RECURRENCE PROPERTIES OF PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

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Let $X_1, X_2, \cdots, X_n, \cdots$ be independent and identically distributed random variables, and write

$$(1) \qquad Z_n = X_1 + X_2 + \cdots + X_n.$$

In [2] Chung and Fuchs have established necessary and sufficient conditions for the random walk $\{Z_n\}$ to be *recurrent*, i.e. for Z_n to return infinitely often to every neighbourhood of the origin. The object of this paper is to obtain similar results for the corresponding process in continuous time.

Thus we shall be concerned with processes with stationary independent increments (for which see [3], ch. VIII, or [6], § 37, where they are called decomposable processes). If $Z(t)$ $(t \geqq 0)$ is any such process with $Z(0) = 0$, then there exists a function $\psi(\theta)$ such that, for all $t \geqq 0$ and all real θ,

$$(2) \qquad E\{e^{i\theta Z(t)}\} = e^{t\psi(\theta)}.$$

The function $\psi(\theta)$ determines all the finite-dimensional distributions of the process $Z(t)$.

It follows easily from (2) that $Z(t)$ is continuous in probability, and therefore ([3], Theorem 2.6) has a version which is separable and measurable. We shall assume without further comment that such a version has been taken.

The problem then is to decide whether or not $Z(t)$ returns to every neighbourhood of the origin for arbitrarily large values of t. In other words, if, for $a > 0$, we define the random set

$$(3) \qquad S(a) = \{t \geqq 0; \quad |Z(t)| < a\},$$

then we have to decide whether or not $S(a)$ is unbounded for all $a > 0$. Notice that, since Z is assumed to be measurable, $S(a)$ is, with probability one, a measurable subset of the non-negative real line R^+.

For any $h > 0$, the random variables

$$(4) \qquad X_n(h) = Z(nh) - Z((n-1)h), \qquad (n = 1, 2, \cdots),$$

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are independent and identically distributed, so that the process

$$\{Z(nh); \quad n = 1, 2, \cdots\}$$

is a random walk of the type considered by Chung and Fuchs. It is rather obvious that, if this walk is recurrent, then $S(a)$ is almost certainly unbounded. The converse is also true, but much more difficult to prove.

If A is any measurable subset of R^+, we shall denote by $||A||$ the Lebesgue measure of A.

Now fix the positive number a, and consider the following propositions:

I. $\boldsymbol{P}\{S(a) \text{ is unbounded}\} = 1$,

II. $\boldsymbol{P}\{||S(2a)|| = \infty\} = 1$,

III. $\int_0^\infty \boldsymbol{P}\{|Z(t)| < 2a\}dt = \infty$,

IV. For some $h > 0$, $\sum_{n=1}^\infty \boldsymbol{P}\{|Z(nh)| < 2a\} = \infty$,

V. For some $h > 0$, the random walk $\{Z(nh); n = 1, 2, \cdots\}$ is recurrent (or more precisely has recurrent values) in the sense of Chung and Fuchs [2].

We first prove the chain of implications

$$\mathrm{I} \Rightarrow \mathrm{II} \Rightarrow \mathrm{III} \Rightarrow \mathrm{IV} \Rightarrow \mathrm{V} \Rightarrow \mathrm{I}. \tag{5}$$

The difficult part of the proof is the implication (I $\Rightarrow$ II), for which we use an argument similar to one used in ([1], § II.10) and ([5], Lemma 4).

Proof of (I $\Rightarrow$ II): If Y is any random variable with $0 \leqq Y \leqq 1$, then

$$\boldsymbol{P}\{Y > \tfrac{1}{2}\} \geqq 2\boldsymbol{E}\{Y\} - 1. \tag{6}$$

Now let τ, $\varepsilon > 0$, and let Λ be any event defined on $\{Z(t);\ t \leqq \tau\}$ with the property that $|Z(\tau)| < a$ on Λ. Then, applying (6) to

$$Y = \left(\frac{1}{2\varepsilon}\right) ||S(2a) \cap (\tau, \tau+2\varepsilon)||,$$

we have

$$\begin{aligned} &\boldsymbol{P}\{||S(2a) \cap (\tau,\ \tau+2\varepsilon)|| > \varepsilon \,|\Lambda\} \\ &\quad \geqq \varepsilon^{-1} \boldsymbol{E}\{||S(2a) \cap (\tau,\ \tau+2\varepsilon)|| \,|\Lambda\} - 1 \\ &\quad = \varepsilon^{-1} \int_0^{2\varepsilon} \boldsymbol{P}\{|Z(\tau+t)| < 2a \,|\Lambda\}dt - 1 \\ &\quad \geqq \varepsilon^{-1} \int_0^{2\varepsilon} \boldsymbol{P}\{|Z(\tau+t) - Z(\tau)| < a \,|\Lambda\}dt - 1, \\ &\qquad\qquad \text{since } |Z(\tau)| < a \text{ on } \Lambda, \\ &\quad = \varepsilon^{-1} \int_0^{2\varepsilon} \boldsymbol{P}\{|Z(t)| < a\}dt - 1, \\ &\qquad\qquad \text{since } Z \text{ has stationary independent increments,} \\ &\quad = \gamma(\varepsilon), \ \text{(say)}. \end{aligned}$$

