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PAPER

Not every countable complete distributive lattice is sober

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Abstract

The study of the sobriety of Scott spaces has got a relatively long history in domain theory. Lawson and Hoffmann independently proved that the Scott space of every continuous directed complete poset (usually called domain) is sober. Johnstone constructed the first directed complete poset whose Scott space is non-sober. Soon after, Isbell gave a complete lattice with a non-sober Scott space. Based on Isbell's example, Xu, Xi, and Zhao showed that there is even a complete Heyting algebra whose Scott space is non-sober. Achim Jung then asked whether every countable complete lattice has a sober Scott space. The main aim of this paper is to answer Jung's problem by constructing a countable complete lattice whose Scott space is non-sober. This lattice is then modified to obtain a countable distributive complete lattice with a non-sober Scott space. In addition, we prove that the topology of the product space $\Sigma P \times \Sigma Q$ coincides with the Scott topology of the product poset $P \times Q$ if the set $Id(P)$ and $Id(Q)$ of all incremental ideals of posets P and Q are both countable. Based on this, it is deduced that a directed complete poset P has a sober Scott space, if $Id(P)$ is countable and ΣP is coherent and well filtered. In particular, every complete lattice L with $Id(L)$ countable has a sober Scott space.

Keywords: Countability; Sobriety; Scott topology; Complete lattices

1. Introduction

Sobriety is one of the earliest studied major properties of T_0 topological spaces. It has been used in the characterization of spectra spaces of commutative rings (Hochster 1969). In recent years, this property and some of its weaker forms have been extensively investigated from various different perspectives. The Scott topology is the most important topology in domain theory which bridges a strong link between topological and order structures. Lawson (1979) and Hoffmann (1981) proved independently that the Scott space of every domain (continuous directed complete poset) is sober. At the early time, it was an open problem whether the Scott space of every directed complete poset (dcpo, for short) is sober. Johnstone constructed the first counterexample to give a negative answer (Johnstone 1981). Soon, Isbell (1982) came up with a complete lattice whose Scott space is non-sober. However, Isbell's complete lattice is neither distributive nor countable.

A poset P will be called sober if its Scott space ΣP is sober. In Jung (2018), Achim Jung posed two problems. One of them is whether every distributive complete lattice is sober.

Using Isbell's complete lattice, Xu, Xi, and Zhao gave a negative answer to this problem (Xu et al. 2021).

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The second problem by Jung (also mentioned by Xu and Zhao in 2020) is the following one:

Problem 1.1. Is there a non-sober countable complete lattice?

In the current paper, we will give an answer to this problem. The main structure we are going to use is the poset $\mathbb{N}^{<\mathbb{N}}$ of all words (or, nonempty finite sequences) in the set $\mathbb{N}$ of all positive integers. In Section 2, we shall list some properties of $\mathbb{N}^{<\mathbb{N}}$ to be used. In Section 3, we construct a countable complete distributive lattice whose Scott space is non-sober; thus, we give an answer to Problem 1.1.

In Section 4, we prove some positive results on the sobriety of Scott spaces. First, we prove that the topology of the product space $\Sigma P \times \Sigma Q$ coincides with the Scott topology on the product poset $P \times Q$ if the set $Id(P)$ and $Id(Q)$ of all incremental ideals of posets P and Q are both countable. Based on this result, we deduce that a directed complete poset P is sober if $Id(P)$ is countable and the space ΣP is coherent and well filtered. In particular, every complete lattice L with $Id(L)$ countable is sober.

2. Preliminaries

In this section, we recall some basic definitions and results to be used later. For more details on them, we refer the reader to Gierz et al. (2003) and Goubault-Larrecq (2013).

