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A NOTE ON QUOTIENT FIELDS OF POWER SERIES RINGS

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ABSTRACT. Let R be an integral domain with quotient field K. If R has an overring $S \neq K$, such that $S[[X]]$ is integrally closed, then the "algebraic degree" of $K((X))$ over the quotient field of $R[[X]]$ is infinite. In particular, it holds for completely integrally closed domain or Noetherian domain R.

In this note, any integral domain R is commutative with identity. It is well known that if R is an integral domain with quotient field K, then the quotient field of $R[X]$ is $K(X)$. But it is not the case for the power series rings.

If K is a field, the quotient field of the power series ring $K[[X]]$ is the Laurent series ring $K((X))$. In general, the quotient field $Q(R[[X]])$ of $R[[X]]$ is properly contained in $K((X))$. Gilmer [2] gave a necessary and sufficient condition for the ring $Q(R[[X]]) = K((X))$: For any sequence $\{(a_i)\}_{i=1}^{\infty}$ of nonzero principal ideals of R, $\bigcap_{i=1}^{\infty}(a_i) \neq (0)$. In particular, if there exists $a \in R \setminus \{0\}$ such that $\bigcap_{i=1}^{\infty}(a)^i = (0)$, then $Q(R[[X]])$ is properly contained in $K((X))$. Sheldon [5, Theorem 2.1] showed that the transcendental degree of $K((X))$ over $Q(R[[X]])$ is infinite.

We shall prove that if R has an overring S such that $S[[X]]$ is integrally closed, then the "algebraic degree" of $K((X))$ over $Q(R[[X]])$ is infinite. In particular, if R is completely integrally closed or Noetherian, the algebraic degree is infinite. (For a discussion of rings R such that $R[[X]]$ be integrally closed, we refer to Ohm [4].) We also remark that R is completely integrally closed if and only if $Q(R[[X]]) \neq Q(S[[X]])$ for any subring $S, R \underset{\neq}{\subset} S \subseteq K$ [5, Theorem 3.4].

Let R be an integral domain which is not equal to its quotient field K. A ring S is called an *overring* of R if $R \subset S \subset K$. Let $R[[X]]$ be the power series ring over R and $K((X))$ the Laurent series ring over K. If $Q(R[[X]])$ is the quotient field of $R[[X]]$, then $Q(R[[X]]) \subset K((X))$. Let L be the algebraic closure of $Q(R[[X]])$ in $K((X))$.

THEOREM. *If R has an overring S such that $S \neq K$ and $S[[X]]$ is integrally closed, then the algebraic degree $\big[L : Q(R[[X]])\big]$ is infinite.*

PROOF. Since $S \neq K$, we can choose $a \in R$ which is not a unit in S and let

$$f(T) = T^n - aT^{n-1} + X \in S[[X]][T] \subset S[T][[X]].$$

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FIRST STEP. $f(T)$ is irreducible in $S[T][[X]]$.

Suppose that

$$(1)\qquad (T^n - aT^{n-1}) + X = \big(p_0(T) + p_1(T)X + p_2(T)X^2 + \cdots\big)\big(q_0(T) + q_1(T)X + q_2(T)X^2 + \cdots\big)$$

is a nontrivial factorization of $f(T)$ in $S[T][[X]]$. Since $p_0(T) + p_1(T)X + \cdots$ and $q_0(T) + q_1(T)X + \cdots$ are not units in $S[T][[X]]$, $p_0(T)$ and $q_0(T)$ are not units in $S[T]$. From (1), we have

$$p_0(T)q_0(T) = T^n - aT^{n-1}.$$

So we may assume that

$$p_0(T) = (T-a)T^i,$$
$$q_0(T) = T^j, \quad i+j = n-1, j \geq 1.$$

Considering the coefficient of X in (1), we have

$$p_0(T)q_1(T) + p_1(T)q_0(T) = 1$$
$$(T-a)T^iq_1(T) + p_1(T)T^j = 1.$$

Since $j \geq 1$, $i = 0$. Let $T = 0$. Then $-aq_1(0) = 1$ and a is a unit in S. A contradiction.

SECOND STEP. $f(T)$ is irreducible in $S[[X]][T]$.

Suppose that

$$(2)\qquad T^n - aT^{n-1} + X = \big(T^\ell + f_{\ell-1}(X)T^{\ell-1} + \cdots + f_0(X)\big)\big(T^m + g_{m-1}(X)T^{m-1} + \cdots + g_0(X)\big)$$

is a nontrivial factorization of $f(T)$ in $S[[X]][T]$. Since $f_0(X)g_0(X) = X$, one of f_0 and g_0 is a unit in $S[[X]]$; say $f_0(X)$ is a unit. Because $T^\ell + f_{\ell-1}(X)T^{\ell-1} + \cdots + f_0(X)$ is not a unit in $S[[X]][T]$, $\ell \geq 1$. We regard $T^\ell + f_{\ell-1}(X)T^{\ell-1} + \cdots + f_0(X)$ as an element in $S[T][[X]]$. It is not a unit since $\ell \geq 1$.

If $T^m + g_{m-1}(X)T^{m-1} + \cdots + g_0(X)$ is a unit in $S[T][[X]]$, then $T^m + g_{m-1}(0)T^{m-1} + \cdots + g_0(0)$ is a unit in $S[T]$. But $X \mid g_0(X)$ in $S[[X]]$, so $g_0(0) = 0$. This is impossible. Hence (2) is also a nontrivial factorization of $f(T)$ in $S[T][[X]]$. This contradicts the First Step.

THIRD STEP. $f(T)$ is irreducible in $Q(S[[X]])[T]$, and hence it is irreducible in $Q(R[[X]])[T]$.

Since $S[[X]]$ is integrally closed and the monic polynomial $f(T)$ is irreducible over $S[[X]]$, it is irreducible over $Q(S[[X]])$ by [6, p. 260, Theorem 5]. The second statement is obvious.

FOURTH STEP. $f(T)$ has a root in $K((X))$.

Let $\sigma\colon K[[X]] \longrightarrow K[[X]]/(X) \cong K$ be the canonical homomorphism. Then $\sigma f(T) = T^n - aT^{n-1} = T^{n-1}(T-a)$. Since T^{n-1} and $T-a$ are comaximal in $K[T]$, by Hensel's

Lemma [1, p. 215, Theorem 1; 3, p. 189, Theorem (44.4)], there exist monic polynomials $g(T), h(T) \in K[[X]][T]$ such that $\sigma g = T^{n-1}$, $\sigma h = T - a$ and $f(T) = g(T)h(T)$. Thus h has degree one and it gives a root of $f(T)$ in $K[[X]]$.

CONCLUSION. For any n, the root is an element in $K((X))$ which is algebraic over $Q(R[[X]])$ of degree n. Thus $\left[L : Q(R[[X]])\right] = \infty$. ■

COROLLARY. *If R is completely integrally closed or Noetherian, then the degree $\left[L : Q(R[[X]])\right]$ is infinite.*

PROOF. If R is completely integrally closed, then $R[[X]]$ is integrally closed [1, p. 313, Proposition 14; 4]. Hence the corollary follows by theorem.

If R is Noetherian, then the integral closure S of R is a Krull domain [3, p. 118, Theorem (33.10)]. Hence S is completely integrally closed [1, p. 480, Theorem 2] and $S[[X]]$ is integrally closed. Hence the result. ■

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