE

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In this paper we study both the implicit and the explicit complementarity problem using some special and interesting connections between the complementarity problem and fixed point theory in Hilbert space.

1. Introduction

The complementarity problem is one of the interesting and important problems defined since 1964 and it has been much studied in the last fifteen years.

The extensive literature on the (explicit or implicit) complementarity problem (at least three hundred papers) is motivated by its interesting and deep connections with nonlinear analysis and by its interesting applications in areas such as: Optimisation Theory, Engineering, Structural Mechanics, Elasticity Theory, Lubrication Theory, Economics, Variational Calculus, Equilibrium Theory on Networks, Stochastic Optimal Control etcetera [*20*].

We consider in this paper the complementarity problem (explicit and implicit) in Hilbert spaces.

Some natural connections between the complementarity problem and

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some special fixed point theorems are used to prove several existance theorems.

We observe that Theorem 2 is a substantial improvement on our recent result ([16], Theorem 1).

This improvement is based on an incomplete remark of Professor W. Oettli, but our result is more general.

2. Definitions

Let $\langle E,E^*\rangle$ be a dual system of locally convex spaces and let $K \subseteq E$ be a closed convex cone.

We denote by K^* the dual cone of K, that is, $K^* = \{u \in E^* \mid \langle x,u\rangle \geq 0 \ ; \ \forall x \in K\}$.

Given the mappings, $f:K \to E^*$ and $g:K \to E$ we consider the following complementarity problems:

$E.C.P.(f,K)$: find $x_o \in K$ such that $f(x_o) \in K^*$ and $\langle x_o, f(x_o)\rangle = 0$,

$I.C.P.(f,g,K)$: find $x_o \in E$ such that $g(x_o) \in K$, $f(x_o) \in K^*$ and $\langle g(x_o), f(x_o)\rangle = 0$

We say that $E.C.P.(f,K)$ is the *explicit complementarity problem and* $I.C.P.(f,g,K)$ *the implicit complementarity problem.*

The reader can find more details on these problems in [1], [2], [14-19], [23-27], [30], [32] and particularly in [20].

The implicit complementarity problem arises in Stochastic Optimal Control Theory and it was considered by Bensoussan, Lions, Dolcetta, Mosco etcetera [3-7], [13], [29].

We remark that $I.C.P.(f,g,K)$ has not been studied very much in infinite dimensional spaces.

In 1969 Karamardian [25], proved the following result:

THEOREM [Karamardian]. *The problem* $E.C.P.(f,K)$, *where* $E = R^n$ *and* $K = R^n_+$, *has a unique solution if* f *is a continuous and strongly monotone mapping.* □

Now, it is well-known [2], [27] that this result is true in an ordered reflexive Banach space if f is a hemicontinuous and α-monotone mapping.

Another generalisation of Karamardian's theorem is a result proved in 1984 by Dash and Nanda [12] which states that the problem $E.C.P.(f,K)$ has a solution if it is feasible $(\{x \in K \mid f(x) \in K^*\} \neq \phi)$ and f is a hemicontinuous and strictly monotone mapping.

We prove in this paper a similar result (Theorem 2), but for the implicit complementarity problem.

Finally, we remark that our results are considered in a Hilbert space, since in this case we have several interesting connections with fixed point theory.

3. Main results

Let $(H,\langle,\rangle)$ be a Hilbert space and let $K \subseteq H$ be a closed convex cone.

If $D \subseteq H$ is a subset and $f,g:D \to H$ are two mappings, we consider the following implicit complementarity problem:

$(I.C.P.)$: find $x_* \in D$ such that $g(x_*) \in K$, $f(x_*) \in K^*$ and $\langle g(x_*), f(x_*)\rangle = 0$

We recall that if P_K denotes the projection onto K, that is, for every $x \in H$, $P_K(x)$ is the unique element satisfying:

$$\|x - P_K(x)\| = \min_{y \in K} \|x-y\| ,$$

then we have the following classical result.