Let P be a poset. A nonempty subset D of P is *directed* if every two elements of D have an upper bound in D. If D is also a lower set ($D = \downarrow D = \{x \in P : x \leq d \text{ for some } d \in D\}$), then D is called an *ideal*. A poset is called a *directed complete poset* (dcpo, for short) if its every directed subset has a supremum. A *complete lattice* is a poset in which every subset has a supremum and an infimum. A subset U of a poset P is *Scott open* if (i) it is an upper set ($U = \uparrow U = \{x \in P : u \leq x \text{ for some } u \in U\}$) and (ii) for every directed subset D of P with $\sup D$ existing and $\sup D \in U$, it follows that $D \cap U \neq \emptyset$. The complements of Scott open sets are called *Scott closed* sets. The collection of all Scott open subsets of P forms a topology on P, called the *Scott topology* of P, and is denoted by $\sigma(P)$. The collection of all Scott closed subsets of P is denoted by $\Gamma(P)$. The space $(P, \sigma(P))$ called the Scott space of P is written as ΣP.

For two elements x, y in a poset P, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P for which $\sup D$ exists, $y \leq \sup D$ implies $D \cap \uparrow x \neq \emptyset$. A poset P is *continuous* if for each $x \in P$, $\Downarrow x = \{y \in L : y \ll x\}$ is directed and $x = \sup \Downarrow x$. A continuous dcpo is usually called a domain.

An element x of P is *compact* if $x \ll x$. The set of all compact elements of P is denoted by $\mathrm{K}(P)$. A poset P is *algebraic* if for every $x \in P$, the set $\mathrm{K}(P) \cap \downarrow x$ is directed and $x = \sup\,(\mathrm{K}(P) \cap \downarrow x)$. For any compact element $x \in P$, $\uparrow x \in \sigma(P)$. Every algebraic poset is continuous. If L is a complete lattice such that $\mathrm{K}(L) = L$ (all elements are compact), then L is algebraic.

A subset K of a topological space X is *compact* if every open cover of K has a finite subcover. A set K of a topological space is called *saturated* if it is the intersection of its open neighborhood ($K = \uparrow K$ in its specialization order). The *saturation* $\mathrm{sat}A$ of a set A is the intersection of all its open neighborhoods.

Definition 2.1. (Gierz et al. 2003) (1) A topological space X is *sober* if it is T_0 and every irreducible closed subset of X is the closure of a (unique) point.

(2) A T_0 space X is *well filtered* if for each filter base $\mathcal{C}$ of compact saturated sets and each open set U with $\bigcap \mathcal{C} \subseteq U$, there is a $K \in \mathcal{C}$ with $K \subseteq U$.

(3) A space X is *coherent* if the intersection of any two compact saturated sets is again compact.

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Remark 2.2. (1) The Scott space of every continuous dcpo is sober.
(2) Every sober space is well filtered. A retraction of a sober space is sober.
(3) If X is well filtered and $\{F_i : i \in I\}$ is a filtered family of compact saturated subsets of X, then $\bigcap\{F_i : i \in I\}$ is a compact saturated set.

Lemma 2.3. *Assume that L is a complete lattice such that $K(L) = L$. Let*

$$L^* = \{\uparrow F : F \subseteq L \text{ is finite}\}.$$

Then $(L^*, \subseteq)$ is a distributive complete lattice.

Proof. Since L is a complete lattice with $\mathrm{K}(L) = L$, L is an algebraic lattice. Thus, ΣL is sober; hence, it is well filtered.

For any $\uparrow F, \uparrow G \in L^*$, it is easy to verify that in L^*,

$$\sup\{\uparrow F, \uparrow G\} = \uparrow(F \cup G), \inf\{\uparrow F, \uparrow G\} = \uparrow F \wedge \uparrow G = (\uparrow F) \cap (\uparrow G) = \uparrow\{x \vee y : x \in F, y \in G\}.$$

Thus, L^* is a lattice.

(1) L^* is a distributive lattice.

For any two elements $\uparrow F, \uparrow G$ in L^*, it is easy to see that $\inf\{\uparrow F, \uparrow G\} = \uparrow F \cap \uparrow G$ and $\sup\{\uparrow F, \uparrow G\} = \uparrow F \cup \uparrow G$. Hence, the finite sup (inf) in L^* is the set union (intersection), which means that L^* is a sublattice of the distributive powerset lattice $\mathcal{P}(L)$; thus, L^* is distributive.

