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ARTICLE

A New Condition for Agglomeration in Bayesian Confirmation

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Abstract

Bayesian confirmation does not generally *agglomerate* over conjunction. That is, whenever a piece of evidence E confirms two hypotheses H_1 and H_2 *individually*, it does not follow that E also confirms them *conjunctively*. Here, I present a condition under which the latter *does* follow from the former. But this new condition reveals a surprising fact: Bayesian confirmation agglomerates over conjunction whenever the evidence in question also confirms that *both* target hypotheses are *false*.

1. Introduction

According to Bayesian confirmation theory, a piece of evidence E *confirms* two hypotheses H_1 and H_2 individually if and only if E makes each of them *more likely* to be true (Fitelson 2001; Strevens 2017). That is, the following two inequalities are satisfied:

$$P(H_1|E) > P(H_1) \quad \text{and} \quad P(H_2|E) > P(H_2). \tag{1}$$

This conception of confirmation is perhaps *the* most popular currently on the market. But it is a well-known fact, presumably first noted by Carnap (1950), that confirmation, thus understood, does not always *agglomerate* over conjunction (the label is from Leitgeb 2013). That is, condition (1) does not entail that

$$P(H_1 \wedge H_2|E) > P(H_1 \wedge H_2). \tag{2}$$

To see this more clearly, consider sampling a card from a standard deck. Let E be that the card is red, H_1 that it is a heart, and H_2 that it is a diamond (Roche 2012). Here, the agglomeration antecedent (1) is satisfied but the consequent (2) is not.[1]

[1] Conditional probability $P(H|E)$ is defined as usual by $P(H \wedge E)/P(E)$ provided $P(E) > 0$. To ensure well-definedness, I will tacitly assume that the relevant probabilities are non-extreme. Notice that the card example also shows that (1) neither entails that H_1 and H_2 are positively correlated *unconditionally* nor *conditional on* E. That is, it neither follows from (1) that $P(H_1 \wedge H_2) > P(H_1)P(H_2)$ nor that $P(H_1 \wedge H_2|E) > P(H_1|E)P(H_2|E)$. The example also helps us to see that (1) does not entail that E confirms

Still, there are conditions under which Bayesian confirmation *does* agglomerate over conjunction, i.e., conditions under which (1) does entail (2). And in this paper, I would like to present a new one. This new condition will, however, turn out somewhat puzzling. I will introduce it in section 2 and point out in section 3 that a precursor can already be found in the work of Carnap and Salmon. I then discuss an objection in section 4 and examine how the new condition relates to previous agglomeration conditions in section 5. Finally, I conclude in section 6.

2. NOR-confirmation

The new agglomeration condition I would like to present is the following:

$$P(\neg H_1 \wedge \neg H_2 | E) > P(\neg H_1 \wedge \neg H_2). \tag{3}$$

Less formally, it states that the evidence in question E confirms that neither hypothesis H_1 nor H_2 is true, or, equivalently, that *both* target hypotheses are *false*. Due to the obvious relationship to Peirce's (1933) NOR-connective $H_1 \downarrow H_2$, I will call condition (3) *NOR-confirmation*.

Intuitively, NOR-confirmation (3) is at odds with both the agglomeration antecedent (1) and the consequent (2). After all, the two latter conditions state that the evidence in question E confirms that *both* target hypotheses H_1 and H_2 are *true*, namely individually and conjunctively. It might therefore come as a surprise that NOR-confirmation (3) guarantees that agglomeration is *valid* for Bayesian confirmation. That is, whenever (3) holds, (1) entails (2). For a proof, first observe that (3) is equivalent to[2]

$$P(H_1 \vee H_2 | E) < P(H_1 \vee H_2),$$

which, by general additivity, expands to

$$P(H_1|E) + P(H_2|E) - P(H_1 \wedge H_2|E) < P(H_1) + P(H_2) - P(H_1 \wedge H_2).$$