PROPOSITION 1. [34]. *For every element* $x \in H$, $P_K(x)$ *is characterised by the following properties:*

$1^\circ)$ $\langle P_K(x)-x, y\rangle \geq 0; \ \forall y \in K$,

$2^\circ)$ $\langle P_K(x)-x, P_K(x)\rangle = 0$. □

Our next result on the problem $(I.C.P.)$ is based on the following fixed point theorem. (Theorem 1)

We recall that a metric space (X,ρ) is said to be metrically convex, if for each $x, y \in X$, $(x \neq y)$ there is a $z \neq x,y$ for which $\rho(x,y) = \rho(x,z) + \rho(z,y)$.

We write $P = \{\rho(x,y) \mid x,y \in X\}$.

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THEOREM 1 [Boyd and Wong] [9]. *Let* (X, ρ) *be a complete metrically convex metric space. If for the mapping* $T: X \to X$ *there is a mapping* $\phi: P \to P_+$ *satisfying,*

1°) $\rho(T(x), T(y)) \leq \phi(\rho(x,y))$,

2°) $\phi(t) < t$, *for all* $t \in \bar{P}|\{0\}$,

then T *has a unique fixed point* x_0 *and* $T^n(x) \to x_0$ *for each* $x \in X$.□

DEFINITION 1. *Given a subset* $D \subseteq H$, *we consider the mappings* $f, g: D \to H$; Φ, $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ *and we say that:*

a°) f *is a* Φ*-Lipschitz mapping with respect to* g *if* ,
$\|f(x) - f(y)\| \leq \|g(x) - g(y)\| \, \Phi(\|g(x) - g(y)\|)$; $\forall x, y \in D$,

b°) f *is a* Ψ*-strongly monotone mapping with respect to* g *if*
$\langle f(x) - f(y), g(x) - g(y)\rangle \geq \|g(x) - g(y)\|^2 \, \Psi(\|g(x) - g(y)\|)$;
$\forall x, y \in D$.

If in Definition 1, $g(x) = x$; $\forall x \in D$ then we say that f is a Φ-Lipschitz mapping (respectively, f is a Ψ-strongly monotone mapping).

Obviously, if Φ and Ψ are strictly positive constants, we obtain from Definition a° (respectively b°) that f is a Lipschitz (respectively strongly monotone) mapping.

THEOREM 2. *Let* $(H, \langle , \rangle)$ *be a Hilbert space and let* $K \subset H$ *be a closed convex cone. If, for a subset* $D \subseteq H$, *the mappings* $f, g: D \longrightarrow H$ *satisfy the following assumptions:*

1°) f *is a* Φ*-Lipschitz mapping with respect to* g ,

2°) f *is a* Ψ*-strongly monotone with respect to* g ,

3°) *there exists a real number* $\tau > 0$ *such that,* $\tau\Phi^2(t) < 2\Psi(t) < \frac{1}{\tau} + \tau\Phi^2(t)$; $\forall\, t \in \mathbb{R}_+$,

4°) $K \subseteq g(D)$,

then the problem (I.C.P.) is soluble.

Moreover, if g *is one to one, then the problem (I.C.P.) has a unique solution.*

Proof. Using assumption 4°, we consider the mapping $h: K \longrightarrow H$ (which is not unique) defined by $h(u) := f(x)$, where x is an arbitrary element of $g^{-1}(u)$ and $u \in K$.

From this definition we observe that h has the following properties:

$5°)$ $\|h(u) - h(v)\| \leq \|u - v\| \ \Phi(\|u - v\|); \ \forall \ u, v \in K$,

$6°)$ $< h(u) - h(v), u - v > \geq \|u - v\|^2 \ \Psi(\|u - v\|); \ \forall \ u, v \in K$.

We observe now that the problem $(I.C.P.)$ is equivalent to the following explicit complementarity problem:

$E.C.P.(h,K)$: find $u_* \in K$ such that $h(u_*) \in K^*$ and

$$< u_*, h(u_*)> = 0 \ .$$

But, from Proposition 1, we deduce that problem $E.C.P. \ (h,K)$ has a solution if and only if the mapping $T:K \longrightarrow K$ defined by

$$T(u) = P_K(u - \tau h(u)); \quad \forall \ u \in K \ ,$$

has a fixed point (where τ is the real number used in assumption $3°$).

