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ERROR BOUNDS FOR AUGMENTED TRUNCATIONS OF DISCRETE-TIME BLOCK-MONOTONE MARKOV CHAINS UNDER GEOMETRIC DRIFT CONDITIONS

HIROYUKI MASUYAMA,* *Kyoto University*

Abstract

In this paper we study the augmented truncation of discrete-time block-monotone Markov chains under geometric drift conditions. We first present a bound for the total variation distance between the stationary distributions of an original Markov chain and its augmented truncation. We also obtain such error bounds for more general cases, where an original Markov chain itself is not necessarily block monotone but is blockwise dominated by a block-monotone Markov chain. Finally, we discuss the application of our results to GI/G/1-type Markov chains.

Keywords: Augmented truncation; block monotonicity; blockwise dominance; pathwise ordering; geometric drift condition; level-dependent QBD; M/G/1-type Markov chain; GI/M/1-type Markov chain; GI/G/1-type Markov chain

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1. Introduction

Various semi-Markovian queues and their state-dependent extensions can be analyzed through block-structured Markov chains characterized by an infinite number of block matrices, such as level-dependent quasi-birth-and-death processes (LDQBDs), M/G/1-, GI/M/1-, and GI/G/1-type Markov chains (see, e.g. [8]).

For LDQBDs, there exist some numerical procedures based on the RG-factorization, though their implementation requires the truncation of the infinite sequence of block matrices in a heuristic way [2], [4], [19]. Such 'truncation in implementation' is also necessary for *level-independent* M/G/1- and GI/M/1-type Markov chains (see, e.g. [21, Section 4]), and, thus, for GI/G/1-type Markov chains. To the best of the author's knowledge, there are no studies on the computation of the stationary distributions of *level-dependent* M/G/1- and GI/M/1-type Markov chains and more general Markov chains. For these Markov chains, the RG-factorization method does not seem effective in developing numerical procedures with *good* properties, such as space and time saving and the guarantee of accuracy, since the resulting expression of the stationary distribution is characterized by an infinite number of R- and G-matrices [24]. As for the transient distribution, Masuyama and Takine [16] proposed a stable and accuracy-guaranteed algorithm based on the uniformization technique (see, e.g. [22]).

As mentioned above, it is challenging to develop a numerical procedure for computing the stationary distributions of block-structured Markov chains characterized by an infinite number of block matrices. A practical and simple solution to this problem is to truncate the transition probability matrix so that it is of a finite dimension. The stationary distribution of the resulting

Received 29 June 2012; revision received 27 March 2014.
* Postal address: Department of Systems Science, Graduate School of Informatics, Kyoto University, Kyoto, 606-8501, Japan. Email address: masuyama@sys.i.kyoto-u.ac.jp

finite Markov chain can be computed by a general purpose algorithm, in principle. However, the obtained stationary distribution includes an error caused by truncating the original transition probability matrix. Therefore, from a practical point of view, it is significant to estimate the 'truncation error'.

Tweedie [23] and Liu [13] studied the estimation of the error caused by truncating (stochastically) monotone Markov chains (see, e.g. [6]). Tweedie [23] gave error bounds for the last-column-augmented truncation of a monotone Markov chain with geometric ergodicity. The last-column-augmented truncation is constructed by augmenting the last column of the *northwest corner truncation* of a transition probability matrix so that the resulting finite matrix is stochastic. On the other hand, Liu [13] assumed that a monotone Markov chain is subgeometrically ergodic and derived error bounds for the last-column-augmented truncation.

Unfortunately, block-structured Markov chains are not monotone in general. Li and Zhao [12] extended the notion of monotonicity to block-structured Markov chains. The new notion is called '(stochastic) block monotonicity'. Block-monotone Markov chains (BMMCs) arise from queues in Markovian environments, such as queues with batch Markovian arrival processes (BMAPs) [14]. Li and Zhao [12] proved that if an original Markov chain is block monotone then the stationary distributions of its augmented truncations converge to that of the original Markov chain, which is the motivation for this study.

In what follows, we give an overview of the work of Li and Zhao [12]. To this end, we introduce some symbols. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Let $\mathbb{Z}_+^{\le n} = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$ and $\mathbb{Z}_+^{\le \infty} := \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Furthermore, let $\mathbb{F}^{\le n} = \mathbb{Z}_+^{\le n} \times \mathbb{D}$ for $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, where $\mathbb{D} = \{1, 2, \dots, d\}$. For simplicity, we write $\mathbb{F}$ for $\mathbb{F}^{\le \infty}$.

The following is the definition of block monotonicity for stochastic matrices.

Definition 1.1. (*[12, Definition 2.5].*) Let

$$\boldsymbol{S} = (s(k, i; l, j))_{(k,i),(l,j) \in \mathbb{F}^{\le n}}$$

denote a stochastic matrix, where $n \in \overline{\mathbb{N}}$. A Markov chain characterized by $\boldsymbol{S}$ and $\boldsymbol{S}$ itself are said to be (stochastically) block monotone with block size d if, for all $k \in \mathbb{Z}_+^{\le n-1}$ and $l \in \mathbb{Z}_+^{\le n}$,

$$\sum_{m=l}^{n} s(k, i; m, j) \le \sum_{m=l}^{n} s(k+1, i; m, j), \qquad i, j \in \mathbb{D}.$$

We denote by BM_d the set of block-monotone stochastic matrices with block size d.

Let $\boldsymbol{P} = (p(k, i; l, j))_{(k,i),(l,j) \in \mathbb{F}}$ denote a stochastic matrix. Let $\{(X_\nu, J_\nu); \nu \in \mathbb{Z}_+\}$ denote a bivariate Markov chain with state space $\mathbb{F}$ and transition probability matrix $\boldsymbol{P}$. The following result is obvious from the definition. Its proof is thus omitted.

Proposition 1.1. *If $\boldsymbol{P} \in \mathsf{BM}_d$ then $\psi(i, j) := \sum_{l=0}^{\infty} p(k, i; l, j)$, $i, j \in \mathbb{D}$, is constant with respect to $k \in \mathbb{Z}_+$ and $\{J_\nu; \nu \in \mathbb{Z}_+\}$ is a Markov chain whose transition probability matrix is given by $\boldsymbol{\Psi} := (\psi(i, j))_{i,j \in \mathbb{D}}$, i.e. $\psi(i, j) = \mathbb{P}(J_{\nu+1} = j \mid J_\nu = i)$ for $i, j \in \mathbb{D}$.*

Proposition 1.1 implies the following *pathwise ordered property* of BMMCs (see Lemma A.1): if $\boldsymbol{P} \in \mathsf{BM}_d$ then there exist two BMMCs, $\{(X'_\nu, J'_\nu); \nu \in \mathbb{Z}_+\}$ and $\{(X''_\nu, J''_\nu); \nu \in \mathbb{Z}_+\}$, with transition probability matrix $\boldsymbol{P}$ on a common probability $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X'_\nu \le X''_\nu$ and $J'_\nu = J''_\nu$ for all $\nu \in \mathbb{N}$ if $X'_0 \le X''_0$ and $J'_0 = J''_0$.

Let ${}_{(n)}\boldsymbol{P}_* = ({}_{(n)}p_*(k, i; l, j))_{(k,i),(l,j)\in\mathbb{F}}$, $n \in \mathbb{N}$, denote a stochastic matrix such that, for $i, j \in \mathbb{D}$,

$$\begin{aligned}
{}_{(n)}p_*(k, i; l, j) &\ge p(k, i; l, j), && k \in \mathbb{Z}_+, && l \in \mathbb{Z}_+^{\le n}, \\
{}_{(n)}p_*(k, i; l, j) &= 0, && k \in \mathbb{Z}_+, && l \in \mathbb{Z}_+ \setminus \mathbb{Z}_+^{\le n}, \\
\sum_{l=0}^{n} {}_{(n)}p_*(k, i; l, j) &= \sum_{l=0}^{\infty} p(k, i; l, j), && k \in \mathbb{Z}_+.
\end{aligned}$$

The stochastic matrix ${}_{(n)}\boldsymbol{P}_*$ is called *a block-augmented first-n-block-column truncation* (for short, block-augmented truncation) of $\boldsymbol{P}$.

Remark 1.1. The block-augmented truncation ${}_{(n)}\boldsymbol{P}_*$ can be partitioned as

$$ {}_{(n)}\boldsymbol{P}_* = \begin{array}{c} \mathbb{F}^{\le n} \\ \mathbb{F} \setminus \mathbb{F}^{\le n} \end{array} \overset{\begin{array}{cc} \mathbb{F}^{\le n} & \mathbb{F} \setminus \mathbb{F}^{\le n} \end{array}}{\begin{pmatrix} {}_{(n)}\boldsymbol{P}_*^{\le n} & \boldsymbol{O} \\ * & \boldsymbol{O} \end{pmatrix}}, \tag{1.1}$$

where $\boldsymbol{O}$ is the zero matrix and ${}_{(n)}\boldsymbol{P}_*^{\le n}$ is equivalent to the block-augmented truncation defined by Li and Zhao [12]. Our definition facilitates the algebraic operation for the original stochastic matrix $\boldsymbol{P}$ and its block-augmented truncation ${}_{(n)}\boldsymbol{P}_*$ since they are of the same dimension.

Throughout this paper, unless otherwise stated, we assume that $\boldsymbol{P}$ is irreducible and positive recurrent, and we denote its unique stationary probability vector by $\boldsymbol{\pi} = (\pi(k, i))_{(k,i)\in\mathbb{F}} > \boldsymbol{0}$ (see, e.g. [3, Theorem 3.1, Section 3.1]). However, ${}_{(n)}\boldsymbol{P}_*$ may have more than one positive recurrent (communication) class in $\mathbb{F}^{\le n}$.

Let ${}_{(n)}\boldsymbol{\pi}_* = ({}_{(n)}\pi_*(k, i))_{(k,i)\in\mathbb{F}}$, $n \in \mathbb{N}$, denote a stationary probability vector of ${}_{(n)}\boldsymbol{P}_*$. Equation (1.1) implies that ${}_{(n)}\pi_*(k, i) = 0$ for all $(k, i) \in \mathbb{F} \setminus \mathbb{F}^{\le n}$ (see, e.g. [5, Theorem 1, Section I.7]) and ${}_{(n)}\boldsymbol{\pi}_*^{\le n} := ({}_{(n)}\pi_*(k, i))_{(k,i)\in\mathbb{F}^{\le n}}$ is a solution of ${}_{(n)}\boldsymbol{\pi}_*^{\le n}\,{}_{(n)}\boldsymbol{P}_*^{\le n} = {}_{(n)}\boldsymbol{\pi}_*^{\le n}$ and ${}_{(n)}\boldsymbol{\pi}_*^{\le n}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ denotes a column vector of 1s with an appropriate dimension. It is also known that if $\boldsymbol{P} \in \mathsf{BM}_d$ then $\lim_{n\to\infty} {}_{(n)}\boldsymbol{\pi}_* = \boldsymbol{\pi}$, where the convergence is elementwise (see [12, Theorem 3.4]).

Let ${}_{(n)}\boldsymbol{P}_n = ({}_{(n)}p_n(k, i; l, j))_{(k,i),(l,j)\in\mathbb{F}}$, $n \in \mathbb{N}$, denote a block-augmented truncation of $\boldsymbol{P}$ such that, for $i, j \in \mathbb{D}$,

$$ {}_{(n)}p_n(k, i; l, j) = \begin{cases} p(k, i; l, j), & k \in \mathbb{Z}_+,\ l \in \mathbb{Z}_+^{\le n-1}, \\ \displaystyle\sum_{m=n}^{\infty} p(k, i; m, j), & k \in \mathbb{Z}_+,\ l = n, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

which is called *the last-column-block-augmented first-n-block-column truncation* (for short, the last-column-block-augmented truncation). Let ${}_{(n)}\boldsymbol{\pi}_n = ({}_{(n)}\pi_n(k, i))_{(k,i)\in\mathbb{F}}$, $n \in \mathbb{N}$, denote a stationary probability vector of ${}_{(n)}\boldsymbol{P}_n$, where ${}_{(n)}\pi_n(k, i) = 0$ for all $(k, i) \in \mathbb{F} \setminus \mathbb{F}^{\le n}$. We obtain the following result.

Proposition 1.2. ([12, Theorem 3.6].) *If* $\boldsymbol{P} \in \mathsf{BM}_d$ *and* ${}_{(n)}\boldsymbol{\pi}_n$ *is the unique stationary distribution of* ${}_{(n)}\boldsymbol{P}_n$*, then there exists an infinite increasing sequence* $\{n_k \in \mathbb{N}; k \in \mathbb{Z}_+\}$ *such that, for all* $k \in \mathbb{Z}_+$,

$$0 \le \sum_{l=0}^{n_k} \sum_{i\in\mathbb{D}} ({}_{(n)}\pi_n(l, i) - \pi(l, i)) \le \sum_{l=0}^{n_k} \sum_{i\in\mathbb{D}} ({}_{(n)}\pi_*(l, i) - \pi(l, i)).$$

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Based on Proposition 1.2, Li and Zhao [12] stated that the last-column-block-augmented truncation ${}_{(n)}\boldsymbol{P}_n$ is the *best* approximation to $\boldsymbol{P}$ among the block-augmented truncations of $\boldsymbol{P}$, though they did not estimate the distance between ${}_{(n)}\boldsymbol{\pi}_n$ and $\boldsymbol{\pi}$.

In this paper we consider some cases where $\boldsymbol{P}$ satisfies the geometric drift condition (see Section 15.2.2 of [17]) but may be periodic. We first assume that $\boldsymbol{P} \in \mathsf{BM}_d$ and then present a bound for the total variation distance between ${}_{(n)}\boldsymbol{\pi}_n$ and $\boldsymbol{\pi}$, which is expressed as follows:

$$\|{}_{(n)}\boldsymbol{\pi}_n - \boldsymbol{\pi}\| := \sum_{(k,i)\in\mathbb{F}} |{}_{(n)}\pi_n(k,i) - \pi(k,i)| \le C_m(n),$$

where C_m is some function on $\mathbb{Z}_+$ with a supplementary parameter $m \in \mathbb{N}$ such that C_m is nonincreasing for any fixed m. The bound presented in this paper is a generalization of that of Tweedie [23] (see Theorem 4.2 therein). We also obtain such error bounds for more general cases, where $\boldsymbol{P}$ itself is not necessarily block monotone but is blockwise dominated by a block-monotone stochastic matrix.