By simple algebra, this is equivalent to

$$P(H_1|E) - P(H_1) + P(H_2|E) - P(H_2) < P(H_1 \wedge H_2|E) - P(H_1 \wedge H_2),$$

and, by condition (1), the following equivalent of (2) follows:

$$0 < P(H_1 \wedge H_2|E) - P(H_1 \wedge H_2).$$

To see that NOR-confirmation (3) is a *non-trivial* condition for agglomeration, i.e., (3) is *consistent* with (1), consider an urn containing ten balls with three binary attributes, distributed as shown in Table 1. Let the evidence E be that a randomly drawn ball is blue, H_1 that it is small, and H_2 that it is clean. Then, NOR-confirmation (3) is satisfied, i.e., the evidence confirms that the drawn ball is *not* small and *not* clean:

the disjunction $H_1 \vee H_2$. That is, it does not follow from (1) that $P(H_1 \vee H_2|E) > P(H_1 \vee H_2)$. Simply let E be that the drawn card is black, H_1 that it is *not* a heart, and H_2 that is *not* a diamond.

[2] Notice that I am not arguing that the sufficiency of NOR-confirmation (3) for agglomeration is *mathematically* surprising. It is surprising from a confirmation-theoretic perspective. Thanks to an anonymous referee for pushing me to be more explicit here. Also notice that NOR-confirmation (3) and the agglomeration antecedent (1) entail more than just (2). For instance, they also entail that E confirms the two material conditionals $H_1 \supset H_2$, $H_2 \supset H_1$ and their conjunction $H_1 \leftrightarrow H_2$. And it also follows that E confirms each hypothesis H_1 and H_2 conditional on the other, and that E confirms each negated hypothesis $\neg H_1$ and $\neg H_2$ conditional on the other. See also section 5.

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Table 1. Urn model under which NOR-confirmation (3), (1), and thus (2) are jointly satisfied

	Dirty		Clean		
	Big	Small	Big	Small	Total
Red	1	2	2	0	5
Blue	2	0	0	3	5
Total	3	2	2	3	10

$$P(\neg H_1 \wedge \neg H_2|E) = 4/10 > P(\neg H_1 \wedge \neg H_2) = 3/10.$$

The agglomeration antecedent (1) is satisfied, i.e., the evidence confirms that the drawn ball is small and clean *individually*,

$$\forall i \in 1, 2 : P(H_i|E) = 6/10 > (H_i) = 5/10,$$

and hence the agglomeration consequent (2) is satisfied, i.e., the evidence confirms that the drawn ball is small and clean *conjunctively*:

$$P(H_1 \wedge H_2|E) = 6/10 > P(H_1 \wedge H_2) = 3/10.$$

This shows that NOR-confirmation (3) is a non-trivial condition for agglomeration.

3. Carnap and Salmon

The observation that NOR-confirmation (3) and the agglomeration antecedent (1) are consistent is not entirely new: this fact was already noted *implicitly* by Carnap (1950) and Salmon (1983). The two authors discussed examples in which a piece of evidence E confirms two hypotheses H_1 and H_2 individually while *disconfirming* their disjunction $H_1 \vee H_2$, the latter being equivalent to NOR-confirmation (3). What *is* new, however, is that this makes Carnap's and Salmon's examples rather peculiar instances of agglomeration. To see this more clearly, consider Salmon's example:

> [A] medical researcher finds evidence confirming the hypothesis that Jones is suffering from viral pneumonia and also confirming the hypothesis that Jones is suffering from bacterial pneumonia—yet this very same evidence disconfirms the hypothesis that Jones has pneumonia! It is difficult to entertain such a state of affairs, even as an abstract possibility. (Salmon 1983, section 3)[3]

Salmon found the fact that such situations can arise "shocking and counterintuitive" (Salmon 1983, section 3). But he overlooked that being an instance of agglomeration, it follows that the evidence also confirms the hypothesis that Jones has viral *and* bacterial pneumonia. Just imagine the following dialogue:

[3] Atkinson et al.'s (2009) so-called *Alan Author Effect* is structurally equivalent to the phenomenon described by Salmon. The effect occurs when a piece of evidence E confirms a conjunction $H_1 \wedge H_2$ while disconfirming its conjuncts H_1 and H_2 individually. This is equivalent to confirming the negated hypotheses individually while disconfirming their disjunction.