We prove now that in fact T has a fixed point.

Indeed we have,

$$\begin{aligned}
\|T(u) - T(v)\|^2 &= \|P_K(u - \tau h(u)) - P_K(v - \tau h(v))\|^2 \leq \\
&\leq \|(u - \tau h(u)) - (v - \tau h(v))\|^2 = \|(u - v) - \tau(h(u) - h(v))\|^2 = \\
&= \|u - v\|^2 - 2\tau<u-v,h(u) - h(v)> + \tau^2\|h(u) - h(v)\|^2 \leq \\
&\leq \|u - v\|^2 - 2\tau\|u-v\|^2\Psi(\|u-v\|) + \tau^2\|u-v\|^2\Phi^2(\|u-v\|) = \\
&= \|u - v\|^2[1 - 2\tau\Psi(\|u-v\|) + \tau^2\Phi^2(\|u-v\|)] \ ,
\end{aligned}$$

which implies

$$\|T(u) - T(v)\| \leq \|u-v\|[1-2\tau\Psi(\|u-v\|) + \tau^2\Phi^2(\|u-v\|)]^{1/2}; \ \forall u,v \in K.$$

If we write

$$\phi(t) = t[1-2\tau\Psi(t) + \tau^2\Phi^2(t)]^{\frac{1}{2}}; \quad \forall t \in \mathbb{R}_+ \ ,$$

we observe, using assumption $3°$ and the fact that a Hilbert space is a complete metrically convex metric space, that all assumptions of Theorem 1 are satisfied. $(\overline{P} = \mathbb{R}_+)$.

Hence, T has a unique fixed point u_* and for every $u \in K$, $T^n(u) \to u_*$.

Obviously, if g is one to one then the problem $(I.C.P.)$ has a unique solution. □

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Remark. From the proof of Theorem 2 we obtain that, a solution of the problem *(I.C.P.)* is a solution of equation, $g(x) = u_*$; $x \in D$, where u_* is obtained by successive approximations using the operator T.

The next Corollary is a substantial improvement of our result (Theorem 1, [16]).

COROLLARY 1. *Let* $(H,\langle , \rangle)$ *be a Hilbert space and let* $K \subseteq H$ *be a closed convex cone. If for a subset* $D \subseteq H$ *the mappings,* $f,g:D \to H$ *satisfy the following assumptions:*

1°) f *is* k*-Lipschitz with respect to* g,

2°) f *is* c*-strongly monotone with respect to* g,

3°) $K \subseteq g(D)$,

then the problem I.C.P.) has a solution and this solution is unique if g *is one to one.*

Proof. We observe that, by replacing the constant c by a smaller constant $c_1 (0 < c_1 < c)$ and noting that f is still c_1-strongly monotone with respect to g, we may find a real number $\tau > 0$ such that

$$\tau k^2 < 2c < \frac{1}{\tau} + \tau k^2 ,$$

and we apply Theorem 2.

Indeed, as in the proof of Theorem 2 we consider the mapping $T(u) = P_K(u - \tau h(u))$, where $0 < \tau < \frac{2c}{k^2}$ and replacing c by $c_1 (0 < c_1 < c)$ such that $\tau k^2 < 2c_1 < \min(\frac{1}{\tau} + \tau k^2, 2c)$, we obtain that assumption $3°)$ of Theorem 2 is satisfied with f c_1-strongly monotone with respect to g.

COROLLARY 2. *Let* $(H,\langle , \rangle)$ *be a Hilbert space and let* $K \subseteq H$ *be a closed cone. If* $f:K \to H$ *satisfies the following assumptions:*

1°) f *is* Φ*-Lipschitz,*

2°) f *is* Ψ*-strongly monotone,*

3°) there exists a real number $\tau > 0$ *such that,*

$$\tau \Phi^2(t) < 2\Psi(t) < \frac{1}{\tau} + \tau\Phi^2(t) \; ; \; \forall t \in \mathbb{R}_+ ,$$

then the problem E.C.P.(f,K) has a solution and this solution is unique. □

COROLLARY 3. *If* $(H,<,>)$ *is a Hilbert space,* $K \subseteq H$ *a closed convex cone and* $f:K \to H$ *satisfies the following assumptions:*