The rest of this paper is divided into four sections. In Section 2 we provide preliminary results on block-monotone stochastic matrices. The main result of this paper is presented in Section 3, and some extensions are discussed in Section 4. As an example, these results are applied to GI/G/1-type Markov chains in Section 5.

2. Preliminaries

In this section we first introduce some definitions and notation, and then provide some basic results on block-monotone stochastic matrices.

2.1. Definitions and notation

Let $\boldsymbol{I}$ denote an identity matrix whose dimension depends on the context (we may write $\boldsymbol{I}_m$ to represent the $m \times m$ identity matrix). For any square matrix $\boldsymbol{M}$, let $\boldsymbol{M}^0 = \boldsymbol{I}$. Define

$$\boldsymbol{T}_d = \begin{pmatrix} \boldsymbol{I}_d & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{I}_d & \boldsymbol{I}_d & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{I}_d & \boldsymbol{I}_d & \boldsymbol{I}_d & \boldsymbol{O} & \cdots \\ \boldsymbol{I}_d & \boldsymbol{I}_d & \boldsymbol{I}_d & \boldsymbol{I}_d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \boldsymbol{T}_d^{-1} = \begin{pmatrix} \boldsymbol{I}_d & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ -\boldsymbol{I}_d & \boldsymbol{I}_d & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{O} & -\boldsymbol{I}_d & \boldsymbol{I}_d & \boldsymbol{O} & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & -\boldsymbol{I}_d & \boldsymbol{I}_d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\boldsymbol{T}_d\boldsymbol{T}_d^{-1} = \boldsymbol{T}_d^{-1}\boldsymbol{T}_d = \boldsymbol{I}$. Let $\boldsymbol{T}_d^{\le n}$, $n \in \overline{\mathbb{N}}$, denote the $|\mathbb{F}^{\le n}| \times |\mathbb{F}^{\le n}|$ northwest corner truncation of $\boldsymbol{T}_d$, where $|\cdot|$ denotes set cardinality. Note that $\boldsymbol{T}_d = \boldsymbol{T}_d^{\le\infty}$ and $(\boldsymbol{T}_d^{\le n})^{-1}$, $n \in \overline{\mathbb{N}}$, is equal to the $|\mathbb{F}^{\le n}| \times |\mathbb{F}^{\le n}|$ northwest corner truncation of $\boldsymbol{T}_d^{-1}$.

We now introduce the following definitions.

Definition 2.1. (*[12, Definition 2.1].*) For $n \in \overline{\mathbb{N}}$, let $\boldsymbol{f} = (f(k,i))_{(k,i)\in\mathbb{F}^{\le n}}$ denote a column vector with block size d. The vector $\boldsymbol{f}$ is said to be block increasing if $(\boldsymbol{T}_d^{\le n})^{-1}\boldsymbol{f} \ge \boldsymbol{0}$, i.e. $f(k,i) \le f(k+1,i)$ for all $(k,i) \in \mathbb{Z}_+^{\le n-1} \times \mathbb{D}$. We denote by BI_d the set of block-increasing column vectors with block size d.

Definition 2.2. For $n \in \overline{\mathbb{N}}$, let $\boldsymbol{\mu} = (\mu(k,i))_{(k,i)\in\mathbb{F}^{\le n}}$ and $\boldsymbol{\eta} = (\eta(k,i))_{(k,i)\in\mathbb{F}^{\le n}}$ denote probability vectors with block size d. The vector $\boldsymbol{\mu}$ is said to be (stochastically) blockwise dominated by $\boldsymbol{\eta}$ (denoted by $\boldsymbol{\mu} \prec_d \boldsymbol{\eta}$) if $\boldsymbol{\mu}\boldsymbol{T}_d^{\le n} \le \boldsymbol{\eta}\boldsymbol{T}_d^{\le n}$.

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Definition 2.3. For $n \in \overline{\mathbb{N}}$, let $\boldsymbol{P}_h = (p_h(k,i;l,j))_{(k,i),(l,j)\in\mathbb{F}^{\le n}}$, $h = 1, 2$, denote a stochastic matrix with block size d. The matrix $\boldsymbol{P}_1$ is said to be (stochastically) blockwise dominated by $\boldsymbol{P}_2$ (denoted by $\boldsymbol{P}_1 \prec_d \boldsymbol{P}_2$) if $\boldsymbol{P}_1 \boldsymbol{T}_d^{\le n} \le \boldsymbol{P}_2 \boldsymbol{T}_d^{\le n}$.

Remark 2.1. The columns of $\boldsymbol{T}_d^{\le n}$ are linearly independent vectors in BI_d, and, thus, every vector $\boldsymbol{f} \in \mathsf{BI}_d$ is expressed as a linear combination of columns of $\boldsymbol{T}_d^{\le n}$. Therefore, $\boldsymbol{\mu} \prec_d \boldsymbol{\eta}$ and $\boldsymbol{P}_1 \prec_d \boldsymbol{P}_2$ if and only if $\boldsymbol{\mu f} \le \boldsymbol{\eta f}$ and $\boldsymbol{P}_1 \boldsymbol{f} \le \boldsymbol{P}_2 \boldsymbol{f}$, respectively, for any $\boldsymbol{f} \in \mathsf{BI}_d$. According to this equivalence, we can establish the definition of the blockwise dominance relation '$\prec_d$' (see [12, Definitions 2.2 and 2.7]).

2.2. Basic results on block-monotone stochastic matrices

In this subsection we present three propositions on $|\mathbb{F}^{\le n}| \times |\mathbb{F}^{\le n}|$ stochastic matrices, where $n \in \overline{\mathbb{N}}$. The first two (Propositions 2.1 and 2.2 below) hold for any $|\mathbb{F}^{\le n}| \times |\mathbb{F}^{\le n}|$ stochastic matrix $\boldsymbol{S} = (s(k,i;l,j))$ in BM_d. The first proposition is immediate from Definition 1.1 and, thus, we omit the proof. The second proposition is an extension of Theorem 1.1 of [10]. The third proposition (Proposition 2.3 below) is a fundamental result for any two $|\mathbb{F}^{\le n}| \times |\mathbb{F}^{\le n}|$ stochastic matrices $\boldsymbol{P}_1 = (p_1(k,i;l,j))$ and $\boldsymbol{P}_2 = (p_2(k,i;l,j))$ such that $\boldsymbol{P}_1 \prec_d \boldsymbol{P}_2$, which is an extension of Lemma 1 of [7].

Proposition 2.1. *It holds that* $\boldsymbol{S} \in \mathsf{BM}_d$ *if and only if* $(\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S} \boldsymbol{T}_d^{\le n} \ge \boldsymbol{O}$.

Proposition 2.2. *The following conditions are equivalent.*

(a) $\boldsymbol{S} \in \mathsf{BM}_d$.

(b) $\boldsymbol{\mu S} \prec_d \boldsymbol{\eta S}$ *for any two probability vectors* $\boldsymbol{\mu}$ *and* $\boldsymbol{\eta}$ *such that* $\boldsymbol{\mu} \prec_d \boldsymbol{\eta}$.

(c) $\boldsymbol{S f} \in \mathsf{BI}_d$ *for any* $\boldsymbol{f} \in \mathsf{BI}_d$.

Proof. The equivalence of conditions (a) and (c) was shown in Theorem 3.8 of [12]. However, for the readers' convenience, we give a complete proof. First, we prove that condition (a) implies condition (b). To this end, we assume that $\boldsymbol{S} \in \mathsf{BM}_d$ and $\boldsymbol{\mu} \prec_d \boldsymbol{\eta}$. It then follows from Proposition 2.1 and Definition 2.2 that $(\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S} \boldsymbol{T}_d^{\le n} \ge \boldsymbol{O}$ and $\boldsymbol{\mu} \boldsymbol{T}_d^{\le n} \le \boldsymbol{\eta} \boldsymbol{T}_d^{\le n}$. Thus, it follows that

$$\boldsymbol{\mu S T}_d^{\le n} = \boldsymbol{\mu T}_d^{\le n} (\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S T}_d^{\le n} \le \boldsymbol{\eta T}_d^{\le n} (\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S T}_d^{\le n} = \boldsymbol{\eta S T}_d^{\le n},$$

which shows that $\boldsymbol{\mu S} \prec_d \boldsymbol{\eta S}$, i.e. condition (b) holds.

Next, we prove that condition (b) implies condition (a). For $(k,i) \in \mathbb{F}^{\le n}$, let $\boldsymbol{\xi}_{(k,i)} = (\xi_{(k,i)}(l,j))_{(l,j)\in\mathbb{F}^{\le n}}$ denote a $1 \times |\mathbb{F}^{\le n}|$ unit vector whose (k,i)th element is equal to 1. Let $\boldsymbol{\eta} = \boldsymbol{\xi}_{(k,i)}$ and $\boldsymbol{\mu} = \boldsymbol{\xi}_{(k-1,i)}$ for any fixed $(k,i) \in (\mathbb{Z}_+^{\le n} \setminus \{0\}) \times \mathbb{D}$. It then follows that $\boldsymbol{\mu} \prec_d \boldsymbol{\eta}$ and, thus, condition (b) yields $(\boldsymbol{\eta} - \boldsymbol{\mu}) \boldsymbol{S T}_d^{\le n} \ge \boldsymbol{0}$, where $\boldsymbol{\eta} - \boldsymbol{\mu}$ is equal to the (k,i)th row of $(\boldsymbol{T}_d^{\le n})^{-1}$. Furthermore, $\boldsymbol{\xi}_{(0,i)} \boldsymbol{S T}_d^{\le n} \ge \boldsymbol{0}$ for $i \in \mathbb{D}$, where $\boldsymbol{\xi}_{(0,i)}$ is equal to the $(0,i)$th row of $(\boldsymbol{T}_d^{\le n})^{-1}$. Consequently, condition (b) implies that $(\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S T}_d^{\le n} \ge \boldsymbol{O}$, i.e. condition (a) is satisfied (see Proposition 2.1).

To complete the proof, we prove the equivalence of conditions (a) and (c). We now assume that condition (a) holds. According to Definition 2.1, $(\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{f} \ge \boldsymbol{0}$ for any $\boldsymbol{f} \in \mathsf{BI}_d$. Combining this with $(\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S T}_d^{\le n} \ge \boldsymbol{O}$ (by condition (a)), we obtain

$$(\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S f} = (\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{S T}_d^{\le n} (\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{f} \ge \boldsymbol{0},$$

and, thus, $\boldsymbol{S f} \in \mathsf{BI}_d$, which shows that condition (c) is satisfied. Finally, we assume that condition (c) holds, and we fix $\boldsymbol{f} \in \mathsf{BI}_d$ to be a column of $\boldsymbol{T}_d^{\le n}$ (see Remark 2.1).

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It then follows that $\boldsymbol{S}\boldsymbol{f} \in \mathsf{BI}_d$, i.e. $(\boldsymbol{T}_d^{\le n})^{-1}\boldsymbol{S}\boldsymbol{f} \ge \boldsymbol{0}$. Therefore, $(\boldsymbol{T}_d^{\le n})^{-1}\boldsymbol{S}\boldsymbol{T}_d^{\le n} \ge \boldsymbol{O}$. The proof is completed.

Proposition 2.3. *If $\boldsymbol{P}_1 \prec_d \boldsymbol{P}_2$ and either $\boldsymbol{P}_1 \in \mathsf{BM}_d$ or $\boldsymbol{P}_2 \in \mathsf{BM}_d$, then the following statements hold.*

(a) *For all $k \in \mathbb{Z}_+^{\le n}$ and $i, j \in \mathbb{D}$,*

$$\sum_{l \in \mathbb{Z}_+^{\le n}} p_1(k, i; l, j) = \sum_{l \in \mathbb{Z}_+^{\le n}} p_2(k, i; l, j),$$

which is constant with respect to k.

(b) *$\boldsymbol{P}_1^m \prec_d \boldsymbol{P}_2^m$ for all $m \in \mathbb{N}$.*

(c) *Suppose that $\boldsymbol{P}_2$ is irreducible. If $\boldsymbol{P}_2$ is recurrent or positive recurrent, then $\boldsymbol{P}_1$ has exactly one recurrent or, respectively, positive recurrent class that includes the states $\{(0, i); i \in \mathbb{D}\}$, which is reachable from all the other states with probability 1. Thus, if $\boldsymbol{P}_2$ is positive recurrent then $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ have the unique stationary distributions $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$, respectively, and $\boldsymbol{\pi}_1 \prec_d \boldsymbol{\pi}_2$.*

Proof. We consider only the case in which $\boldsymbol{P}_1 \in \mathsf{BM}_d$ because the case in which $\boldsymbol{P}_2 \in \mathsf{BM}_d$ is discussed in a very similar way. We first prove statement (a). It follows from $\boldsymbol{P}_1 \in \mathsf{BM}_d$ and Proposition 1.1 that $\sum_{l \in \mathbb{Z}_+^{\le n}} p_1(k, i; l, j)$ is constant with respect to k for each $(i, j) \in \mathbb{D}^2$, which is denoted by $\psi_1(i, j)$. Furthermore, from $\boldsymbol{P}_1 \prec_d \boldsymbol{P}_2$, it follows that

$$\psi_1(i, j) = \sum_{l \in \mathbb{Z}_+^{\le n}} p_1(k, i; l, j) \le \sum_{l \in \mathbb{Z}_+^{\le n}} p_2(k, i; l, j), \qquad k \in \mathbb{Z}_+^{\le n},\ i, j \in \mathbb{D}. \tag{2.1}$$

Since $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are stochastic matrices, $\sum_{j \in \mathbb{D}} \psi_1(i, j) = \sum_{j \in \mathbb{D}} \sum_{l \in \mathbb{Z}_+^{\le n}} p_2(k, i; l, j) = 1$ for all $(k, i) \in \mathbb{F}^{\le n}$. From this and (2.1), we obtain $\psi_1(i, j) = \sum_{l \in \mathbb{Z}_+^{\le n}} p_2(k, i; l, j)$ for all $k \in \mathbb{Z}_+^{\le n}$ and $i, j \in \mathbb{D}$.