(2) L^* is a complete lattice

Since L^* is a lattice and has a top element $\uparrow 0_L$, in order to prove L^* is a complete lattice, it remains to show that every filtered subset of L^* has an infimum. Let $\mathcal{D} = \{\uparrow F_i : i \in I\}$ be a filtered subset of L^*. Then each $\uparrow F_i$ is a compact saturated subset of ΣL. Hence, by Remarks 2.2(3), the intersection $A = \bigcap\{\uparrow F_i : i \in I\}$ is a compact saturated subset of ΣL. For each $x \in A$, x is compact, so $\uparrow x \in \sigma(L)$.

Now $A = \bigcup\{\uparrow x : x \in A\}$ (note that A is an upper set). As A is compact, there is a finite subset $G \subseteq A$, such that $A = \bigcup\{\uparrow y : y \in G\} = \uparrow G$, which is in L^*. Clearly $\inf \mathcal{D} = \uparrow G$.

It follows that $(L^*, \subseteq)$ is a complete lattice. □

An ideal I of a poset P is *incremental* if I is not a principal ideal ($I \neq \downarrow x$ for any $x \in P$). We use $Id(P)$ to denote the set of all incremental ideals of a poset P.

Proposition 2.4. *(Gierz et al. 2003, Corollary II-1.12) If L is a dcpo and a sup semilattice such that the sup operation is jointly Scott continuous, then ΣL is sober.*

Let $\mathbb{N}$ be the set of all nonnegative integers. Then, $\mathbb{N}$ is a poset with the ordinary order $\leq$ of numbers. Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all nonempty finite words (or, finite strings) over $\mathbb{N}$. The *prefix order* "$\leq$" on $\mathbb{N}^{<\mathbb{N}}$ is defined as follows:

For any $x = a_1 a_2 \cdots a_n, y = b_1 b_2 \cdots b_m$ in $\mathbb{N}^{<\mathbb{N}}$,

$$x \leq y \text{ if and only if } n \leq m \text{ and } a_i = b_i \text{ for all } 1 \leq i \leq n.$$

The poset $\mathbb{N}^{<\mathbb{N}}$ is countable and does not have infinite decreasing sequences. One can arrange all the elements from $\mathbb{N}^{<\mathbb{N}}$ in a sequence such that larger elements appear later. Thus, we can define a monotone injective function from $\mathbb{N}^{<\mathbb{N}}$ to $\mathbb{N}$.

We shall make use of the following result, for its proof, see Remark 6.1.

Proposition 2.5. *There is a monotone bijective function $f : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$.*

Remark 2.6. Consider the set $\mathbb{N}$. There exists a sequence $\{E_k\}_{k \in \mathbb{N}}$ of disjoint infinite subsets of $\mathbb{N}$.

For example, let $E_1 = \{2m : m \in \mathbb{N}\}$. Assume that we have defined disjoint subsets $E_k (k = 1, 2, \cdots, m)$ such that each E_i is an infinite subset of $\mathbb{N} - \bigcup\{E_j : j \leq i - 1\}$ and $\mathbb{N} - \bigcup\{E_j : j \leq i\}$ is

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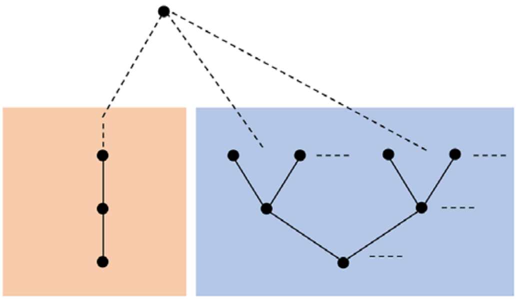

Figure 1. The basic gadget of P.

infinite. Then choose E_{m+1} to be an infinite subset of $\mathbb{N} - \bigcup\{E_i : i \leq m\}$ such that $\mathbb{N} - \bigcup\{E_i : i \leq m+1\}$ is infinite.

By induction, we can have a sequence $\{E_k\}_{k\in\mathbb{N}}$ of disjoint infinite subsets of $\mathbb{N}$.