1°) f *is k-Lipschitz ,*

2°) f *is c-strongly monotone,*

then the problem E.C.P.(f,K) has a solution and this solution is unique. □

COROLLARY 4. *Let* $(H,<,>)$ *be a Hilbert space and let* $K \subseteq H$ *be a closed convex cone. If for a subset* $D \subseteq H$ *and* $f,g:D \to H$ *the following assumptions are satisfied:*

1°) f *is a* Ψ*-strongly monotone mapping with respect to* g *,*

2°) g *is an expansive mapping, that is,*

$(\exists\lambda\geq 1)(\forall x,y \in D)(\|g(x)-g(y)\| \geq \lambda\|x-y\|)$,

3°) $\|f(x) - f(y)\| \leq \|x-y\|\Phi(\|g(x) - g(y)\|)$; $\forall x,y \in D$,

4°) there exists a real number $\tau > 0$ *such that,*

$$\tau\Phi^2(t) < 2\Psi(t) < \frac{1}{\tau} + \tau\Phi^2(t) \ ; \ \forall t \in \mathbb{R}_+ \ ,$$

5°) $K \subseteq g(D)$,

then the problem (I.C.P.) has a unique solution. □

We recall that a mapping $h:D \to H$ is said to be accretive if and only if

$\|x-y\| \leq \|(x-y) + \lambda(h(x) - h(y)\|$; for all $x,y \in D$ and all $\lambda \geq 0$.

Also, $U:D \to H$ is said to be pseudo-contractive if and only if, for all $x,y \in D$ and all $\lambda > 0$ we have ,

$$\|x-y\| \leq \|(1+\lambda)(x-y) - (U(x) - U(y))\| \ .$$

A classical result proved by Kato and Browder is the following.

If $g = Id-U$ *, where* $U:H \to H$ *, then the mapping* U *is pseudo-contractive if and only if* g *is accretive.*

COROLLARY 5. *Let* $(H,<,>)$ *be a Hilbert space and let* $K \subseteq H$ *be a closed convex cone. For a subset* $D \subseteq H$ *and* $f,g:D \to H$ *, suppose the following assumptions are satisfied:*

1°) f *is* Ψ*-strongly monotone with respect to* g *,*

2°) $g-\rho Id$ *is accretive for some* $\rho > 0$ *on* D *,*

3°) $\|f(x)-f(y)\| \leq \|x-y\|\Phi(\|g(x)-g(y)\|)$; $\forall x, y \in D$

4°) $K \subseteq g(D)$,

5°) there exists a real number $\tau > 0$ *such that,* $\tau\Phi^2(t) <$

$< 2\,\rho\Phi(t) < \frac{\rho}{\tau} + \tau\Phi^2(t)$; $\forall\, t \in R_+$ *, then the problem (I.C.P.)*

has a solution which is unique if $\rho \geq 1$.

Proof. As in our paper [16] we obtain,

$$\|x-y\| \leq \rho^{-1}\|g(x)-g(y)\| \ ; \forall\, x,\, y \in D \ ,$$

and consequently from assumption $3°$ we deduce,

$$\|f(x) - f(y)\| \leq \rho^{-1}\|g(x) - g(y)\|\Phi(\|g(x) - g(y)\|)$$

and we can apply Theorem 2. □

Given $f,\ g{:}D \to H$ we say that f is α-monotone with respect to g , if there exists a strictly increasing function $\alpha{:}[0, +\infty) \to [0, +\infty)$ with $\alpha(0) = 0$ and $\lim\limits_{t\to+\infty} \alpha(t) = +\infty$ such that,

$$< f(x)-f(y),\ g(x)-g(y)> \ \geq \|g(x)-g(y)\|\alpha(\|g(x)-g(y)\|);\ \forall\, x,\, y \in D\ .$$

If $g(x) = x$, for every $x \in D$ we say that f is α-monotone.

The following result is a direct consequence of Luna's Theorem [27].