Next, we prove statement (b) by induction. Suppose that, for some $m \in \mathbb{N}$, $\boldsymbol{P}_1^m \prec_d \boldsymbol{P}_2^m$, i.e. $\boldsymbol{P}_1^m \boldsymbol{T}_d^{\le n} \le \boldsymbol{P}_2^m \boldsymbol{T}_d^{\le n}$ (which is true at least for $m = 1$). Combining this with $(\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{P}_1 \boldsymbol{T}_d^{\le n} \ge \boldsymbol{O}$ (due to $\boldsymbol{P}_1 \in \mathsf{BM}_d$) yields

$$\begin{aligned}
\boldsymbol{P}_1^{m+1} \boldsymbol{T}_d^{\le n} &= \boldsymbol{P}_1^m \boldsymbol{T}_d^{\le n} (\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{P}_1 \boldsymbol{T}_d^{\le n} \\
&\le \boldsymbol{P}_2^m \boldsymbol{T}_d^{\le n} (\boldsymbol{T}_d^{\le n})^{-1} \boldsymbol{P}_1 \boldsymbol{T}_d^{\le n} \\
&= \boldsymbol{P}_2^m \boldsymbol{P}_1 \boldsymbol{T}_d^{\le n} \\
&\le \boldsymbol{P}_2^m \boldsymbol{P}_2 \boldsymbol{T}_d^{\le n} \\
&= \boldsymbol{P}_2^{m+1} \boldsymbol{T}_d^{\le n},
\end{aligned}$$

and, thus, $\boldsymbol{P}_1^{m+1} \prec_d \boldsymbol{P}_2^{m+1}$. Therefore statement (b) is true.

Finally, we prove statement (c). Note that there exist two Markov chains characterized by $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, called Markov chains 1 and 2, which are pathwise ordered by the blockwise dominance of $\boldsymbol{P}_2$ over $\boldsymbol{P}_1$ (see Lemma A.2). Since $\boldsymbol{P}_2$ is irreducible and recurrent, Markov chain 2 and, thus, Markov chain 1 can reach any state $(0, i)$, $i \in \mathbb{D}$, from all the states in the state space $\mathbb{F}^{\le n}$ with probability 1, and the mean first passage time to each state $(0, i)$, $i \in \mathbb{D}$, is finite if $\boldsymbol{P}_2$ is positive recurrent. These facts show that the first part of statement (c) holds. Finally, we prove

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that $\boldsymbol{\pi}_1 \prec_d \boldsymbol{\pi}_2$. Note here that $(\boldsymbol{I} + \boldsymbol{P}_h)/2$, $h = 1, 2$, is aperiodic and has the same stationary distribution as that of $\boldsymbol{P}_h$. Thus, we assume without loss of generality that $\boldsymbol{P}_h$, $h = 1, 2$, is aperiodic. It then follows, from statement (b) and the dominated convergence theorem, that $\boldsymbol{e}\boldsymbol{\pi}_1 \boldsymbol{T}_d^{\le n} \le \boldsymbol{e}\boldsymbol{\pi}_2 \boldsymbol{T}_d^{\le n}$ (see [5, Theorem 4, Section I.6]) and, thus, $\boldsymbol{\pi}_1 \boldsymbol{T}_d^{\le n} \le \boldsymbol{\pi}_2 \boldsymbol{T}_d^{\le n}$.

3. Main result

In this section we present a bound for $\| {}_{(n)}\boldsymbol{\pi}_n - \boldsymbol{\pi} \|$, which is the main result of this paper. To establish the bound, we use the $\boldsymbol{v}$-norm, where $\boldsymbol{v} = (v(k,i))_{(k,i)\in\mathbb{F}}$ is any nonnegative column vector. The $\boldsymbol{v}$-norm is defined as follows: for any $1 \times |\mathbb{F}|$ vector $\boldsymbol{x} = (x(k,i))_{(k,i)\in\mathbb{F}}$,

$$\|\boldsymbol{x}\|_{\boldsymbol{v}} = \sup_{|\boldsymbol{g}|\le \boldsymbol{v}} \left| \sum_{(k,i)\in\mathbb{F}} x(k,i) g(k,i) \right| = \sup_{\boldsymbol{0}\le \boldsymbol{g}\le \boldsymbol{v}} \sum_{(k,i)\in\mathbb{F}} |x(k,i)| g(k,i),$$

where $|\boldsymbol{g}|$ is a column vector obtained by taking the absolute value of each element of $\boldsymbol{g}$. By definition, $\|\cdot\|_{\boldsymbol{e}} = \|\cdot\|$, i.e. the $\boldsymbol{e}$-norm is equivalent to the total variation distance.

We need some further definitions. For $m \in \mathbb{Z}_+$ and $(k,i) \in \mathbb{F}$, let

$$\boldsymbol{p}^m(k,i) = (p^m(k,i;l,j))_{(l,j)\in\mathbb{F}} \quad \text{and} \quad {}_{(n)}\boldsymbol{p}_n^m(k,i) = ({}_{(n)}p_n^m(k,i;l,j))_{(l,j)\in\mathbb{F}}$$

denote probability vectors such that $p^m(k,i;l,j)$ and ${}_{(n)}p_n^m(k,i;l,j)$ represent the $(k,i;l,j)$th elements of $\boldsymbol{P}^m$ and $({}_{(n)}\boldsymbol{P}_n)^m$, respectively (when $m = 1$, the superscript '1' may be omitted). Clearly, $p^m(k,i;l,j) = \mathbb{P}(X_m = l, J_m = j \mid X_0 = k, J_0 = i)$ for $(k,i) \times (l,j) \in \mathbb{F}^2$.

Let $\varpi(i) = \sum_{k=0}^{\infty} \pi(k,i) > 0$ for $i \in \mathbb{D}$. Note that if $\boldsymbol{P} \in \mathsf{BM}_d$ then $\boldsymbol{\varpi} = (\varpi(i))_{i\in\mathbb{D}}$ is the stationary distribution of $\boldsymbol{\Psi}$ (and, thus, the Markov chain $\{J_\nu\}$; see Proposition 1.1). Note also that if $\boldsymbol{P} \in \mathsf{BM}_d$ then ${}_{(n)}\boldsymbol{P}_n \prec_d \boldsymbol{P}$ and, thus, ${}_{(n)}\boldsymbol{\pi}_n \prec_d \boldsymbol{\pi}$ (by Proposition 2.3(c)), which implies that, for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} {}_{(n)}\pi_n(k,i) = \sum_{k=0}^{\infty} \pi(k,i) = \varpi(i), \qquad i \in \mathbb{D}. \tag{3.1}$$

For any function $\varphi(\cdot,\cdot)$ on $\mathbb{F}$, let $\varphi(k,\boldsymbol{\varpi}) = \sum_{i\in\mathbb{D}} \varpi(i)\varphi(k,i)$ for $k \in \mathbb{Z}_+$.

In what follows, we estimate $\| {}_{(n)}\boldsymbol{\pi}_n - \boldsymbol{\pi} \|$. By the triangle inequality, it follows that

$$\begin{aligned} \| {}_{(n)}\boldsymbol{\pi}_n - \boldsymbol{\pi} \| \le \| \boldsymbol{p}^m(0,\boldsymbol{\varpi}) - \boldsymbol{\pi} \| + \| {}_{(n)}\boldsymbol{p}_n^m(0,\boldsymbol{\varpi}) - {}_{(n)}\boldsymbol{\pi}_n \| \\ + \| {}_{(n)}\boldsymbol{p}_n^m(0,\boldsymbol{\varpi}) - \boldsymbol{p}^m(0,\boldsymbol{\varpi}) \|. \end{aligned} \tag{3.2}$$

The third term on the right-hand side of (3.2) is bounded as in the following lemma, which is proved without $\boldsymbol{P} \in \mathsf{BM}_d$.

Lemma 3.1. *For all $m \in \mathbb{N}$,*

$$\begin{aligned} &\| {}_{(n)}\boldsymbol{p}_n^m(k,i) - \boldsymbol{p}^m(k,i) \| \\ &\quad \le \sum_{h=0}^{m-1} \sum_{(l,j)\in\mathbb{F}} {}_{(n)}p_n^h(k,i;l,j) \Delta_n(l,j), \qquad n \in \mathbb{N},\ (k,i) \in \mathbb{F}, \end{aligned} \tag{3.3}$$

where

$$\Delta_n(l,j) = \| \boldsymbol{p}(l,j) - {}_{(n)}\boldsymbol{p}_n(l,j) \| = 2 \sum_{l'>n,\, j'\in\mathbb{D}} p(l,j;l',j'), \qquad (l,j) \in \mathbb{F}. \tag{3.4}$$

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Proof. Clearly, (3.3) holds for $m = 1$. Note that, for $m, n \in \mathbb{N}$,

$$({}_{(n)}\boldsymbol{P}_n)^{m+1} - \boldsymbol{P}^{m+1} = {}_{(n)}\boldsymbol{P}_n[({}_{(n)}\boldsymbol{P}_n)^m - \boldsymbol{P}^m] + ({}_{(n)}\boldsymbol{P}_n - \boldsymbol{P})\boldsymbol{P}^m.$$

It then follows that, for $m = 2, 3, \ldots,$

$$\begin{aligned}
&\|{}_{(n)}\boldsymbol{p}_n^{m+1}(k,i) - \boldsymbol{p}^{m+1}(k,i)\| \\
&\quad\le \sum_{(l,j)\in\mathbb{F}} {}_{(n)}p_n(k,i;l,j)\|{}_{(n)}\boldsymbol{p}_n^m(l,j) - \boldsymbol{p}^m(l,j)\| \\
&\qquad + \sum_{(l,j)\in\mathbb{F}} |{}_{(n)}p_n(k,i;l,j) - p(k,i;l,j)| \sum_{(l',j')\in\mathbb{F}} p^m(l,j;l',j') \\
&\quad= \sum_{(l,j)\in\mathbb{F}} {}_{(n)}p_n(k,i;l,j)\|{}_{(n)}\boldsymbol{p}_n^m(l,j) - \boldsymbol{p}^m(l,j)\| + \Delta_n(k,i), \qquad (3.5)
\end{aligned}$$

where the last equality follows from the fact that $\sum_{(l',j')\in\mathbb{F}} p^m(l,j;l',j') = 1$. Thus, if (3.3) holds for some $m \ge 2$ then (3.5) yields

$$\begin{aligned}
&\|{}_{(n)}\boldsymbol{p}_n^{m+1}(k,i) - \boldsymbol{p}^{m+1}(k,i)\| \\
&\quad\le \sum_{(l,j)\in\mathbb{F}} {}_{(n)}p_n(k,i;l,j)\left[\sum_{h=0}^{m-1}\sum_{(l',j')\in\mathbb{F}} {}_{(n)}p_n^h(l,j;l',j')\Delta_n(l',j')\right] + \Delta_n(k,i) \\
&\quad= \sum_{h=0}^{m-1}\sum_{(l',j')\in\mathbb{F}}\left(\sum_{(l,j)\in\mathbb{F}} {}_{(n)}p_n(k,i;l,j)\,{}_{(n)}p_n^h(l,j;l',j')\right)\Delta_n(l',j') + \Delta_n(k,i) \\
&\quad= \sum_{h=0}^{m-1}\sum_{(l',j')\in\mathbb{F}} {}_{(n)}p_n^{h+1}(k,i;l',j')\Delta_n(l',j') + \Delta_n(k,i) \\
&\quad= \sum_{h=0}^{m}\sum_{(l,j)\in\mathbb{F}} {}_{(n)}p_n^h(k,i;l,j)\Delta_n(l,j).
\end{aligned}$$

The following lemma implies that the first two terms on the right-hand side of (3.2) converge to 0 as $m \to \infty$ without the aperiodicity of $\boldsymbol{P}$.

Lemma 3.2. *Let κ denote the period of $\boldsymbol{P}$. If $\boldsymbol{P} \in \mathsf{BM}_d$ and $\boldsymbol{P}$ is irreducible, then the following statements hold.*

(a) *There exist disjoint nonempty sets $\mathbb{D}_0, \mathbb{D}_1, \ldots, \mathbb{D}_{\kappa-1}$ such that $\mathbb{D} = \bigcup_{h=0}^{\kappa-1}\mathbb{D}_h$ and*

$$\sum_{(l,j)\in\mathbb{Z}_+\times\mathbb{D}_{h+1}} p(k,i;l,j) = 1, \qquad (k,i) \in \mathbb{Z}_+ \times \mathbb{D}_h,\ h \in \mathbb{Z}_+^{\le\kappa-1},$$

where $\mathbb{D}_{h'} = \mathbb{D}_h$ if $h' \equiv h (\text{mod } \kappa)$.

(b) $\kappa \le d = |\mathbb{D}|$. *Thus, every irreducible monotone stochastic matrix (which is in BM_1) is aperiodic.*

(c) *If $\boldsymbol{P}$ is positive recurrent then, for $k \in \mathbb{Z}_+$,*

$$\lim_{m\to\infty} \boldsymbol{p}^m(k,\boldsymbol{\varpi}) = \boldsymbol{\pi}, \qquad \lim_{m\to\infty} {}_{(n)}\boldsymbol{p}_n^m(k,\boldsymbol{\varpi}) = {}_{(n)}\boldsymbol{\pi}_n, \quad n \in \mathbb{N}. \qquad (3.6)$$

Proof. We prove statement (a) by contradiction. From Proposition 5.4.2 of [17], we know that there exist disjoint nonempty sets $\mathbb{F}_0, \mathbb{F}_1, \ldots, \mathbb{F}_{\kappa-1}$ such that $\mathbb{F} = \bigcup_{h=0}^{\kappa-1} \mathbb{F}_h$ and

$$\sum_{(l,j)\in\mathbb{F}_{h+1}} p(k,i;l,j) = 1, \qquad (k,i) \in \mathbb{F}_h,\ h \in \mathbb{Z}_+^{\le \kappa-1}, \tag{3.7}$$

where $\mathbb{F}_{h'} = \mathbb{F}_h$ if $h' \equiv h \pmod{\kappa}$. We suppose that there exist some $(k_*, i_*) \in \mathbb{N} \times \mathbb{D}$ and $h_* \in \mathbb{Z}_+^{\le \kappa-1}$ such that $(0, i_*) \in \mathbb{F}_{h_*}$ and $(k_*, i_*) \notin \mathbb{F}_{h_*}$. We now consider coupled Markov chains $\{(X'_\nu, J'_\nu); \nu \in \mathbb{Z}_+\}$ and $\{(X''_\nu, J''_\nu); \nu \in \mathbb{Z}_+\}$ with transition probability matrix $\boldsymbol{P}$, which are pathwise ordered such that $X'_\nu \le X''_\nu$ and $J'_\nu = J''_\nu$ for all $\nu \in \mathbb{N}$ if $X'_0 \le X''_0$ and $J'_0 = J''_0$ (see Lemma A.1). We also fix $(X'_0, J'_0) = (0, i_*) \in \mathbb{F}_{h_*}$ and $(X''_0, J''_0) = (k_*, i_*) \notin \mathbb{F}_{h_*}$. It then follows from (3.7) that

$$(X'_\nu, J'_\nu) \in \mathbb{F}_h \quad \text{implies that} \quad (X''_\nu, J''_\nu) \notin \mathbb{F}_h \qquad \text{for all } \nu \in \mathbb{N}. \tag{3.8}$$

Furthermore, since $\boldsymbol{P}$ is irreducible, there exists some $\nu_* \in \mathbb{N}$ such that $(X''_{\nu_*}, J''_{\nu_*}) = (0, i_*)$ and, thus, $(X'_{\nu_*}, J'_{\nu_*}) \in \mathbb{N} \times \{i_*\}$ by (3.8). This conclusion, however, contradicts the pathwise ordering of $\{(X'_\nu, J'_\nu)\}$ and $\{(X''_\nu, J''_\nu)\}$, i.e. $X'_\nu \le X''_\nu$ and $J'_\nu = J''_\nu$ for all $\nu \in \mathbb{N}$. Consequently, statement (a) holds, and statement (b) is immediate from statement (a).