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Researcher:	Mr. Jones, good to see you! I just received your lab results. I have some good and some bad news for you. The bad news is that the results confirm that you have viral pneumonia; and they also confirm that you have bacterial pneumonia.
Jones:	Oh dear! So I have both viral *and* bacterial pneumonia?! *That* explains why I feel so miserable!
Researcher:	Well, *that* is not quite what I said, Mr. Jones! In any case, the good news is that the results also confirm that you have *neither* viral *nor* bacterial pneumonia.
Jones:	Wait, didn't you just tell me the opposite? Do the results confirm that I *have* viral and bacterial pneumonia or do they confirm that I *don't*?!
Researcher:	Well, they confirm both, Mr. Jones, albeit in different ways.
Jones:	How can this be? Is there something wrong with the lab results?
Researcher:	No, I can assure you that our lab results are flawless and absolutely reliable. In fact, it follows that they also confirm that you have viral *and* bacterial pneumonia at the same time.

I suspect that most readers will find the researcher's utterances confusing and unhelpful. Perhaps, some will even question the validity of her inference, arguing that the lab results should *disconfirm* the hypothesis that Jones has viral *and* bacterial pneumonia. But the researcher's inference is valid and everything she says is consistent.[4]

4. The rarity objection

One might try to relativize the phenomenon above by arguing that it is probably very *rare*. My response to this objection is twofold: I admit that the phenomenon is not very *prevalent*, but this does not make it less *unsettling*. More precisely, the conjunctive prevalence of cases where NOR-confirmation (3) *and* (1) are jointly satisfied is around 2.5%. And the conditional prevalence of cases where NOR-confirmation (3) is satisfied *if* (1) is satisfied is around 10%. This can be shown using Monte Carlo integration based on 10 million regular probability functions over an algebra generated by three variables (Metropolis and Ulam 1949). The left-hand graph in figure 1 shows how the prevalence stabilizes with increasing number of probability functions.

To put these values into context, compare them with Simpson's (1951) paradox, a different but similarly puzzling probabilistic phenomenon where a piece of evidence E confirms a hypothesis H conditional on some assumption X and conditional on $\neg X$, but E fails to confirm H *unconditionally* (Sprenger and Weinberger 2021). The conjunctive prevalence of such cases is only around 0.83%, and their conditional prevalence is around 3.33%, as shown on the right-hand side of figure 1. But the low prevalence of Simpson's paradox has not kept researchers from finding the

[4] Taking inspiration from Hempel (1960), we might call cases where a single piece of evidence consistently confirms a number of jointly inconsistent hypotheses *evidential inconsistencies*. See also the phenomenon of *floating conclusions* where two contradicting lines of reasoning confirm the same conclusion (Makinson and Schlechta 1991; Horty 2002).

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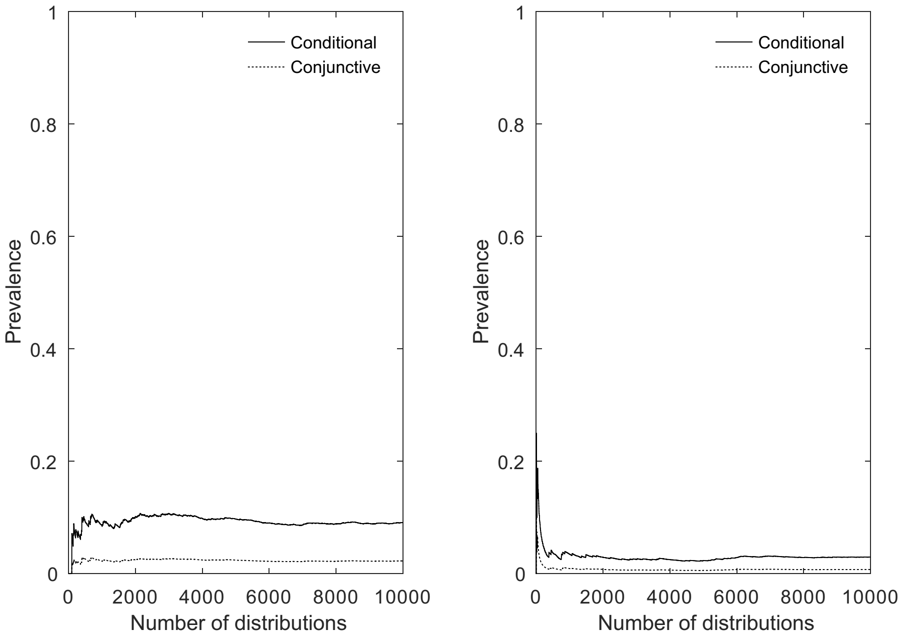