PROPOSITION 2. *Let* $(H,<,>)$ *be a Hilbert space and* $K \subseteq H$ *a closed convex cone. Suppose* $h{:}K \to H$ *is hemicontinuous and* α*-monotone. Then there exists a unique solution of the problem* $E.C.P.(h,K)$. □

Using this result we can prove the following proposition:

PROPOSITION 3. *Let* $(H,<,>)$ *be a Hilbert space and* $K \subseteq H$ *a closed convex cone. If for a subset* $D \subseteq H$ *the mappings* $f,\ g{:}D \to H$ *satisfy the following assumptions:*

$1°)$ f *is a* Φ*-Lipschitz mapping with respect to* g *and* $\lim\limits_{r\to 0} \Phi(r) \neq \infty$,

$2°)$ f *is* α*-monotone with respect to* g ,

$3°)$ $K \subseteq g(D)$,

then the problem (I.C.P.) has a solution.

Proof. We consider the problem $E.C.P.(h,K)$ where the mapping $h{:}K \longrightarrow H$ is defined as in the proof of Theorem 2 and we observe that all the assumptions of Proposition 2 are satisfied. □

4. Another complementarity problem

We consider again a Hilbert space $(H,<,>)$ and let $K \subseteq H$ be a closed convex cone.

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Given a mapping $f:K \to H$ we consider in this section the complementarity problem:

$$(C.P.): \quad \left\| \text{find } x_o \in K \text{ such that } f(x_o) \in K^* \text{ and } \langle x_o, f(x_o)\rangle = 0 \right.$$

PROPOSITION 4. *The problem (C.P.) has a solution if and only if the mapping*

$$\phi(x) = P_K(x) - f(P_K(x)); \; \forall\, x \in H$$

has a fixed point in H.

If x_o *is a fixed point of* ϕ *then* $x_* = P_K(x_o)$ *is a solution of the problem (C.P.).*

Proof. Suppose that ϕ has a fixed point, for example, $x_o = \phi(x_o)$, that is,

$$x_o = P_K(x_o) - f(P_K(x_o)) .$$

We write $x_* = P_K(x_o)$, which implies $x_* \in K$, and $x_o = x_* - f(x_*)$, or $x_* - x_o = f(x_*)$.

From Proposition 1 we obtain $\langle f(x_*), y \rangle \geq 0$; $\forall\, y \in K$, that is, $f(x_*) \in K^*$

Using Proposition 1 again we have, $\langle f(x_*), x_* \rangle = 0$ and hence x_* is a solution of the problem $(C.P.)$.

Conversely, suppose that $x_* \in K$ is a solution of the problem $(C.P.)$.

We write $x_o = x_* - f(x_*)$ and from Moreau's decomposition Theorem [28] (since x_* is a solution of the problem $(C.P.)$), we deduce that, $P_K(x_o) = x_*$ and finally

$$\phi(x_o) = P_K(x_o) - f(P_K(x_o)) = x_* - f(x_*) = x_o ,$$

that is, x_o is a fixed point of ϕ. □

Thus, we can solve the problem $(C.P.)$ if we are able to find a fixed point for the mapping ϕ.

This problem is not simple since many known fixed point theorems are not applicable in this case.

We consider this problem, in this paper, in the particular case when the cone K has the property that P_K is monotone increasing with

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respect to the order defined by K, that is, for every x, y such that $x \leq_K y$, we have, $P_K(x) \leq_K P_K(y)$.

Recently we studied this property in [21], [22].

Let $(R^n, <,>)$ be the Euclidean space ordered by a closed convex cone K.

We write $K^o = \{y \in R^n \mid \langle x, y\rangle \leq 0;\ \forall\, x \in K\}$.

A closed proper and generating cone K in R^n is said to be "thin" if for any two vectors u and v on two different extreme rays of K^o one has $\langle u, v\rangle \leq 0$. [21].

We proved the following result in [21]:

The metric projection P_K onto the proper closed and generating cone $K \subseteq R^n$ is monotone increasing if and only if K is thin.

As yet we do not know a similar result for infinite Hilbert spaces, but we know several sufficient conditions.