Next we prove statement (c). Fix $k \in \mathbb{Z}_+$ arbitrarily. Let $q \colon \mathbb{D} \mapsto \mathbb{Z}_+^{\le \kappa-1}$ denote a surjection function such that $i \in \mathbb{D}_{q(i)}$. It then follows from [5, Theorem 4, Section I.6] that, for $h \in \mathbb{Z}_+^{\le \kappa-1}$,

$$\lim_{m'\to\infty} p^{m'\kappa+h}(k,i;l,j) = \mathbb{I}_{\{h \equiv q(j)-q(i) \pmod{\kappa}\}} \kappa \pi(l,j), \qquad (l,j) \in \mathbb{F}, \tag{3.9}$$

where $\mathbb{I}_{\{\cdot\}}$ denotes a function that takes value 1 if the statement in the braces is true and takes value 0 otherwise. From (3.9), we obtain, for $h \in \mathbb{Z}_+^{\le \kappa-1}$ and $(l,j) \in \mathbb{F}$,

$$\begin{aligned}
\lim_{m'\to\infty} \sum_{i\in\mathbb{D}} \varpi(i) p^{m'\kappa+h}(k,i;l,j) &= \lim_{m'\to\infty} \sum_{h'=0}^{\kappa-1} \sum_{i\in\mathbb{D}_{h'}} \varpi(i) p^{m'\kappa+h}(k,i;l,j) \\
&= \kappa \sum_{h'=0}^{\kappa-1} \sum_{i\in\mathbb{D}_{h'}} \varpi(i) \mathbb{I}_{\{h \equiv q(j)-q(i) \pmod{\kappa}\}} \pi(l,j) \\
&= \kappa \sum_{h'=0}^{\kappa-1} \sum_{i\in\mathbb{D}_{h'}} \varpi(i) \mathbb{I}_{\{h \equiv q(j)-h' \pmod{\kappa}\}} \pi(l,j),
\end{aligned} \tag{3.10}$$

where the last equality is due to the fact that $q(i) = h'$ for $i \in \mathbb{D}_{h'}$. Note that here $\sum_{i\in\mathbb{D}_{h'}} \varpi(i) = \sum_{(k,i)\in\mathbb{F}_{h'}} \pi(k,i) = 1/\kappa$ for any $h' \in \mathbb{Z}_+^{\le \kappa-1}$ (see [5, Theorem 1, Section I.7]). Note also that, for any $h \in \mathbb{Z}_+^{\le \kappa-1}$ and $j \in \mathbb{D}$, there exists the unique $h' \in \mathbb{Z}_+^{\le \kappa-1}$ such that $h \equiv q(j) - h' \pmod{\kappa}$. From (3.10), we then obtain, for $h \in \mathbb{Z}_+^{\le \kappa-1}$,

$$\lim_{m'\to\infty} \sum_{i\in\mathbb{D}} \varpi(i) p^{m'\kappa+h}(k,i;l,j) = \pi(l,j), \qquad (l,j) \in \mathbb{F},$$

which leads to the first limit in (3.6). Furthermore, since ${}_{(n)}\boldsymbol{P}_n \prec_d \boldsymbol{P} \in \mathsf{BM}_d$, it follows from Proposition 2.3(c) that ${}_{(n)}\boldsymbol{P}_n$ has the unique positive recurrent class. Consequently, we can prove the second limit in (3.6) in the same way as the first.

To estimate the first two terms on the right-hand side of (3.2), we assume that the geometric drift condition holds.

Assumption 3.1. *There exists a column vector* $\boldsymbol{v} = (v(k, i))_{(k,i)\in\mathbb{F}} \in \mathsf{BI}_d$ *such that* $\boldsymbol{v} \ge \boldsymbol{e}$ *and, for some* $\gamma \in (0, 1)$ *and* $b \in (0, \infty)$*,*

$$\boldsymbol{P}\boldsymbol{v} \le \gamma \boldsymbol{v} + b\,\mathbf{1}_0, \tag{3.11}$$

where $\mathbf{1}_K = (1_K(k, i))_{(k,i)\in\mathbb{F}}$*,* $K \in \mathbb{Z}_+$*, denotes a column vector such that* $1_K(k, i) = 1$ *for* $(k, i) \in \mathbb{F}^{\le K}$ *and* $1_K(k, i) = 0$ *for* $(k, i) \in \mathbb{F} \setminus \mathbb{F}^{\le K}$*.*

Remark 3.1. Suppose that $\boldsymbol{P}$ is irreducible. Since the state space $\mathbb{F}$ is countable, every subset of $\mathbb{F}$ includes a *small set* and, thus, a *petite set* (see [17, Theorem 5.2.2, Proposition 5.5.3]). Therefore, if the irreducible $\boldsymbol{P}$ is aperiodic and Assumption 3.1 holds, then there exist $r \in (1, \infty)$ and $C \in (0, \infty)$ such that $\sum_{m=1}^{\infty} r^m \|\boldsymbol{p}^m(k, i) - \boldsymbol{\pi}\|_{\boldsymbol{v}} \le Cv(k, i)$ for all $(k, i) \in \mathbb{F}$, which shows that $\boldsymbol{P}$ is $\boldsymbol{v}$-geometrically ergodic (see [17, Theorem 15.0.1]).

The following lemma is an extension of Theorem 2.2 of [15] to discrete-time BMMCs.

Lemma 3.3. *Suppose that* $\boldsymbol{P} \in \mathsf{BM}_d$ *and* $\boldsymbol{P}$ *is irreducible. If Assumption 3.1 holds, then, for all* $k \in \mathbb{Z}_+$ *and* $m \in \mathbb{N}$*,*

$$\|\boldsymbol{p}^m(k, \boldsymbol{\varpi}) - \boldsymbol{\pi}\|_{\boldsymbol{v}} \le 2\gamma^m \left[v(k, \boldsymbol{\varpi})(1 - 1_0(k, \boldsymbol{\varpi})) + \frac{b}{1-\gamma} \right], \tag{3.12}$$

$$\|{}_{(n)}\boldsymbol{p}_n^m(k, \boldsymbol{\varpi}) - {}_{(n)}\boldsymbol{\pi}_n\|_{\boldsymbol{v}} \le 2\gamma^m \left[v(k, \boldsymbol{\varpi})(1 - 1_0(k, \boldsymbol{\varpi})) + \frac{b}{1-\gamma} \right] \quad \textit{for all } n \in \mathbb{N}. \tag{3.13}$$

Proof. We first prove (3.12). To this end, we consider three copies $\{(X_\nu^{(h)}, J_\nu^{(h)}); \nu \in \mathbb{Z}_+\}$, $h = 0, 1, 2$, of the BMMC $\{(X_\nu, J_\nu); \nu \in \mathbb{Z}_+\}$, which are defined on a common probability space in such a way that

$$(X_0^{(0)}, J_0^{(0)}) = (0, J), \qquad (X_0^{(1)}, J_0^{(1)}) = (k, J), \qquad (X_0^{(2)}, J_0^{(2)}) = (X, J),$$

where $k \in \mathbb{Z}_+$ and (X, J) denotes a random vector distributed with $\mathbb{P}(X = l,\ S = j) = \pi(l, j)$ for $(l, j) \in \mathbb{F}$. According to the pathwise-ordered property of BMMCs (see Lemma A.1), we assume without loss of generality that

$$X_\nu^{(0)} \le X_\nu^{(1)}, \qquad X_\nu^{(0)} \le X_\nu^{(2)}, \qquad J_\nu^{(0)} = J_\nu^{(1)} = J_\nu^{(2)}, \quad \text{for all } \nu \in \mathbb{Z}_+. \tag{3.14}$$

For simplicity, let

$$\mathbb{E}_{(k,i)}[\cdot] = \mathbb{E}[\cdot \mid X_0 = k, J_0 = i], \qquad (k, i) \in \mathbb{F},$$
$$\mathbb{E}_{(k,i);(0,j)}[\cdot] = \mathbb{E}[\cdot \mid (X_0^{(h)}, J_0^{(h)}) = (k, i), (X_0^{(0)}, J_0^{(0)}) = (0, j)], \qquad (k, i) \in \mathbb{F},\ j \in \mathbb{D},$$

where $h = 1, 2$. Furthermore, let $\boldsymbol{g} = (g(l, j))_{(l,j)\in\mathbb{F}}$ denote a column vector satisfying $|\boldsymbol{g}| \le \boldsymbol{v}$, i.e. $|g(l, j)| \le v(l, j)$ for $(l, j) \in \mathbb{F}$. It then follows that, for $m = 1, 2, \ldots$,

$$\boldsymbol{p}^m(k, \boldsymbol{\varpi})\boldsymbol{g} = \sum_{i\in\mathbb{D}} \varpi(i) \sum_{(l,j)\in\mathbb{F}} p^m(k, i; l, j) g(l, j) = \mathbb{E}[\mathbb{E}_{(k,J)}[g(X_m, J_m)]],$$

$$\boldsymbol{\pi}\boldsymbol{g} = \boldsymbol{\pi}\boldsymbol{P}^m\boldsymbol{g} = \sum_{(k,i)\in\mathbb{F}} \pi(k, i) \sum_{(l,j)\in\mathbb{F}} p^m(k, i; l, j) g(l, j) = \mathbb{E}[\mathbb{E}_{(X,J)}[g(X_m, J_m)]].$$

Thus, by the triangle inequality, we obtain

$$\begin{aligned}
&|\boldsymbol{p}^m(k,\boldsymbol{\varpi})\boldsymbol{g}-\boldsymbol{\pi}\boldsymbol{g}| \\
&\quad = |\mathbb{E}[\mathbb{E}_{(k,J)}[g(X_m,J_m)]] - \mathbb{E}[\mathbb{E}_{(X,J)}[g(X_m,J_m)]]| \\
&\quad \le |\mathbb{E}[\mathbb{E}_{(k,J);(0,J)}[g(X_m^{(1)},J_m^{(1)})]] - \mathbb{E}[\mathbb{E}_{(k,J);(0,J)}[g(X_m^{(0)},J_m^{(0)})]]| \\
&\qquad + |\mathbb{E}[\mathbb{E}_{(X,J);(0,J)}[g(X_m^{(2)},J_m^{(2)})]] - \mathbb{E}[\mathbb{E}_{(X,J);(0,J)}[g(X_m^{(0)},J_m^{(0)})]]|. \qquad (3.15)
\end{aligned}$$

Let $T_h = \inf\{m \in \mathbb{Z}_+;\ X_\nu^{(h)} = X_\nu^{(0)}$ for all $\nu \ge m\}$ for $h = 1, 2$. It holds that

$$g(X_\nu^{(1)},J_\nu^{(1)}) = g(X_\nu^{(0)},J_\nu^{(0)}), \qquad \nu \ge T_1, \qquad (3.16)$$

$$g(X_\nu^{(2)},J_\nu^{(2)}) = g(X_\nu^{(0)},J_\nu^{(0)}), \qquad \nu \ge T_2. \qquad (3.17)$$

Applying (3.16) and (3.17) to (3.15) and using the fact that $|\boldsymbol{g}| \le \boldsymbol{v}$ (but not $\boldsymbol{P} \in \mathsf{BM}_d$) yields

$$\begin{aligned}
&|\boldsymbol{p}^m(k,\boldsymbol{\varpi})\boldsymbol{g}-\boldsymbol{\pi}\boldsymbol{g}| \\
&\quad \le \mathbb{E}[\mathbb{E}_{(k,J);(0,J)}[|g(X_m^{(1)},J_m^{(1)}) - g(X_m^{(0)},J_m^{(0)})|\mathbb{I}_{\{T_1>m\}}]] \\
&\qquad + \mathbb{E}[\mathbb{E}_{(X,J);(0,J)}[|g(X_m^{(2)},J_m^{(2)}) - g(X_m^{(0)},J_m^{(0)})|\mathbb{I}_{\{T_2>m\}}]] \\
&\quad \le \mathbb{E}[\mathbb{E}_{(k,J);(0,J)}[v(X_m^{(1)},J_m^{(1)})\mathbb{I}_{\{T_1>m\}}]] \\
&\qquad + \mathbb{E}[\mathbb{E}_{(k,J);(0,J)}[v(X_m^{(0)},J_m^{(0)})\mathbb{I}_{\{T_1>m\}}]] \\
&\qquad + \mathbb{E}[\mathbb{E}_{(X,J);(0,J)}[v(X_m^{(2)},J_m^{(2)})\mathbb{I}_{\{T_2>m\}}]] \\
&\qquad + \mathbb{E}[\mathbb{E}_{(X,J);(0,J)}[v(X_m^{(0)},J_m^{(0)})\mathbb{I}_{\{T_2>m\}}]]. \qquad (3.18)
\end{aligned}$$

Combining (3.18) with (3.14) and $\boldsymbol{v} \in \mathsf{BI}_d$, we obtain, for all $|\boldsymbol{g}| \le \boldsymbol{v}$,

$$\begin{aligned}
|\boldsymbol{p}^m(k,\boldsymbol{\varpi})\boldsymbol{g}-\boldsymbol{\pi}\boldsymbol{g}| &\le 2\mathbb{E}[\mathbb{E}_{(k,J);(0,J)}[v(X_m^{(1)},J_m^{(1)})\mathbb{I}_{\{T_1>m\}}]] \\
&\quad + 2\mathbb{E}[\mathbb{E}_{(X,J);(0,J)}[v(X_m^{(2)},J_m^{(2)})\mathbb{I}_{\{T_2>m\}}]]. \qquad (3.19)
\end{aligned}$$