Hence we have

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$$(7) \qquad P\{\|S(2a)\cap(\tau,\ \infty)\| > \varepsilon\ |A\} \geqq \gamma(\varepsilon).$$

Moreover, since Z is continuous in probability,

$$P\{|Z(t)| < a\} \to 1$$

as $t \to 0$, so that

$$(8) \qquad \gamma(\varepsilon) \to 1 \qquad (\varepsilon \to 0).$$

Now for any $T,\ \eta > 0$,

$$\begin{aligned} &P\{\|S(2a)\cap(T,\ \infty)\| > \varepsilon\} \\ &\quad \geqq \sum_{m=0}^{\infty} P\{|Z(T+r\eta)| \geqq a(0 < r < m), \\ &\qquad\qquad |Z(T+m\eta)| < a,\ \|S(2a)\cap(T+m\eta,\ \infty)\| > \varepsilon\} \\ &\quad \geqq \gamma(\varepsilon)\sum_{m=0}^{\infty} P\{|Z(T+r\eta)| \geqq a(0 < r < m),\ |Z(T+m\eta)| < a\}, \\ &\qquad\qquad \text{using (7) with } \tau = T+m\eta, \\ &\quad = \gamma(\varepsilon)P\{|Z(T+m\eta)| < a \text{ for some integer } m\}. \end{aligned}$$

Putting $\eta = 2^{-k}$, letting $k \to \infty$ through integer values, and using the separability of Z, we obtain

$$P\{\|S(2a)\cap(T,\ \infty)\| > \varepsilon\} \geqq \gamma(\varepsilon)P\{|Z(T+t)| < a \text{ for some } t > 0\}.$$

Thus (I) implies that

$$P\{\|S(2a)\cap(T,\ \infty)\| > \varepsilon\} \geqq \gamma(\varepsilon).$$

Hence, letting $T \to \infty$,

$$P\{\|S(2a)\| = \infty\} \geqq \gamma(\varepsilon),$$

and so, letting $\varepsilon \to 0$ and using (8),

$$P\{\|S(2a)\| = \infty\} \geqq 1.$$

Thus I $\Rightarrow$ II.

Proof of (II $\Rightarrow$ III): If (II) holds, then

$$\int_0^{\infty} P\{|Z(t)| < 2a\}dt = E\{\|S(2a)\|\} = \infty,$$

so that (III) holds.

Proof of (III $\Rightarrow$ IV): If we write $f(t) = P\{|Z(t)| < 2a\}$, then the required result follows immediately from the following lemma of Chung ([1], p. 182), of which, for the sake of completeness, we include the proof.

If f is a non-negative, continuous function on R^+, and if

$$\int_0^\infty f(x)dx = \infty,$$

then

$$\sum_{n=1}^\infty f(nh) = \infty$$

for all h in some dense G_δ subset H of R^+.

Proof: Suppose that there exist a, b with $0 < a < b$ and an integer N such that

$$s(h) = \sum_{n=1}^\infty f(nh) \leqq N$$

for all $a < h < b$. Then

$$N(b-a) \geqq \int_a^b s(h)dh = \sum_{n=1}^\infty \int_a^b f(nh)dh = \sum_{n=1}^\infty \frac{1}{n}\int_{na}^{nb} f(x)dx$$

$$= \int_a^\infty \sum_{na \leqq x \leqq nb} \left(\frac{1}{n}\right) f(x)dx = \int_a^\infty \phi(x)f(x)dx,$$

where

$$\phi(x) = \sum_{x/b \leqq n \leqq x/a} \frac{1}{n}.$$

But $\phi(x) \to \log (b/a)$ as $x \to \infty$, and hence, for some $A > 0$,

$$\int_A^\infty f(x)dx < \infty,$$

which is contrary to hypothesis. Thus the set

$$G_N = \{h > 0;\ \sum_{n=1}^\infty f(nh) > N\},$$

which by an easy application of Fatou's lemma is open, is also dense in R^+, and therefore

$$H = \{h > 0;\ \sum_{n=1}^\infty f(nh) = \infty\} = \bigcap_{N=1}^\infty G_N$$

is, by Baire's theorem, a dense G_δ subset of R^+.

Proof of (IV $\Rightarrow$ V): This follows at once by applying the results of [2] to the random walk $\{Z(nh)\}$.

Proof of (V $\Rightarrow$ I): If $\{Z(nh)\}$ is recurrent, there is probability one that $S(a)$ contains infinitely many integer multiples of h, and is thus unbounded.