We say that an ordered Hilbert space $(H, <,>, K)$ is a Hilbert lattice if and only if,

1°) H is a vector lattice, [29]

2°) $(\forall\, x, y \in H)(|x| \leq |y| \Longrightarrow \|x\| \leq \|y\|)$.

We proved the following result in [22].

(A) If $(H, <,>, K)$ is a Hilbert lattice, then P_K is monotone increasing and moreover, $P_K(x) = x^+$.

(B) If $(H, <,>, K)$ is an ordered Hilbert space, then the following statements are equivalent:

1°) $K^* \subseteq K$ and P_K is monotone increasing,

2°) H is a vector lattice and $\| |x| \| = \|x\|$ for all $x \in H$.

A convex cone $K \subseteq H$ is said to be polyhedral if there exists, a_i; $i = 1, 2, \ldots, n$ such that

$$K = \{x \in H \mid \langle a_i, x\rangle \leq 0;\ \forall\, i = 1, 2 \ldots, n\}.$$

(C) If $K \subseteq H$ is polyhedral and $\langle a_i, a_j\rangle = 0;\ \forall\, i \neq j$, then P_K is monotone increasing.

If $(H, <,>)$ is a Hilbert space and $K \subseteq H$ is a closed convex cone, then we say that K is sequentially regular if every increasing ordered bounded sequence of K is convergent.

We can prove that every closed normal cone in H is sequentially regular [17].

If $A \subseteq H$ is a subset, we denote by $\alpha(A)$ the measure of non-compactness of A defined by $\alpha(A) = \inf\{r > 0 | A$ can be covered by a finite family of subsets of H of diameter $< r\}$.

If $D \subseteq H$ is a subset, then a mapping $f:D \to H$ is said to be an α-contraction if:

1°) f is a continuous mapping,

2°) $\forall A \subseteq D$, A bounded ==> $f(A)$

3°) there exists $k \in (0,1)$ such that, for every bounded set $A \subseteq D$ we have $\alpha(f(A)) \leq k\alpha(A)$.

More generally, a mapping $f:D \to H$ is said to be condensing if:

1°) f is a continuous mapping,

2°) for every non-compact bounded set $A \subseteq D$ we have $\alpha(f(A)) < \alpha(A)$.

The next result on the complementarity problem uses the following fixed point theorems.

THEOREM 3. [Browder]. *Let $(E,\| \ \|)$ be a uniformly convex Banach space and let $C \subseteq E$ be a bounded closed convex subset. If $T:C \to C$ is non-expansive then T has a fixed point.* □

THEOREM 4. (Sadovski [33]). *Let $(E,\| \ \|)$ be a Banach space and let $C \subseteq E$ be a closed bounded convex subset. If $T:C \to C$ is a condensing mapping, then T has a fixed point.* □

THEOREM 5. *Let $(H, <,>)$ be a Hilbert space and let $K \subseteq H$ be a closed convex cone. Suppose that P_K is monotone increasing and K sequentially regular. Consider a mapping $f:K \to H$ of the form, $f(x) = f_1(x) + f_2(x) + d$, where f_1 is monotone decreasing, f_2 monotone increasing and $d \in H$.*

Suppose the following assumptions are satisfied:

1°) there exist $x_o, y_o \in H$ such that,

$[x_o, y_o] = \{x \in H | x_o \leq x \leq y_o\}$ *is bounded,* *2°) the sequences* $\{x_n\}_{n\in N}$, $\{y_n\}_{n\in N}$ *defined by*,

$$x_{n+1} = P_K(x_n) - f_1(P_K(x_n)) - f_2(P_K(y_n)) - d\ ;$$

$$y_{n+1} = P_K(y_n) - f_1(P_K(y_n)) - f_2(P_K(x_n)) - d$$

satisfy the conditions, $x_0 \le x_1$ *and* $y_1 \le y_0$.

If the mapping, $\phi(x) = P_K(x) - f_1(P_K(x)) - f_2(P_K(x)) - d$ *is:*

i°) nonexpansive,

or ii°) condensing,

or iii°) continuous and $\dim H < +\infty$,

then there exists a fixed point x_* *of* ϕ *and the complementarity problem* C.P.(f,K) *has a solution of the form* $P_K(x_*)$ *where*

$x_n \le x_* \le y_n$, *for every* $n \in N$.