Furthermore, it follows from (3.14) that $X_m^{(h)} = 0$, $h = 1, 2$, implies that $X_\nu^{(h)} = X_\nu^{(0)}$ for all $\nu \ge m$, which leads to $T_h \le \inf\{\nu \in \mathbb{Z}_+;\ X_\nu^{(h)} = 0\}$ for $h = 1, 2$. Thus, it holds that

$$\mathbb{E}[\mathbb{E}_{(k,J);(0,J)}[v(X_m^{(1)},J_m^{(1)})\mathbb{I}_{\{T_1>m\}}]] \le \mathbb{E}[\mathbb{E}_{(k,J)}[v(X_m,J_m)\mathbb{I}_{\{\tau_0>m\}}]], \qquad (3.20)$$

$$\mathbb{E}[\mathbb{E}_{(X,J);(0,J)}[v(X_m^{(2)},J_m^{(2)})\mathbb{I}_{\{T_2>m\}}]] \le \mathbb{E}[\mathbb{E}_{(X,J)}[v(X_m,J_m)\mathbb{I}_{\{\tau_0>m\}}]], \qquad (3.21)$$

where $\tau_0 = \inf\{\nu \in \mathbb{Z}_+;\ X_\nu = 0\}$. Substituting (3.20) and (3.21) into (3.19) yields

$$\begin{aligned}
\|\boldsymbol{p}^m(k,\boldsymbol{\varpi})-\boldsymbol{\pi}\|_{\boldsymbol{v}} &\le 2\mathbb{E}[\mathbb{E}_{(k,J)}[v(X_m,J_m)\mathbb{I}_{\{\tau_0>m\}}]] \\
&\quad + 2\mathbb{E}[\mathbb{E}_{(X,J)}[v(X_m,J_m)\mathbb{I}_{\{\tau_0>m\}}]]. \qquad (3.22)
\end{aligned}$$

Let $M_m = \gamma^{-m} v(X_m,J_m)\mathbb{I}_{\{\tau_0>m\}}$ for $m \in \mathbb{Z}_+$. If $\tau_0 \le m$ then $M_{m+1} = M_m = 0$. On the other hand, suppose that $\tau_0 > m$ and, thus, $(X_m, J_m) = (k, i) \in \mathbb{N} \times \mathbb{D}$ (from the fact that $\{\tau_0 > m\} \subseteq \{X_m \in \mathbb{N}\}$). We then have, for $(k, i) \in \mathbb{N} \times \mathbb{D}$,

$$\begin{aligned}
\mathbb{E}[M_{m+1} \mid (X_m,J_m) = (k,i), \tau_0 > m] &= \sum_{(l,j)\in\mathbb{N}\times\mathbb{D}} p(k,i;l,j)\gamma^{-m-1}v(l,j) \\
&\le \sum_{(l,j)\in\mathbb{F}} p(k,i;l,j)\gamma^{-m-1}v(l,j) \\
&\le \gamma^{-m} v(k,i),
\end{aligned}$$

where the last inequality follows from (3.11). Thus, $\{M_m\}$ is a supermartingale.

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Let $\{\theta_\nu; \nu \in \mathbb{Z}_+\}$ denote a sequence of stopping times for $\{M_m; m \in \mathbb{Z}_+\}$ such that $0 \le \theta_1 \le \theta_2 \le \cdots$ and $\lim_{\nu\to\infty} \theta_\nu = \infty$. Note that, for any $m' \in \mathbb{Z}_+$, $\min(m', \theta_\nu)$ is a stopping time for $\{M_m; m \in \mathbb{Z}_+\}$. It then follows from Doob's optional sampling theorem that, for $(k, i) \in \mathbb{F}$, $\mathbb{E}_{(k,i)}[M_{\min(m,\theta_\nu)}] \le \mathbb{E}_{(k,i)}[M_0]$, i.e.

$$\mathbb{E}_{(k,i)}[\gamma^{-\min(m,\theta_\nu)} v(X_{\min(m,\theta_\nu)}, J_{\min(m,\theta_\nu)})\mathbb{I}_{\{\tau_0>\min(m,\theta_\nu)\}}] \le v(k, i)(1 - 1_0(k, i)).$$

Thus, letting $\nu \to \infty$ and using Fatou's lemma, we have

$$\mathbb{E}_{(k,i)}[v(X_m, J_m)\mathbb{I}_{\{\tau_0>m\}}] \le \gamma^m v(k, i)(1 - 1_0(k, i)), \tag{3.23}$$

which leads to

$$\begin{aligned}\mathbb{E}[\mathbb{E}_{(k,J)}[v(X_m, J_m)\mathbb{I}_{\{\tau_0>m\}}]] &= \sum_{i\in\mathbb{D}} \varpi(i)\mathbb{E}_{(k,i)}[v(X_m, J_m)\mathbb{I}_{\{\tau_0>m\}}] \\ &\le \gamma^m v(k, \boldsymbol{\varpi})(1 - 1_0(k, \boldsymbol{\varpi})),\end{aligned} \tag{3.24}$$

where we used the fact that $1_0(k, i) = 1_0(k, \boldsymbol{\varpi})$ for all $i \in \mathbb{D}$. Note here that premultiplying both sides of (3.11) by $\boldsymbol{\pi}$ yields $\boldsymbol{\pi}\boldsymbol{v} \le b/(1-\gamma)$, from which, together with (3.23), we obtain

$$\mathbb{E}[\mathbb{E}_{(X,J)}[v(X_m, J_m)\mathbb{I}_{\{\tau_0>m\}}]] \le \gamma^m \sum_{(k,i)\in\mathbb{F}} \pi(k, i)v(k, i) \le \gamma^m \frac{b}{1-\gamma}. \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.22) yields (3.12).

Next we consider (3.13). Since $\boldsymbol{P} \in \mathsf{BM}_d$, we have ${}_{(n)}\boldsymbol{P}_n \in \mathsf{BM}_d$ and ${}_{(n)}\boldsymbol{P}_n \prec_d \boldsymbol{P}$. Thus, since $\boldsymbol{P}$ is irreducible and positive recurrent, Proposition 2.3(c) implies that ${}_{(n)}\boldsymbol{P}_n$ has the unique positive recurrent class, which includes the states $\{(0, i); i \in \mathbb{D}\}$. Furthermore, it follows from $\boldsymbol{v} \in \mathsf{BI}_d$, (3.11), and Remark 2.1 that

$${}_{(n)}\boldsymbol{P}_n\boldsymbol{v} \le \boldsymbol{P}\boldsymbol{v} \le \gamma\boldsymbol{v} + b\,\boldsymbol{1}_0\,. \tag{3.26}$$

Therefore, we can prove (3.13) in the same way as (3.12).

Combining (3.2) with Lemmas 3.1 and 3.3, we obtain the following theorem.

Theorem 3.1. *Suppose that $\boldsymbol{P} \in \mathsf{BM}_d$ and that $\boldsymbol{P}$ is irreducible. If Assumption 3.1 holds, then*

$$\|{}_{(n)}\boldsymbol{\pi}_n - \boldsymbol{\pi}\| \le 4\gamma^m \frac{b}{1-\gamma} + 2m \sum_{i\in\mathbb{D}} {}_{(n)}\pi_n(n, i) \quad \textit{for all } n, m \in \mathbb{N}, \tag{3.27}$$

$$\|{}_{(n)}\boldsymbol{\pi}_n - \boldsymbol{\pi}\| \le \frac{b}{1-\gamma}\left(4\gamma^m + 2m \sum_{i\in\mathbb{D}} \frac{1}{v(n, i)}\right) \quad \textit{for all } n, m \in \mathbb{N}. \tag{3.28}$$

Remark 3.2. If $d = 1$, Theorem 3.1 is reduced to Theorem 4.2 of [23].

Proof of Theorem 3.1. From (3.2) and Lemma 3.3, we have

$$\|{}_{(n)}\boldsymbol{\pi}_n - \boldsymbol{\pi}\| \le 4\gamma^m \frac{b}{1-\gamma} + \|{}_{(n)}\boldsymbol{p}_n^m(0, \boldsymbol{\varpi}) - \boldsymbol{p}^m(0, \boldsymbol{\varpi})\|. \tag{3.29}$$

From Lemma 3.1 (which does not require $\boldsymbol{P} \in \mathsf{BM}_d$), we obtain, for $m \in \mathbb{N}$,

$$\begin{aligned}\|{}_{(n)}\boldsymbol{p}_n^m(0, \boldsymbol{\varpi}) - \boldsymbol{p}^m(0, \boldsymbol{\varpi})\| &\le \sum_{i \in \mathbb{D}} \varpi(i) \|{}_{(n)}\boldsymbol{p}_n^m(0, i) - \boldsymbol{p}^m(0, i)\| \\ &\le \sum_{h=0}^{m-1} \sum_{(l,j)\in\mathbb{F}} \left(\sum_{i\in\mathbb{D}} \varpi(i)_{(n)} p_n^h(0, i; l, j) \right) \Delta_n(l, j). \end{aligned} \tag{3.30}$$

It follows from (3.1) and ${}_{(n)}\boldsymbol{P}_n \in \mathsf{BM}_d$ that $(\boldsymbol{\varpi}, 0, 0, \dots) \prec_d {}_{(n)}\boldsymbol{\pi}_n$ and $({}_{(n)}\boldsymbol{P}_n)^h \in \mathsf{BM}_d$ for $h \in \mathbb{N}$. Thus, Proposition 2.2 yields

$$(\boldsymbol{\varpi}, 0, 0, \dots)({}_{(n)}\boldsymbol{P}_n)^h \prec_d {}_{(n)}\boldsymbol{\pi}_n ({}_{(n)}\boldsymbol{P}_n)^h = {}_{(n)}\boldsymbol{\pi}_n. \tag{3.31}$$

In addition, $\boldsymbol{P} \in \mathsf{BM}_d$ and (3.4) imply that the column vector $\vec{\boldsymbol{\delta}}_n := (\Delta_n(l, j))_{(l,j)\in\mathbb{F}}$ with block size d is block increasing, i.e. $\vec{\boldsymbol{\delta}}_n \in \mathsf{BI}_d$. Combining this and (3.31) with Remark 2.1, it follows that

$$(\boldsymbol{\varpi}, 0, 0, \dots)({}_{(n)}\boldsymbol{P}_n)^h \vec{\boldsymbol{\delta}}_n \le {}_{(n)}\boldsymbol{\pi}_n \vec{\boldsymbol{\delta}}_n.$$

Applying (3.4) to the right-hand side of the above inequality, we obtain

$$\begin{aligned}&\sum_{(l,j)\in\mathbb{F}} \left(\sum_{i\in\mathbb{D}} \varpi(i)_{(n)} p_n^h(0, i; l, j) \right) \Delta_n(l, j) \\ &\quad \le 2 \sum_{(l,j)\in\mathbb{F}} {}_{(n)}\pi_n(l, j) \sum_{l'>n, j'\in\mathbb{D}} p(l, j; l', j') \\ &\quad \le 2 \sum_{(l,j)\in\mathbb{F}} {}_{(n)}\pi_n(l, j) \sum_{j'\in\mathbb{D}} {}_{(n)}p_n(l, j; n, j') \\ &\quad = 2 \sum_{j'\in\mathbb{D}} {}_{(n)}\pi_n(n, j'), \end{aligned} \tag{3.32}$$

where the second inequality follows from (1.2) and the last equality follows from the fact that ${}_{(n)}\boldsymbol{\pi}_n \cdot {}_{(n)}\boldsymbol{P}_n = {}_{(n)}\boldsymbol{\pi}_n$. Substituting (3.32) into (3.30) yields

$$\|{}_{(n)}\boldsymbol{p}_n^m(0, \boldsymbol{\varpi}) - \boldsymbol{p}^m(0, \boldsymbol{\varpi})\| \le 2m \sum_{j'\in\mathbb{D}} {}_{(n)}\pi_n(n, j'),$$

from which, together with (3.29), we obtain (3.27).

Next, we prove (3.28). Premultiplying both sides of (3.26) by ${}_{(n)}\boldsymbol{\pi}_n$ and using the fact that ${}_{(n)}\boldsymbol{\pi}_n \cdot {}_{(n)}\boldsymbol{P}_n = {}_{(n)}\boldsymbol{\pi}_n$, we obtain ${}_{(n)}\boldsymbol{\pi}_n \boldsymbol{v} \le b/(1-\gamma)$, which leads to

$${}_{(n)}\pi_n(n, i) \le \frac{b}{1-\gamma} \frac{1}{v(n, i)}, \qquad i \in \mathbb{D}.$$

Substituting this inequality into (3.27) yields (3.28).

4. Extensions of the main result

In this section we do not necessarily assume that $\boldsymbol{P}$ (i.e. the Markov chain $\{(X_\nu, J_\nu); \nu \in \mathbb{Z}_+\}$) is block monotone, but do assume that $\boldsymbol{P}$ is blockwise dominated by an irreducible and positive recurrent stochastic matrix in BM_d, which is denoted by $\widetilde{\boldsymbol{P}} = (\widetilde{p}(k, i; l, j))_{(k,i),(l,j)\in\mathbb{F}}$.

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Let $\widetilde{\boldsymbol{\pi}} = (\widetilde{\pi}(k,i))_{(k,i)\in\mathbb{F}}$ denote the stationary probability vector of $\widetilde{\boldsymbol{P}}$. It follows, from $\boldsymbol{P} \prec_d \widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$ and Proposition 2.3(c) that $\boldsymbol{\pi} \prec_d \widetilde{\boldsymbol{\pi}}$ and, thus,

$$\sum_{k=0}^{\infty} \widetilde{\pi}(k,i) = \sum_{k=0}^{\infty} \pi(k,i) = \varpi(i), \qquad i \in \mathbb{D}. \tag{4.1}$$

Let $\{(\widetilde{X}_\nu, \widetilde{J}_\nu);\ \nu \in \mathbb{Z}_+\}$ denote a BMMC with state space $\mathbb{F}$ and transition probability matrix $\widetilde{\boldsymbol{P}}$. Since $\boldsymbol{P} \prec_d \widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$, we can assume (without loss of generality) that the pathwise ordering of $\{(\widetilde{X}_\nu, \widetilde{J}_\nu)\}$ and $\{(X_\nu, J_\nu)\}$ holds, i.e. if $X_0 \le \widetilde{X}_0$ and $J_0 = \widetilde{J}_0$, then $X_\nu \le \widetilde{X}_\nu$ and $J_\nu = \widetilde{J}_\nu$ for all $n \in \mathbb{N}$ (see Lemma A.2).

The following result is an extension of Theorem 5.1 of [23].