Figure 1. Prevalence of the NOR-effect (left) and the Simpson-effect (right).

phenomenon unsettling. So, even if cases where NOR-confirmation (3) and the agglomeration antecedent (1) are jointly satisfied are rare, they are prevalent *enough* to care about.

5. Previous agglomeration conditions

NOR-confirmation (3) is not the *only* agglomeration condition for Bayesian confirmation. As Reichenbach (1956) showed in his analysis of common-cause structures, agglomeration is also valid if the evidence *screens off* both hypotheses from each other:

$$P(H_1|E \wedge H_2) = P(H_1|E) \quad \text{and} \quad P(H_1|\neg E \wedge H_2) = P(H_1|\neg E). \tag{4}$$

And, as Falk (1986) pointed out in his discussion of Cohen's (1977) corroboration theorem, agglomeration remains valid even if screening-off is relaxed as follows:[5]

$$P(H_1|E \wedge H_2)P(H_1|E) \quad \text{and} \quad P(H_1|\neg E \wedge H_2) \leq P(H_1|\neg E). \tag{5}$$

[5] Cohen's condition (5) should not be confused with *weak screening-off* $P(H_1|E \wedge H_2) \geq P(H_1|E)$ and $P(H_1|\neg E \wedge H_2) \geq P(H_1|\neg E)$ (Atkinson and Peijnenburg 2021). The two conditions only differ in the second conjunct. But (5) guarantees agglomeration while weak screening-off does not. And weak screening-off guarantees transitivity while (5) does not (Suppes 1986; Roche 2012).

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Salmon (1983) uncovered another interesting condition. While agglomeration can also fail for independent hypotheses, it cannot if, additionally, the two target hypotheses are independent *conditional on the evidence*:

$$P(H_1|H_2 \wedge E) = P(H_1|E) \quad \text{and} \quad P(H_1|H_2) = P(H_1). \tag{6}$$

Finally, there are two more recent conditions from the literature on the problem of irrelevant conjunction (Schurz 2022). The first is part of Fitelson's (2002) *confirmational irrelevance condition*,

$$P(H_2|H_1 \wedge E) = P(H_2) \quad \text{and} \quad P(H_2|H_1) = P(H_2), \tag{7}$$

and the second is Hawthorne and Fitelson's (2004) conditional irrelevance condition which states that the evidence is irrelevant for one hypothesis conditional on the other:

$$P(H_2|H_1 \wedge E) = P(H_2|H_1). \tag{8}$$

Now, interestingly, NOR-confirmation (3) is *logically independent* of each of the aforementioned conditions (4)–(8). That is, NOR-confirmation (3) is consistent with each of them but neither entails nor is entailed by any of them. A proof of this statement is provided in Appendix A.

Notice, however, that most of these logical independence relationships *break down* once the agglomeration antecedent (1) is satisfied. More precisely, if (1) holds, then NOR-confirmation (3) is inconsistent with screening-off (4), full independence (6), confirmational irrelevance (7), and conditional irrelevance (8). A proof of this is provided in Appendix B. With these remarks, I close my discussion of NOR-confirmation (3).

6. Conclusion

In this short paper, I have presented a new condition under which Bayesian confirmation agglomerates over conjunction. One might think that such a condition is helpful because it allows us to establish claims about Bayesian confirmation without tedious case-by-case examination (Shogenji 2003; Roche 2012). But the condition presented here is more puzzling than helpful: it is difficult to see why Bayesian confirmation should agglomerate over conjunction whenever the new condition is satisfied. I hope that Bayesian confirmation theorists can help with an explanation.

Appendix

A. Logical independence

The probability distributions provided in Table 2 show that (3) is logically independent of (4) to (8). Under distribution 1, all conditions (3)–(8) are satisfied and thus none of them entails the negation of the other. Under distribution 2, NOR-confirmation (3) is satisfied while none of the other conditions is. And under distribution 3, NOR-confirmation (3) is violated while the other conditions are satisfied.