We have therefore proved the chain of implications (5), which shows that, for any $a > 0$, the propositions (I)—(V) are all equivalent. Moreover, since (V) does not involve the value of a, each of these propositions is independent of a. If, for some (and then for all) positive a, any one (and then all) of the conditions (I)—(V) is true, then $Z(t)$ will be said to be *recurrent*. Otherwise $Z(t)$ will be said to be *transient*.

It follows from what we have proved already that a recurrent process not only returns to any neighbourhood of the origin, but also spends an infinite time in that neighbourhood. On the other hand, if $Z(t)$ is transient, then $|Z(t)| \geqq a$ for all sufficiently large t, so that $|Z(t)| \to \infty$ as $t \to \infty$.

It remains to find a criterion for recurrence in terms of the distribution of $Z(t)$, i.e. in terms of $\psi(\theta)$. This is best done using the condition (III). For any $\alpha > 0$, let τ_α be a random variable independent of $\{Z(t)\}$ and having a probability density $\alpha e^{-\alpha\tau}$ on $\tau \geqq 0$, and put $\zeta_\alpha = Z(\tau_\alpha)$. Then

$$\int_0^\infty \boldsymbol{P}\{|Z(t)| < 2a\}dt = \lim_{\alpha\to 0} \int_0^\infty \boldsymbol{P}\{|Z(t)| < 2a\}e^{-\alpha t}dt$$
$$= \lim_{\alpha\to 0} \alpha^{-1} \boldsymbol{P}\{|\zeta_\alpha| < 2a\}.$$

Hence $Z(t)$ is recurrent if and only if, for some (and then for all) $a > 0$,

$$\lim_{\alpha\to 0} \alpha^{-1} \boldsymbol{P}\{|\zeta_\alpha| < 2a\} = \infty.$$

Now write
$$\chi(y) = \{1-|y|\}^2, \quad (|y| < 1),$$
$$= 0, \quad (|y| \geqq 1).$$

Then $\frac{1}{4}\boldsymbol{P}\{|\zeta_\alpha| < \frac{1}{2}\} \leqq \boldsymbol{E}\{\chi(\zeta_\alpha)\} \leqq \boldsymbol{P}\{|\zeta_\alpha| < 1\}$, so that $Z(t)$ is recurrent if and only if

$$\lim_{\alpha\to 0} \alpha^{-1} \boldsymbol{E}\{\chi(\zeta_\alpha)\} = \infty.$$

But

$$\int_{-\infty}^\infty \chi(y)e^{-i\theta y}dy = \frac{4}{\theta^2}\left(1-\frac{\sin\theta}{\theta}\right),$$

and therefore

$$\boldsymbol{E}\{\chi(\zeta_\alpha)\} = \frac{2}{\pi}\int_{-\infty}^\infty \boldsymbol{E}\{e^{i\theta\zeta_\alpha}\}\left(1-\frac{\sin\theta}{\theta}\right)\frac{d\theta}{\theta^2}.$$

Now

$$\boldsymbol{E}\{e^{i\theta\zeta_\alpha}\} = \int_0^\infty \alpha e^{-\alpha t}\, e^{t\psi(\theta)}dt = \alpha[\alpha-\psi(\theta)]^{-1},$$

so that, taking real parts, we have

$$\alpha^{-1}\boldsymbol{E}\{\chi(\zeta_\alpha)\} = \frac{4}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{1}{\alpha-\psi(\theta)}\right)\left(1-\frac{\sin\theta}{\theta}\right)\frac{d\theta}{\theta^2}.$$

Hence $Z(t)$ is recurrent if and only if

$$\lim_{\alpha\to 0}\int_0^\infty \mathrm{Re}\left(\frac{1}{\alpha-\psi(\theta)}\right)\left(1-\frac{\sin\theta}{\theta}\right)\frac{d\theta}{\theta^2}=\infty. \tag{9}$$

Since $e^{t\psi(\theta)}$ is a characteristic function, $\psi(\theta)$ has negative real part, and so

$$\mathrm{Re}\left(\frac{1}{\alpha-\psi(\theta)}\right)>0$$

for all θ. From this it follows at once that (9) is equivalent to the condition

$$\lim_{\alpha\to 0}\int_0^\infty \mathrm{Re}\left(\frac{1}{\alpha-\psi(\theta)}\right)\frac{d\theta}{1+\theta^2}=\infty. \tag{10}$$

This necessary and sufficient condition for recurrence is the continuous time analogue of a result of Feller and Orey [4]. Notice that, by Fatou's lemma, (10) is implied by

$$\int_0^\infty \mathrm{Re}\left(-\frac{1}{\psi(\theta)}\right)\frac{d\theta}{1+\theta^2}=\infty, \tag{11}$$

so that (11) is a sufficient condition for recurrence.

The analysis of this paper generalizes, with only very slight modifications, to the case of vector-valued processes with stationary independent increments.

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