Theorem 4.1. *Suppose that*

(i) $\widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$ *and* $\widetilde{\boldsymbol{P}}$ *is irreducible;*

(ii) $\boldsymbol{P} \prec_d \widetilde{\boldsymbol{P}}$*; and*

(iii) *there exists a column vector* $\boldsymbol{v} = (v(k,i))_{(k,i)\in\mathbb{F}} \in \mathsf{BI}_d$ *such that* $\boldsymbol{v} \ge \boldsymbol{e}$ *and*

$$\widetilde{\boldsymbol{P}} \boldsymbol{v} \le \gamma \boldsymbol{v} + b\,\mathbf{1}_0 \tag{4.2}$$

for some $\gamma \in (0,1)$ *and* $b \in (0,\infty)$.

Under these conditions, (3.28) holds.

Proof. We first prove the two bounds (3.12) and (3.13). Let (X, J) and $(\widetilde{X}, \widetilde{J})$ denote two random vectors on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(X = k, J = i) = \pi(k,i)$ and $\mathbb{P}(\widetilde{X} = k, \widetilde{J} = i) = \widetilde{\pi}(k,i)$ for $(k,i) \in \mathbb{F}$. Note that, since $\boldsymbol{\pi} \prec_d \widetilde{\boldsymbol{\pi}}$, it follows that $\sum_{l=k}^{\infty} \pi(l,i)/\varpi(i) \le \sum_{l=k}^{\infty} \widetilde{\pi}(l,i)/\varpi(i)$ for $(k,i) \in \mathbb{F}$. According to this inequality and (4.1), we can assume that $X \le \widetilde{X}$ and $J = \widetilde{J}$ (see [18, Theorem 1.2.4]). We then introduce the copies $\{(\widetilde{X}_\nu^{(h)}, \widetilde{J}_\nu^{(h)})\}$ and $\{(X_\nu^{(h)}, J_\nu^{(h)})\}$, $h = 0,1,2$, of the Markov chains $\{(\widetilde{X}_\nu, \widetilde{J}_\nu)\}$ and $\{(X_\nu, J_\nu)\}$, respectively, on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\begin{aligned}
&(\widetilde{X}_0^{(0)}, \widetilde{J}_0^{(0)}) = (0, \widetilde{J}), \qquad (\widetilde{X}_0^{(1)}, \widetilde{J}_0^{(1)}) = (k, \widetilde{J}), \qquad (\widetilde{X}_0^{(2)}, \widetilde{J}_0^{(2)}) = (\widetilde{X}, \widetilde{J}),\\
&(X_0^{(0)}, J_0^{(0)}) = (0, J), \qquad (X_0^{(1)}, J_0^{(1)}) = (k, J), \qquad (X_0^{(2)}, J_0^{(2)}) = (X, J).
\end{aligned}$$

From the pathwise ordering of $\{(\widetilde{X}_\nu, \widetilde{J}_\nu)\}$ and $\{(X_\nu, J_\nu)\}$, it holds that, for $h = 0,1,2$,

$$X_\nu^{(h)} \le \widetilde{X}_\nu^{(h)}, \qquad J_\nu^{(h)} = \widetilde{J}_\nu^{(h)}, \qquad \text{for all } \nu \in \mathbb{Z}_+. \tag{4.3}$$

In addition, by the pathwise-ordered property of $\widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$ (see Lemma A.1), we assume that

$$\widetilde{X}_\nu^{(0)} \le \widetilde{X}_\nu^{(1)}, \qquad \widetilde{X}_\nu^{(0)} \le \widetilde{X}_\nu^{(2)}, \qquad \widetilde{J}_\nu^{(0)} = \widetilde{J}_\nu^{(1)} = \widetilde{J}_\nu^{(2)}, \qquad \text{for all } \nu \in \mathbb{Z}_+. \tag{4.4}$$

Let $\boldsymbol{g} = (g(l,j))_{(l,j)\in\mathbb{F}}$ denote a column vector satisfying $|\boldsymbol{g}| \le \boldsymbol{v}$. It then follows that (3.18) holds under the assumptions of Theorem 4.1 because (3.18) does not require that $\{(X_\nu, J_\nu)\}$ is block monotone. Furthermore, applying (4.3), (4.4), and $\boldsymbol{v} \in \mathsf{BI}_d$ to (3.18), we obtain, for all $|\boldsymbol{g}| \le \boldsymbol{v}$,

$$\begin{aligned}
|\boldsymbol{p}^m(k, \boldsymbol{\varpi})\boldsymbol{g} - \boldsymbol{\pi}\boldsymbol{g}| &\le 2\mathbb{E}[\mathbb{E}_{(k,\widetilde{J});(0,\widetilde{J})}[v(\widetilde{X}_m^{(1)}, \widetilde{J}_m^{(1)})\mathbb{I}_{\{T_1 > m\}}]]\\
&\quad + 2\mathbb{E}[\mathbb{E}_{(\widetilde{X},\widetilde{J});(0,\widetilde{J})}[v(\widetilde{X}_m^{(2)}, \widetilde{J}_m^{(2)})\mathbb{I}_{\{T_2 > m\}}]],
\end{aligned} \tag{4.5}$$

where $T_h = \inf\{m \in \mathbb{Z}_+;\ X_\nu^{(h)} = X_\nu^{(0)} \text{ for all } \nu \ge m\}$ for $h = 1,2$.

It follows from (4.3) and (4.4) that, for each $h \in \{1, 2\}$, $\widetilde{X}_m^{(h)} = 0$ implies that $X_m^{(h)} = X_m^{(0)} = 0$ and, thus, $X_\nu^{(h)} = X_\nu^{(0)}$ for all $\nu \ge m$, which leads to $T_h \le \inf\{\nu \in \mathbb{Z}_+; \widetilde{X}_\nu^{(h)} = 0\}$. Therefore, from (4.5), we can obtain the following inequality (see the derivation of (3.22) from (3.19)):

$$\|\boldsymbol{p}^m(k, \boldsymbol{\varpi}) - \boldsymbol{\pi}\|_{\boldsymbol{v}} \le 2\mathbb{E}[\mathbb{E}_{(k,\widetilde{J})}[v(\widetilde{X}_m, \widetilde{J}_m)\mathbb{I}_{\{\widetilde{\tau}_0 > m\}}]] + 2\mathbb{E}[\mathbb{E}_{(\widetilde{X},\widetilde{J})}[v(\widetilde{X}_m, \widetilde{J}_m)\mathbb{I}_{\{\widetilde{\tau}_0 > m\}}]].$$

Here $\widetilde{\tau}_0 = \inf\{\nu \in \mathbb{Z}_+; \widetilde{X}_\nu = 0\}$. Furthermore, following the discussion after (3.22), we can show that, for all $k \in \mathbb{Z}_+$ and $m \in \mathbb{N}$,

$$\|\boldsymbol{p}^m(k, \boldsymbol{\varpi}) - \boldsymbol{\pi}\|_{\boldsymbol{v}} \le 2\gamma^m\left[v(k, \boldsymbol{\varpi})(1 - 1_0(k, \boldsymbol{\varpi})) + \frac{b}{1-\gamma}\right],$$

$$\|{}_{(n)}\boldsymbol{p}_n^m(k, \boldsymbol{\varpi}) - {}_{(n)}\boldsymbol{\pi}_n\|_{\boldsymbol{v}} \le 2\gamma^m\left[v(k, \boldsymbol{\varpi})(1 - 1_0(k, \boldsymbol{\varpi})) + \frac{b}{1-\gamma}\right] \quad \text{for all } n \in \mathbb{N}.$$

Consequently, we obtain the two bounds (3.12) and (3.13).

It remains to prove that

$$\|{}_{(n)}\boldsymbol{p}_n^m(0, \boldsymbol{\varpi}) - \boldsymbol{p}^m(0, \boldsymbol{\varpi})\| \le \frac{2mb}{1-\gamma}\sum_{i\in\mathbb{D}}\frac{1}{v(n,i)}.$$

Let $\widetilde{\Delta}_n(l, j) = 2\sum_{l'>n, j'\in\mathbb{D}}\widetilde{p}(l, j; l', j')$ for $(l, j) \in \mathbb{F}$. Since $\boldsymbol{P} \prec_d \widetilde{\boldsymbol{P}}$, it holds that $\Delta_n(l, j) \le \widetilde{\Delta}_n(l, j)$ for $(l, j) \in \mathbb{F}$. Note here that (3.30) still holds and, thus,

$$\|{}_{(n)}\boldsymbol{p}_n^m(0, \boldsymbol{\varpi}) - \boldsymbol{p}^m(0, \boldsymbol{\varpi})\| \le \sum_{h=0}^{m-1}\sum_{(l,j)\in\mathbb{F}}\left(\sum_{i\in\mathbb{D}}\varpi(i)_{(n)}p_n^h(0, i; l, j)\right)\widetilde{\Delta}_n(l, j). \tag{4.6}$$

We now define ${}_{(n)}\widetilde{\boldsymbol{P}}_n$ as the last-column-block-augmented first-n-block-column truncation of $\widetilde{\boldsymbol{P}}$ and ${}_{(n)}\widetilde{\boldsymbol{\pi}}_n = ({}_{(n)}\widetilde{\pi}_n(k, i))_{(n,i)\in\mathbb{F}}$ as the stationary distribution of ${}_{(n)}\widetilde{\boldsymbol{P}}_n$. We also define ${}_{(n)}\widetilde{\boldsymbol{p}}_n^m(k, i) = ({}_{(n)}\widetilde{p}_n^m(k, i; l, j))_{(l,j)\in\mathbb{F}}$ as a probability vector such that ${}_{(n)}\widetilde{p}_n^m(k, i; l, j)$ represents the $(k, i; l, j)$th element of $({}_{(n)}\widetilde{\boldsymbol{P}}_n)^m$. It then follows from ${}_{(n)}\boldsymbol{P}_n \prec_d {}_{(n)}\widetilde{\boldsymbol{P}}_n$ and Proposition 2.3(b) that $({}_{(n)}\boldsymbol{P}_n)^h \prec_d ({}_{(n)}\widetilde{\boldsymbol{P}}_n)^h$ for $h \in \mathbb{N}$. Therefore, Remark 2.1 and $(\widetilde{\Delta}_n(l, j))_{(l,j)\in\mathbb{F}} \in \mathsf{BI}_d$ (due to the fact that $\widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$) yield

$$\sum_{(l,j)\in\mathbb{F}} {}_{(n)}p_n^h(0, i; l, j)\widetilde{\Delta}_n(l, j) \le \sum_{(l,j)\in\mathbb{F}} {}_{(n)}\widetilde{p}_n^h(0, i; l, j)\widetilde{\Delta}_n(l, j). \tag{4.7}$$

Substituting (4.7) into (4.6), we have

$$\|{}_{(n)}\boldsymbol{p}_n^m(0, \boldsymbol{\varpi}) - \boldsymbol{p}^m(0, \boldsymbol{\varpi})\| \le \sum_{h=0}^{m-1}\sum_{(l,j)\in\mathbb{F}}\left(\sum_{i\in\mathbb{D}}\varpi(i)_{(n)}\widetilde{p}_n^h(0, i; l, j)\right)\widetilde{\Delta}_n(l, j).$$

In addition, since ${}_{(n)}\widetilde{\boldsymbol{P}}_n \prec_d \widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$, Proposition 2.3(c) implies that ${}_{(n)}\widetilde{\boldsymbol{\pi}}_n \prec_d \widetilde{\boldsymbol{\pi}}$ and, thus, $\sum_{k=0}^\infty {}_{(n)}\widetilde{\pi}_n(k, i) = \sum_{k=0}^\infty \widetilde{\pi}(k, i)$ for $i \in \mathbb{D}$. Combining this with (4.1), we have $\varpi(i) = \sum_{k=0}^\infty {}_{(n)}\widetilde{\pi}_n(k, i)$ for $i \in \mathbb{D}$. As a result, according to the discussion following (3.30) in the proof of Theorem 3.1, we can prove that

$$\|{}_{(n)}\boldsymbol{p}_n^m(0, \boldsymbol{\varpi}) - \boldsymbol{p}^m(0, \boldsymbol{\varpi})\| \le 2m\sum_{i\in\mathbb{D}} {}_{(n)}\widetilde{\pi}_n(n, i) \le \frac{2mb}{1-\gamma}\sum_{i\in\mathbb{D}}\frac{1}{v(n,i)}.$$

We can relax (4.2) if the direct path to the states $\{(0, i); i \in \mathbb{D}\}$ is ‘large’ enough.

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Theorem 4.2. *Suppose that conditions (i) and (ii) of Theorem 4.1 are satisfied. Furthermore, suppose that there exists a column vector* $\boldsymbol{v}' = (v'(k,i))_{(k,i)\in\mathbb{F}} \in \mathsf{BI}_d$ *such that* $\boldsymbol{v}' \ge \boldsymbol{e}$ *and, for some* $\gamma' \in (0,1)$*,* $b' \in (0,\infty)$*, and* $K \in \mathbb{Z}_+$*,*

$$\widetilde{\boldsymbol{P}}\boldsymbol{v}' \le \gamma'\boldsymbol{v}' + b'\,\boldsymbol{1}_K, \tag{4.8}$$

$$\widetilde{\boldsymbol{P}}(K;0)\boldsymbol{e} > \boldsymbol{0}, \tag{4.9}$$

where $\widetilde{\boldsymbol{P}}(k;l)$*,* $k,l \in \mathbb{Z}_+$*, denotes a* $d \times d$ *matrix such that* $\widetilde{\boldsymbol{P}}(k;l) = (\widetilde{p}(k,i;l,j))_{(i,j)\in\mathbb{D}}$*. Under these conditions, (3.28) holds for all* $n \in \mathbb{N}$*, where*

$$\gamma = \frac{\gamma' + B}{1 + B}, \tag{4.10}$$

$$b = b' + B, \tag{4.11}$$

$$v(k,i) = \begin{cases} v'(0,i), & k = 0,\ i \in \mathbb{D}, \\ v'(k,i) + B, & k \in \mathbb{N},\ i \in \mathbb{D}, \end{cases} \tag{4.12}$$

$$\textit{and} \quad B \in (0,\infty) \quad \textit{such that} \quad B\widetilde{\boldsymbol{P}}(K;0)\boldsymbol{e} \ge b'\boldsymbol{e}. \tag{4.13}$$

Remark 4.1. Condition (4.9) ensures that there exists some $B \in (0,\infty)$ that satisfies (4.13). Furthermore, since $\widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$, (4.9) implies that $\widetilde{\boldsymbol{P}}(k;0)\boldsymbol{e} > \boldsymbol{0}$ for all $k = 0,1,\ldots,K$.