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Table 2. Probability distributions showing that NOR-confirmation (3) is logically independent of (4)–(8)

E	H_1	H_2	Distribution 1	Distribution 2	Distribution 3
0	0	0	1/16	1/16	2/16
0	0	1	1/16	1/16	2/16
0	1	0	2/16	2/16	1/16
0	1	1	2/16	2/16	1/16
1	0	0	2/16	2/16	2/16
1	0	1	2/16	3/16	2/16
1	1	0	3/16	3/16	3/16
1	1	1	3/16	2/16	3/16

Distribution 1

Under this distribution, NOR-confirmation (3) is satisfied:

$$P(H_1 \vee H_2 | E) = 8/10 < P(H_1 \vee H_2) = 13/16.$$

Screening-off (4), and thus relaxed screening-off (5), are satisfied:

$$P(H_1 | H_2 \wedge E) = P(H_1 | E) = 3/5 \quad \text{and} \quad P(H_1 | H_2 \wedge \neg E) = P(H_1 | \neg E) = 1/2.$$

We also have

$$P(H_2 | H_1 \wedge E) = P(H_2) = 1/2 \quad \text{and} \quad P(H_1 | H_2) = P(H_1) = 10/16.$$

Thus, full independence (6), confirmational irrelevance (7), and also conditional irrelevance (8) are satisfied.

Distribution 2

NOR-confirmation (3) is satisfied:

$$P(H_1 \vee H_2 | E) = 8/10 < P(H_1 \vee H_2) = 13/16.$$

Screening-off (4), relaxed screening-off (5), and full independence (6) are violated:

$$P(H_1 | H_2 \wedge E) = 2/5 < P(H_1 | E) = 1/2.$$

Conditional irrelevance (8), and thus confirmational irrelevance (7), are violated:

$$P(H_2 | H_1 \wedge E) = 2/5 < P(H_2 | H_1) = 5/9.$$

Distribution 3

NOR-confirmation (3) is violated:

$$P(H_1 \vee H_2 | E) = 16/20 > P(H_1 \vee H_2) = 12/16.$$

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Screening-off (4), and thus relaxed screening-off (5), are satisfied:

$$P(H_1|H_2 \wedge E) = P(H_1|E) = 3/5 \quad \text{and} \quad P(H_1|H_2 \wedge \neg E) = P(H_1|\neg E) = 1/3.$$

We also have

$$P(H_2|H_1 \wedge E) = P(H_2) = 1/2 \quad \text{and} \quad P(H_1|H_2) = P(H_1) = 1/2.$$

Thus, full independence (6), confirmational irrelevance (7), and also conditional irrelevance (8) are satisfied.

B. Breakdown of logical independence

If the agglomeration antecedent (1) is satisfied, then (3) is no longer logically independent of (4)–(8). More precisely, if (1) holds, then NOR-confirmation (3) and the agglomeration antecedent (1) jointly entail that the evidence E confirms H_1 conditional on H_2 and that E confirms H_2 conditional on H_1:

$$P(H_1|H_2 \wedge E) > P(H_1|H_2) \quad \text{and} \quad P(H_2|H_1 \wedge E) > P(H_2|H_1).$$

The two conditions also entail that the evidence E disconfirms H_1 conditional on $\neg H_2$ and that E confirms H_2 conditional on $\neg H_1$:

$$P(H_1|\neg H_2 \wedge E) < P(H_1|\neg H_2) \quad \text{and} \quad P(H_2|\neg H_1 \wedge E) < P(H_2|\neg H_1).$$

Together with screening-off (4) or full-independence (6), the second condition yields a contradiction. And together with confirmational irrelevance (7), and thus with conditional irrelevance (8), the first condition yields a contradiction.

References

Atkinson, David and Jeanne Peijnenburg. 2021. "A new condition for transitivity of probabilistic support." *Erkenntnis* 85 (2):273–300. doi: https://doi.org/10.1007/s10670-020-00349-7

Atkinson, David, Jeanne Peijnenburg, and Theo Kuipers. 2009. "How to confirm the conjunction of disconfirmed hypotheses." *Philosophy of Science* 76 (1):1–21. doi: https://doi.org/10.1086/598164

Carnap, Rudolf. 1950. *Logical Foundations of Probability*. Chicago: University of Chicago Press.