Proof. We consider the mapping $\phi : H \to H$ defined by $\phi(x) = P_K(x) - f_1(P_K(x)) - f_2(P_K(x)) - d;\ x \in H$ and by recurrence we prove that

$$(\alpha_1) : (\forall n \in N)(x_n \le x_{n+1} \le y_{n+1} \le y_n).$$

Indeed, since for $n = 0$ we have

$$x_0 \le x_1 \le y_1 \le y_0\ ,$$

supposing (α_1) true for n we obtain,

$$x_{n+2} \ge x_{n+1}\ ;\ y_{n+2} \le y_{n+1} \quad \text{and}$$

$$\begin{aligned} x_{n+2} &= P_K(x_{n+1}) - f_1(P_K(x_{n+1})) - f_2(P_K(y_{n+1})) - d \\ &\le P_K(y_{n+1}) - f_1(P_K(y_{n+1})) - f_2(P_K(x_{n+1})) - d = y_{n+2}\ . \end{aligned}$$

Hence, we have,

$$x_0 \le x_1 \le \dots \le x_n \le \dots \le y_n \le \dots \le y_1 \le y_0\ .$$

Moreover, we have,

$$(\alpha_2) : \phi([x_n, y_n]) \subset [x_n, y_n]\ ;\ \text{for every } n \in N\ .$$

Indeed, let $x \in [x_n, y_n]$ be an arbitrary element. We have,

$$\begin{aligned} x_{n+1} &= P_K(x_n) - f_1(P_K(x_n) - f_2(P_K(y_n)) - d \\ &\le P_K(x) - f_1(P_K(x)) - f_2(P_K(x)) - d \\ &\le P_K(y_n) - f_1(P_K(y_n)) - f_2(P_K(x_n)) - d = y_{n+1} \end{aligned}$$

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that is, $\phi([x_n,y_n]) \subset [x_{n+1},y_{n+1}] \subset [x_n,y_n]$.

Since K is regular, there exist $u = \lim_{n\to\infty} x_n$, $v = \lim_{n\to\infty} y_n$ and we have $u \le v$.

Now, we remark that,

$$(\alpha_3): \quad \phi([u,v]) \subset [u,v] .$$

Indeed, if we consider $x \in [u,v]$ we observe that, $x_n \le x \le y_n$; for every $n \in N$, which implies,

$$\begin{aligned} x_{n+1} &= P_K(x_n) - f_1(P_K(x_n)) - f_2(P_K(y_n)) - d \\ &\le \phi(x) \le P_K(y_n) - f_1(P_K(y_n)) - f_2(P_K(x_n)) - d \\ &= y_{n+1} \ ; \ \forall n \in N , \end{aligned}$$

and hence, $u \le \phi(x) \le v$.

Finally, we observe that for $[u,v]$ and ϕ we can apply Browder's Theorem or Sadovski's Theorem or Brower's Theorem and the proof is finished. □

Using again the mapping $\phi(x) = P_K(x) - f(P_K(x)$ we obtain a very simple and nice result on the complementarity problem.

PROPOSITION 5. *Let* $(H, <,>)$ *be a Hilbert space and let* $K \subseteq H$ *be a closed convex cone. If* $f(x) = x-h(x); \forall x \in K$, *where* $h:K \to H$ *is a contraction, then the complementarity problem* $C.P.(f,K)$ *has a solution. (This solution is different from zero if* $h(0) \notin - K^*)$.

Proof. Indeed, if we consider the mapping, $\phi(x) = P_K(x) - f(P_K(X))$; for every $x \in H$, we obtain, $\phi(x) = P_K(x) - P_K(x) + h(P_K(x)) = h(P_K(x))$, which is a contraction from H into H .

From Banach's contraction Theorem, ϕ has a fixed point $x_o \in H$ and $x_* = P_K(x_o)$ is a solution of the problem $C.P. (f,K)$. □

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