Proof of Theorem 4.2. According to Theorem 4.1, it suffices to prove that (4.2) holds for some $\gamma \in (0,1)$, $b \in (0,\infty)$, and $\boldsymbol{v} \in \mathsf{BI}_d$ with $\boldsymbol{v} \ge \boldsymbol{e}$. Let $\boldsymbol{v}(k)$ and $\boldsymbol{v}'(k)$, $k \in \mathbb{Z}_+$, denote $d \times 1$ vectors such that $\boldsymbol{v}(k) = (v(k,i))_{i\in\mathbb{D}}$ and $\boldsymbol{v}'(k) = (v'(k,i))_{i\in\mathbb{D}}$. Clearly, it holds that $\boldsymbol{v} = (\boldsymbol{v}(0)^\top, \boldsymbol{v}(1)^\top, \ldots)^\top$ and $\boldsymbol{v}' = (\boldsymbol{v}'(0)^\top, \boldsymbol{v}'(1)^\top, \ldots)^\top$, where '$^\top$' represents the transpose operator. Thus, (4.8), (4.11), and (4.12) yield

$$\begin{aligned}
\sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(0;l)\boldsymbol{v}(l) &\le \sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(0;l)\boldsymbol{v}'(l) + B\boldsymbol{e} \\
&\le \gamma'\boldsymbol{v}'(0) + (b' + B)\boldsymbol{e} \\
&= \gamma'\boldsymbol{v}(0) + b\boldsymbol{e} \\
&\le \gamma\boldsymbol{v}(0) + b\boldsymbol{e},
\end{aligned} \tag{4.14}$$

where the last inequality follows from $\gamma \ge \gamma'$ (by (4.10)).

Furthermore, since $\widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$, $\sum_{l=1}^{\infty} \widetilde{\boldsymbol{P}}(k;l) \le \sum_{l=1}^{\infty} \widetilde{\boldsymbol{P}}(K;l)$ for $k = 1,2,\ldots,K$. From this result and (4.12), it follows that, for $k = 1,2,\ldots,K$,

$$\begin{aligned}
\sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k;l)\boldsymbol{v}(l) &\le \sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k;l)\boldsymbol{v}'(l) + B\sum_{l=1}^{\infty} \widetilde{\boldsymbol{P}}(K;l)\boldsymbol{e} \\
&= \sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k;l)\boldsymbol{v}'(l) + B\{\boldsymbol{e} - \widetilde{\boldsymbol{P}}(K;0)\boldsymbol{e}\}.
\end{aligned} \tag{4.15}$$

Applying (4.8) and (4.13) to the right-hand side of (4.15), we obtain, for $k = 1,2,\ldots,K$,

$$\sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k;l)\boldsymbol{v}(l) \le \gamma'\boldsymbol{v}'(k) + B\boldsymbol{e} + \{b'\boldsymbol{e} - B\widetilde{\boldsymbol{P}}(K;0)\boldsymbol{e}\} \le \gamma'\boldsymbol{v}'(k) + B\boldsymbol{e}. \tag{4.16}$$

Note that, (4.10) implies that $\sup_{x\ge 1}(\gamma' x + B)/(x+B) = \gamma$. Thus, since $\boldsymbol{v}' \ge \boldsymbol{e}$, it holds that $\gamma' v'(k,i) + B \le \gamma(v'(k,i) + B)$. Combining this with (4.12) yields

$$\gamma' \boldsymbol{v}'(k) + B\boldsymbol{e} \le \gamma(\boldsymbol{v}'(k) + B\boldsymbol{e}) = \gamma \boldsymbol{v}(k), \qquad k \in \mathbb{N}. \tag{4.17}$$

Substituting (4.17) into (4.16) yields

$$\sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k;l)\boldsymbol{v}(l) \le \gamma \boldsymbol{v}(k), \qquad k = 1,2,\dots,K. \tag{4.18}$$

Similarly, for $k = K+1, K+2, \dots,$

$$\sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k;l)\boldsymbol{v}(l) \le \sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k;l)\boldsymbol{v}'(l) + B\boldsymbol{e} \le \gamma' \boldsymbol{v}'(k) + B\boldsymbol{e} \le \gamma \boldsymbol{v}(k), \tag{4.19}$$

where the last inequality follows from (4.17). Finally, (4.14), (4.18), and (4.19) yield (4.2).

5. Applications

In this section we discuss the application of our results to GI/G/1-type Markov chains. To this end, we make the following assumption.

Assumption 5.1. (i) *$\boldsymbol{P}$ is of the form*

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{B}(0) & \boldsymbol{B}(1) & \boldsymbol{B}(2) & \boldsymbol{B}(3) & \cdots \\ \boldsymbol{B}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{B}(-2) & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \boldsymbol{B}(-3) & \boldsymbol{A}(-2) & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.1}$$

where $\boldsymbol{A}(k)$ and $\boldsymbol{B}(k)$, $k = 0, \pm 1, \pm 2, \dots,$ are $d \times d$ matrices;

(ii) *$\boldsymbol{P} \in \mathsf{BM}_d$;*

(iii) *$\boldsymbol{P}$ is irreducible and positive recurrent;*

(iv) *$\boldsymbol{A} := \sum_{k=-\infty}^{\infty} \boldsymbol{A}(k)$ is irreducible and stochastic; and*

(v) *$r_{A_+} = \sup\{z > 0; \sum_{k=0}^{\infty} z^k \boldsymbol{A}(k)$ is finite$\} > 1$.*

It follows from Assumption 5.1(i), (ii), and (iv), and Proposition 1.1 that $\boldsymbol{\Psi} = \sum_{l=0}^{\infty}\boldsymbol{B}(l) = \boldsymbol{B}(-k) + \sum_{l=-k+1}^{\infty}\boldsymbol{A}(l)$ for all $k \in \mathbb{N}$, which implies that $\lim_{k\to\infty}\boldsymbol{B}(-k) = \boldsymbol{O}$ and, thus, $\boldsymbol{A} = \boldsymbol{\Psi}$.

Let $\widehat{\boldsymbol{A}}(z)$ denote

$$\widehat{\boldsymbol{A}}(z) = \sum_{k=-\infty}^{\infty} z^k \boldsymbol{A}(k), \qquad z \in (1/r_{A_-}, r_{A_+}) \cap \{1\} =: \mathbb{I}_A, \tag{5.2}$$

where $r_{A_-} = \sup\{z > 0; \sum_{k=1}^{\infty} z^k \boldsymbol{A}(-k)$ is finite$\} \ge 1$. Let $\delta_A(z)$, $z \in \mathbb{I}_A$, denote the real and maximum-modulus eigenvalue of $\widehat{\boldsymbol{A}}(z)$ (see, e.g. [9, Theorems 8.3.1 and 8.4.4]).

Let $\boldsymbol{\mu}_A(z) = (\mu_A(z,i))_{i\in\mathbb{D}}$ and $\boldsymbol{v}_A(z) = (v_A(z,i))_{i\in\mathbb{D}}$, $z \in \mathcal{I}_A$, denote the left and right eigenvectors of $\widehat{\boldsymbol{A}}(z)$ corresponding to the eigenvalue $\delta_A(z)$, i.e.

$$\boldsymbol{\mu}_A(z)\widehat{\boldsymbol{A}}(z) = \delta_A(z)\boldsymbol{\mu}_A(z), \qquad \widehat{\boldsymbol{A}}(z)\boldsymbol{v}_A(z) = \delta_A(z)\boldsymbol{v}_A(z), \tag{5.3}$$

which are normalized such that $\boldsymbol{\mu}_A(z)\boldsymbol{v}_A(z) = 1$ and $\boldsymbol{v}_A(z) \ge \boldsymbol{e}$ for $z \in \mathcal{I}_A$. We then have $\delta_A(z) = \boldsymbol{\mu}_A(z)\widehat{\boldsymbol{A}}(z)\boldsymbol{v}_A(z)$. It also follows from $\boldsymbol{A} = \boldsymbol{\Psi}$ and Assumption 5.1(iv) that $\delta_A(1) = 1$, $\boldsymbol{\mu}_A(1) = c\boldsymbol{\varpi}$, and $\boldsymbol{v}_A(1) = c^{-1}\boldsymbol{e}$ for some $c \in (0, 1]$.

Lemma 5.1. *Under Assumption 5.1, there exists an $\alpha \in (1, r_A)$ such that $\delta_A(\alpha) < 1$.*

Proof. Since $\delta_A(1) = 1$ and $\delta_A(z)$ is differentiable for $z \in \mathcal{I}_A$ (see [1, Theorem 2.1]), it suffices to show that $\delta'_A(1) < 0$. Indeed, $\delta'_A(1) = \boldsymbol{\mu}_A(1)\sum_{k=-\infty}^{\infty} k\boldsymbol{A}(k)\boldsymbol{v}_A(1) = \boldsymbol{\varpi}\sum_{k=-\infty}^{\infty} k\boldsymbol{A}(k)\boldsymbol{e}$, which is equal to the mean drift of the process $\{X_\nu;\ \nu \in \mathbb{Z}_+\}$ away from the boundary and is strictly negative under Assumption 5.1 (see, e.g. [11, Proposition 2.2.1]).

We now define $\boldsymbol{P}(k;l)$, $k, l \in \mathbb{Z}_+$, as a $d \times d$ matrix such that $\boldsymbol{P}(k;l) = (p(k,i;l,j))_{i,j\in\mathbb{D}}$. We also fix $\boldsymbol{v}' = (\boldsymbol{v}'(0)^\top, \boldsymbol{v}'(1)^\top, \ldots)^\top$ such that

$$\boldsymbol{v}'(k) = \alpha^k \boldsymbol{v}_A(\alpha), \qquad k \in \mathbb{Z}_+, \tag{5.4}$$

which leads to $\boldsymbol{v}' \in \mathsf{BI}_d$. It then follows from (5.1) and (5.4) that

$$\sum_{l=0}^{\infty} \boldsymbol{P}(0;l)\boldsymbol{v}'(l) = \sum_{l=0}^{\infty} \alpha^l \boldsymbol{B}(l)\boldsymbol{v}_A(\alpha) =: \boldsymbol{w}(0), \tag{5.5}$$

$$\sum_{l=0}^{\infty} \boldsymbol{P}(k;l)\boldsymbol{v}'(l) = \boldsymbol{B}(-k)\boldsymbol{v}_A(\alpha) + \alpha^k \sum_{l=-k+1}^{\infty} \alpha^l \boldsymbol{A}(l)\boldsymbol{v}_A(\alpha) =: \boldsymbol{w}(k), \qquad k \in \mathbb{N}, \tag{5.6}$$

where $\boldsymbol{w}(0) \le \boldsymbol{w}(0) \le \boldsymbol{w}(1) \le \cdots$ due to the facts that $\boldsymbol{P} \in \mathsf{BM}_d$ and $\boldsymbol{v}' \in \mathsf{BI}_d$ (see Proposition 2.2). Furthermore, using (5.2) and (5.3), we can estimate the right-hand side of (5.6) as

$$\begin{aligned}\sum_{l=0}^{\infty} \boldsymbol{P}(k;l)\boldsymbol{v}'(l) = \boldsymbol{w}(k) &\le \boldsymbol{B}(-k)\boldsymbol{v}_A(\alpha) + \alpha^k \widehat{\boldsymbol{A}}(\alpha)\boldsymbol{v}_A(\alpha) \\ &= \boldsymbol{B}(-k)\boldsymbol{v}_A(\alpha) + \alpha^k \delta_A(\alpha)\boldsymbol{v}_A(\alpha) \\ &< \infty, \qquad k \in \mathbb{N}.\end{aligned} \tag{5.7}$$

Therefore, $\boldsymbol{w}(k)$ is finite for all $k \in \mathbb{Z}_+$. In addition, combining (5.7), $\lim_{k\to\infty} \boldsymbol{B}(-k) = \boldsymbol{O}$, $\boldsymbol{v}_A(\alpha) \ge \boldsymbol{e}$, and Lemma 5.1, we can show that there exist some $\gamma' \in (0, 1)$ and $k_* \in \mathbb{N}$ such that

$$\sum_{l=0}^{\infty} \boldsymbol{P}(k;l)\boldsymbol{v}'(l) \le \gamma' \alpha^k \boldsymbol{v}_A(\alpha) = \gamma' \boldsymbol{v}'(k) \quad \text{for all } k \ge k_*, \tag{5.8}$$

where the last equality is due to (5.4). Consequently, from Theorem 4.2, we have the following result.

Theorem 5.1. *Suppose that Assumption 5.1 holds, and fix $\gamma' \in (0, 1)$ and $k_* \in \mathbb{N}$ to satisfy (5.8). Furthermore, if $\boldsymbol{B}(-K)\boldsymbol{e} > \boldsymbol{0}$ for some nonnegative integer $K \ge k_* - 1$, then the bound (3.28) holds for $\gamma \in (0, 1)$, $b \in (0, \infty)$, and $\boldsymbol{v} \in \mathsf{BI}_d$ such that (4.10)–(4.13) are satisfied, where $\boldsymbol{v}'$ is given by (5.4), $\widetilde{\boldsymbol{P}}(K; 0) = \boldsymbol{B}(-K)$, and*

$$b' = \inf\{x > 0;\ x\boldsymbol{e} \ge \boldsymbol{w}(k) - \gamma' \alpha^k \boldsymbol{v}_A(\alpha) \text{ for all } k = 0, 1, \ldots, K\}. \tag{5.9}$$

Proof. Fix $\widetilde{\boldsymbol{P}} = \boldsymbol{P} \in \mathsf{BM}_d$. From (5.4)–(5.6) and (5.9), it holds that

$$\sum_{l=0}^{\infty} \widetilde{\boldsymbol{P}}(k; l)\boldsymbol{v}'(l) = \gamma' \boldsymbol{v}'(k) + \{\boldsymbol{w}(k) - \gamma' \alpha^k \boldsymbol{v}_A(\alpha)\} \le \gamma' \boldsymbol{v}'(k) + b'\boldsymbol{e}, \qquad k = 0, 1, \ldots, K.$$

This inequality and (5.8) yield (4.8). Furthermore, (4.9) holds due to the fact that $\widetilde{\boldsymbol{P}}(K; 0)\boldsymbol{e} = \boldsymbol{B}(-K)\boldsymbol{e} > \boldsymbol{0}$. Consequently, all the conditions of Theorem 4.2 are satisfied and, thus, the bound (3.28) holds.

Finally, we consider the special case in which $\boldsymbol{B}(-k) = \boldsymbol{A}(-k) = \boldsymbol{O}$ for $k \ge 2$, $\boldsymbol{B}(-1) = \boldsymbol{A}(-1)$ and $\boldsymbol{B}(k) = \boldsymbol{A}(k-1)$, for $k \in \mathbb{Z}_+$, i.e.