Cohen, L. Jonathan. 1977. *The Probable and the Provable*. Oxford: Clarendon Press.

Falk, Arthur. 1986. "Cohen on corroboration." *Mind* 95 (377):110–15. doi: https://doi.org/10.1093/mind/XCV.377.110

Fitelson, Branden. 2001. "Studies in Bayesian confirmation theory." PhD thesis, University of Wisconsin, Madison.

Fitelson, Branden. 2002. "Putting the irrelevance back into the problem of irrelevant conjunction." *Philosophy of Science* 69 (4):611–22. doi: https://doi.org/10.1086/344624

Hawthorne, James and Branden Fitelson. 2004. "Discussion: Re-solving irrelevant conjunction with probabilistic independence." *Philosophy of Science* 71 (4):505–14. doi: https://doi.org/10.1086/423626

Hempel, Carl G. 1960. "Inductive inconsistencies." *Synthese* 12 (4):439–69. doi: https://doi.org/10.1007/BF00485428

Horty, John F. 2002. "Skepticism and floating conclusions." *Artificial Intelligence* 135 (1–2):55–72. doi: https://doi.org/10.1016/S0004-3702(01)00160-6

Leitgeb, Hannes. 2013. "A lottery paradox for counterfactuals without agglomeration." *Philosophy and Phenomenological Research* 89 (3):605–36. doi: https://doi.org/10.1111/phpr.12035

Makinson, David and Karl Schlechta. 1991. "Floating conclusions and zombie paths: Two deep difficulties in the 'directly skeptical' approach to defeasible inheritance nets." *Artificial Intelligence* 48 (2):199–209. doi: https://doi.org/10.1016/0004-3702(91)90061-N

Metropolis, Nicholas and Stanislaw Ulam. 1949. "The Monte Carlo method." *Journal of the American Statistical Association* 44 (247):335–41. doi: https://doi.org/10.1080/01621459.1949.10483310

Peirce, Charles Sanders. 1933. "A Boolian algebra with one constant." In *Collected Papers of Charles Sanders Peirce*, vol. IV, *The Simplest Mathematics*, edited by Charles Hartshorne and Paul Weiss, pp. 13–18. Cambridge, MA: Harvard University Press.

Reichenbach, Hans. 1956. *The Direction of Time*. Berkeley, CA: University of California Press.

Roche, William A. 2012. "A weaker condition for transitivity in probabilistic support." *European Journal for Philosophy of Science* 2 (1):111–18. doi: https://doi.org/10.1007/s13194-011-0033-7

Salmon, Wesley C. 1983. "Confirmation and relevance." In *The Concept of Evidence*, edited by Peter Achinstein. Oxford: Oxford University Press.

Schurz, Gerhard. 2022. "Tacking by conjunction, genuine confirmation and convergence to certainty." *European Journal for Philosophy of Science* 12 (3):1–18. doi: https://doi.org/10.1007/s13194-022-00470-0

Shogenji, Tomoji. 2003. "A condition for transitivity in probabilistic support." *The British Journal for the Philosophy of Science* 54 (4):613–16. doi: https://doi.org/10.1093/bjps/54.4.613

Simpson, Edward H. 1951. "The interpretation of interaction in contingency tables." *Journal of the Royal Statistical Society. Series B* 13 (2):238–41. doi: https://doi.org/10.1111/j.2517-6161.1951.tb00088.x

Sprenger, Jan and Naftali Weinberger. 2021. "Simpson's paradox." In *The Stanford Encyclopedia of Philosophy*, edited by Edward N. Zalta. Stanford, CA: Stanford University Press.

Strevens, Michael. 2017. "Notes on Bayesian confirmation theory." Available at: http://www.strevens.org/bct/BCT.pdf

Suppes, Patrick. 1986. "Non-Markovian causality in the social sciences with some theorems on transitivity." *Synthese* 68:129–40. doi: https://doi.org/10.1007/BF00413969

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