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{O} & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}(-1) & \boldsymbol{A}(0) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.10}$$

which is block monotone with block size d. Note that $\boldsymbol{P}$ in (5.10) is an M/G/1-type transition probability matrix and appears in the analysis of the stationary queue length distribution in the BMAP/GI/1 queue (see [20]). From Theorem 3.1, we then have the following theorem.

Theorem 5.2. *Suppose that Assumption 5.1 holds. Furthermore, if* $\boldsymbol{B}(-k) = \boldsymbol{A}(-k) = \boldsymbol{O}$ *for* $k \ge 2$, $\boldsymbol{B}(-1) = \boldsymbol{A}(-1)$, *and* $\boldsymbol{B}(k) = \boldsymbol{A}(k-1)$ *for* $k \in \mathbb{Z}_+$, *then bound (3.28) holds for* $\gamma = \delta_A(\alpha)$, $b = (\alpha - 1)\max_{i\in\mathbb{D}} v_A(\alpha, i)$, *and* $\boldsymbol{v} = \boldsymbol{v}'$ *given in (5.4).*

Proof. Fixing $\boldsymbol{v} = \boldsymbol{v}'$ and applying (5.2)–(5.4), Lemma 5.1, and the conditions on $\{\boldsymbol{B}(k)\}$ to (5.5) and (5.6), we obtain

$$\sum_{l=0}^{\infty} \boldsymbol{P}(0; l)\boldsymbol{v}(l) = \alpha\delta_A(\alpha)\boldsymbol{v}_A(\alpha) \le \boldsymbol{v}(0) + (\alpha - 1)\boldsymbol{v}_A(\alpha),$$

$$\sum_{l=0}^{\infty} \boldsymbol{P}(k; l)\boldsymbol{v}(l) = \alpha^k\delta_A(\alpha)\boldsymbol{v}_A(\alpha) = \delta_A(\alpha)\boldsymbol{v}(k), \qquad k \in \mathbb{N},$$

which imply that all the conditions of Theorem 3.1 are satisfied, and thus, (3.28) holds.

Appendix A. Pathwise ordering

In this appendix we present two lemmas on the pathwise ordering associated with BMMCs. As in the previous sections, $\boldsymbol{P} = (p(k, i; l, j))_{(k,i),(l,j)\in\mathbb{F}}$ and $\widetilde{\boldsymbol{P}} = (\widetilde{p}(k, i; l, j))_{(k,i),(l,j)\in\mathbb{F}}$ represent $|\mathbb{F}| \times |\mathbb{F}|$ stochastic matrices, though they are not necessarily assumed to be irreducible or recurrent in this appendix.

Let $\{U_\nu; \nu \in \mathbb{N}\}$ and $\{S_\nu; \nu \in \mathbb{N}\}$ denote two independent sequences of independent and identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that U_ν and S_ν are uniformly distributed in (0, 1). Let J_0^* denote a $\mathbb{D}$-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is independent of both $\{U_\nu; \nu \in \mathbb{N}\}$ and $\{S_\nu; \nu \in \mathbb{N}\}$. Furthermore, let $J_\nu^* = G^{-1}(S_\nu \mid J_{\nu-1}^*)$ for $\nu \in \mathbb{N}$, where

$$G^{-1}(s \mid i) = \inf\left\{ j \in \mathbb{D}; \sum_{j'=1}^{j} \psi(i, j') \ge s \right\}, \qquad 0 < s < 1,\ i \in \mathbb{D}.$$

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It then follows that $\{J_\nu^*;\, \nu \in \mathbb{Z}_+\}$ is a $\mathbb{D}$-valued Markov chain on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(J_{\nu+1}^* = j \mid J_\nu^* = i) = \psi(i, j)$ for $i, j \in \mathbb{D}$ and $\nu \in \mathbb{Z}_+$, where $\psi(i, j)$ is defined in Proposition 1.1.

Lemma A.1. (Pathwise-ordered property of BMMCs.) *Suppose that $\boldsymbol{P} \in \mathsf{BM}_d$. Let X_0' and X_0'' denote nonnegative integer-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which are independent of both $\{U_\nu;\, \nu \in \mathbb{N}\}$ and $\{S_\nu;\, \nu \in \mathbb{N}\}$. Furthermore, let $X_\nu' = F^{-1}(U_\nu \mid X_{\nu-1}', J_{\nu-1}^*, J_\nu^*)$ and $X_\nu'' = F^{-1}(U_\nu \mid X_{\nu-1}'', J_{\nu-1}^*, J_\nu^*)$ for $\nu \in \mathbb{N}$, where $F^{-1}(u \mid k, i, j)$, $0 < u < 1$, $k \in \mathbb{Z}_+$, $i, j \in \mathbb{D}$, is defined as*

$$F^{-1}(u \mid k, i, j) = \inf\left\{ l \in \mathbb{Z}_+;\ \sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)} \ge u \right\}. \tag{A.1}$$

Under these conditions, $\{(X_\nu', J_\nu^*);\, \nu \in \mathbb{Z}_+\}$ and $\{(X_\nu'', J_\nu^*);\, \nu \in \mathbb{Z}_+\}$ are Markov chains with a transition probability matrix $\boldsymbol{P}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_\nu' \le X_\nu''$ for all $\nu \in \mathbb{N}$ if $X_0' \le X_0''$.

Proof. Suppose that $X_\nu' \le X_\nu''$ for some $\nu \in \mathbb{Z}_+$. It then follows from $\boldsymbol{P} \in \mathsf{BM}_d$ that

$$\sum_{m=0}^{l} p(X_\nu', J_\nu^*; m, J_{\nu+1}^*) \ge \sum_{m=0}^{l} p(X_\nu'', J_\nu^*; m, J_{\nu+1}^*), \qquad l \in \mathbb{Z}_+.$$

Thus, from the definitions of $\{X_\nu'\}$ and $\{X_\nu''\}$, we have

$$\begin{aligned} X_{\nu+1}'' &= \inf\left\{ l \in \mathbb{Z}_+;\ \sum_{m=0}^{l} \frac{p(X_\nu'', J_\nu^*; m, J_{\nu+1}^*)}{\psi(J_\nu^*, J_{\nu+1}^*)} \ge U_{\nu+1} \right\} \\ &\ge \inf\left\{ l \in \mathbb{Z}_+;\ \sum_{m=0}^{l} \frac{p(X_\nu', J_\nu^*; m, J_{\nu+1}^*)}{\psi(J_\nu^*, J_{\nu+1}^*)} \ge U_{\nu+1} \right\} \\ &= F^{-1}(U_{\nu+1} \mid X_\nu', J_\nu^*, J_{\nu+1}^*) \\ &= X_{\nu+1}'. \end{aligned}$$

Therefore, we have proved by induction that $X_\nu' \le X_\nu''$ for all $\nu \in \mathbb{N}$.

Next, we prove that the dynamics of $\{(X_\nu', J_\nu^*);\, \nu \in \mathbb{Z}_+\}$ are determined by $\boldsymbol{P}$. Let $\sigma(\,\cdot\,)$ denote the sigma-algebra generated by the random variables in the parentheses. From the definition of $\{(X_\nu', J_\nu^*)\}$, it follows that, for $\nu \in \mathbb{N}$,

$$\begin{aligned} &\sigma(X_0', X_1', \ldots, X_{\nu-1}', J_0^*, J_1^*, \ldots, J_{\nu-1}^*) \\ &\qquad \subseteq \sigma(X_0', J_0^*, U_1, U_2, \ldots, U_{\nu-1}, S_1, S_2, \ldots, S_{\nu-1}) \\ &\qquad =: \mathcal{G}_{\nu-1}. \end{aligned}$$

Note that, for $(k, i) \in \mathbb{F}$ and $j \in \mathbb{D}$,

$$\mathcal{G}_{\nu-1} \cap \{X_\nu' = k, J_\nu^* = i, J_{\nu+1}^* = j\} \subseteq \sigma(X_0', J_0^*, U_1, U_2, \ldots, U_\nu, S_1, S_2, \ldots, S_{\nu+1}),$$

which implies that $U_{\nu+1}$ is independent of both $\mathcal{G}_{\nu-1}$ and $\{X_\nu' = k, J_\nu^* = i, J_{\nu+1}^* = j\}$ for

$(k, i) \in \mathbb{F}$ and $j \in \mathbb{D}$. Thus, it follows from the definition of $\{X'_\nu\}$ that

$$\begin{aligned}
&\mathbb{P}(X'_{\nu+1} \le l \mid \mathcal{G}_{\nu-1},\ X'_\nu = k,\ J^*_\nu = i,\ J^*_{\nu+1} = j) \\
&\quad = \mathbb{P}\left(\sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)} \ge U_{\nu+1} \,\middle|\, \mathcal{G}_{\nu-1},\ X'_\nu = k,\ J^*_\nu = i,\ J^*_{\nu+1} = j\right) \\
&\quad = \mathbb{P}\left(\sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)} \ge U_{\nu+1}\right) \\
&\quad = \sum_{m=0}^{l} \frac{p(k, i; m, j)}{\psi(i, j)}, \qquad (k, i) \times (l, j) \in \mathbb{F}^2.
\end{aligned} \tag{A.2}$$

Also, note that $S_{\nu+1}$ is independent of $\mathcal{G}_\nu \supseteq \mathcal{G}_{\nu-1} \cap \{X'_\nu = k,\ J^*_\nu = i\}$ for $(k, i) \in \mathbb{F}$. Therefore, from the definition of $\{J^*_\nu\}$, it holds that, for $(k, i) \in \mathbb{F}$ and $j \in \mathbb{D}$,

$$\begin{aligned}
&\mathbb{P}(J^*_{\nu+1} = j \mid \mathcal{G}_{\nu-1},\ X'_\nu = k,\ J^*_\nu = i) \\
&\quad = \mathbb{P}\left(\sum_{j'=1}^{j-1} \psi(i, j') < S_{\nu+1} \le \sum_{j'=1}^{j} \psi(i, j') \,\middle|\, \mathcal{G}_{\nu-1},\ X'_\nu = k,\ J^*_\nu = i\right) \\
&\quad = \mathbb{P}\left(\sum_{j'=1}^{j-1} \psi(i, j') < S_{\nu+1} \le \sum_{j'=1}^{j} \psi(i, j')\right) \\
&\quad = \psi(i, j).
\end{aligned} \tag{A.3}$$

Combining (A.2) and (A.3) yields

$$\begin{aligned}
&\mathbb{P}(X'_{\nu+1} \le l,\ J^*_{\nu+1} = j \mid \mathcal{G}_{\nu-1},\ X'_\nu = k,\ J^*_\nu = i) \\
&\quad = \mathbb{P}(X'_{\nu+1} \le l \mid \mathcal{G}_{\nu-1},\ X'_\nu = k,\ J^*_\nu = i,\ J^*_{\nu+1} = j) \\
&\qquad \times \mathbb{P}(J^*_{\nu+1} = j \mid \mathcal{G}_{\nu-1},\ X'_\nu = k,\ J^*_\nu = i) \\
&\quad = \sum_{m=0}^{l} p(k, i; m, j), \qquad (k, i) \times (l, j) \in \mathbb{F}^2,
\end{aligned}$$

which shows that $\{(X'_\nu, J^*_\nu);\ \nu \in \mathbb{Z}_+\}$ is a Markov chain with transition probability matrix $\boldsymbol{P}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The same argument holds for $\{(X''_\nu, J^*_\nu); \nu \in \mathbb{Z}_+\}$. We omit the details.

Lemma A.2. (Pathwise ordering by the blockwise dominance.) *Suppose that $\boldsymbol{P} \prec_d \widetilde{\boldsymbol{P}}$ and either $\boldsymbol{P} \in \mathsf{BM}_d$ or $\widetilde{\boldsymbol{P}} \in \mathsf{BM}_d$. Let X^*_0 and $\widetilde{X}^*_0$ denote nonnegative integer-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which are independent of both $\{U_\nu;\ \nu \in \mathbb{N}\}$ and $\{S_\nu;\ \nu \in \mathbb{N}\}$. Furthermore, let $X^*_\nu = F^{-1}(U_\nu \mid X^*_{\nu-1}, J^*_{\nu-1}, J^*_\nu)$ and $\widetilde{X}^*_\nu = \widetilde{F}^{-1}(U_\nu \mid \widetilde{X}^*_{\nu-1}, J^*_{\nu-1}, J^*_\nu)$ for $\nu \in \mathbb{N}$, where $F^{-1}(u \mid k, i, j)$, $0 < u < 1$, $k \in \mathbb{Z}_+, i, j \in \mathbb{D}$, is defined in (A.1) and $\widetilde{F}^{-1}(u \mid k, i, j)$, $0 < u < 1$, $k \in \mathbb{Z}_+, i, j \in \mathbb{D}$, is defined as*

$$\widetilde{F}^{-1}(u \mid k, i, j) = \inf\left\{l \in \mathbb{Z}_+;\ \sum_{m=0}^{l} \frac{\widetilde{p}(k, i; m, j)}{\psi(i, j)} \ge u\right\}.$$

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Under these conditions, $\{(X_\nu^*, J_\nu^*); \nu \in \mathbb{Z}_+\}$ *and* $\{(\widetilde{X}_\nu^*, J_\nu^*); \nu \in \mathbb{Z}_+\}$ *are Markov chains with transition probability matrices* $\boldsymbol{P}$ *and* $\widetilde{\boldsymbol{P}}$*, respectively, on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $X_\nu^* \le \widetilde{X}_\nu^*$ *for all* $\nu \in \mathbb{N}$ *if* $X_0^* \le \widetilde{X}_0^*$.

Proof. In Proposition 2.3(a) we showed that, for all $k \in \mathbb{Z}_+$ and $i, j \in \mathbb{D}$,

$$\psi(i, j) = \sum_{l=0}^{\infty} p(k, i; l, j) = \sum_{l=0}^{\infty} \widetilde{p}(k, i; l, j).$$

Therefore, following the proof of Lemma A.1, we can show that $\{(X_\nu^*, J_\nu^*); \nu \in \mathbb{Z}_+\}$ and $\{(\widetilde{X}_\nu^*, J_\nu^*); \nu \in \mathbb{Z}_+\}$ are Markov chains with transition probability matrices $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$, respectively, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Similarly, we can prove by induction that if $X_0^* \le \widetilde{X}_0^*$, then $X_\nu^* \le \widetilde{X}_\nu^*$ for all $\nu \in \mathbb{N}$. We omit the details.

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