Since $\mathbb{N} \times \mathbb{N}$ is a countable set, there exists a bijection $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Using h and the above remarks, we see that there is an injective function $i : \mathbb{N} \times \mathbb{N} \to \mathcal{P}(\mathbb{N})$ such that each $i(m, n)$ is an infinite set and $i(m_1, n_1) \cap i(m_2, n_2) = \emptyset$ whenever $(m_1, n_1) \neq (m_2, n_2)$. In addition, by subtracting $\{1, 2, \cdots, m\}$ from $i(m, n)$, we can guarantee that all numbers in $i(m, n)$ are strictly greater than m.

Now for each $(m, n) \in \mathbb{N} \times \mathbb{N}$, there is a monotone injection $h_{m,n} : \mathbb{N} \to i(m, n)$. Let $f_{m,n} = h_{m,n} \circ f$. Then, $f_{m,n}$ is a monotone injective function from $\mathbb{N}^{<\mathbb{N}}$ into $i(m, n)$.

3. A Countable Complete Distributive Lattice Whose Scott Space is Non-sober

We now construct a countable complete distributive lattice whose Scott space is non-sober.

Let $L = \mathbb{N} \cup \mathbb{N}^{<\mathbb{N}} \cup \{\top\}$ with the order $\leq$ such that both $\mathbb{N}^{<\mathbb{N}}$ and $\mathbb{N}$ are sub posets and T is the top element.

The poset L can be depicted as Fig. 1.

Next, let $P = \mathbb{N} \times L$.

Let $L_n = \{(n, x) \in P : x \in L\}$. In this section, for $s \in \mathbb{N}^{<\mathbb{N}}$ with length equaling 1 sometimes is considered as a natural number s. We first define the relations $<_1, <_2, <_3$ and $<_4$ on P as follows:

- $(n, x) <_1 (m, y)$ if $n = m$ and $x < y$ holds in L;
- $(n, x) <_2 (m, y)$ if $y = \top$, $x \in \mathbb{N}^{<\mathbb{N}}$ and there exists $k \in \mathbb{N}$ with $k > n$ such that $m \in i(n, k)$ and $m = f_{n,k}(x)$. (In other words, $(n, x) <_2 (f_{n,k}(x), \top)$ for all $n < k$).
- $(n, x) <_3 (m, y)$ if $y = \top$, $x \in \mathbb{N}$ and there exists $d \in \mathbb{N}$ with $d < n$ such that $m \in i(d, n)$ and $m = f_{d,n}(x)$. (In other words, $(n, x) <_3 (f_{d,n}(x), \top)$ for all $d < n$).
  Here $(n, x) \in \mathbb{N} \times \mathbb{N}$, and in the definition of $f_{d,n}(x)$, $x \in \mathbb{N}$ is taken as an element of $\mathbb{N}^{<\mathbb{N}}$ with length equaling 1.
- $(n, x) <_4 (m, y)$ if $y = \top$, $x \in \mathbb{N}$ and there exists $a, b \in \mathbb{N}$, $s \in \mathbb{N}$ with $a < b$ such that $f_{a,b}(s) = n$ and $f_{a,b}(s.x) = m$. (In other words, $(f_{a,b}(s), x) <_4 (f_{a,b}(s.x), \top)$ for all $(a, b) \in \mathbb{N} \times \mathbb{N}$ with $a < b$).

By the above definitions, it is clear that as subsets of $P \times P$, $<_1, <_2, <_3$ and $<_4$ are disjoint. Here we explain, in particular, why $<_3$ and $<_4$ are disjoint. In fact, suppose that $((f_{a,b}(s), x), (f_{a,b}(s.x), \top))$ is in $<_4$. Let $n = f_{a,b}(s)$. Then by the definition of $f_{a,b}$, $n > b$, hence, $f_{a,b}(s.x) \neq f_{d,n}(x)$ for any $d \in \mathbb{N}$ with $d < n$. Hence, $((f_{a,b}(s), x), (f_{a,b}(s.x), \top))$ is not in $<_3$.

Example 3.1. The following are some concrete examples illustrating the strict orders $<_i$ $(i = 1, 2, 3, 4)$.

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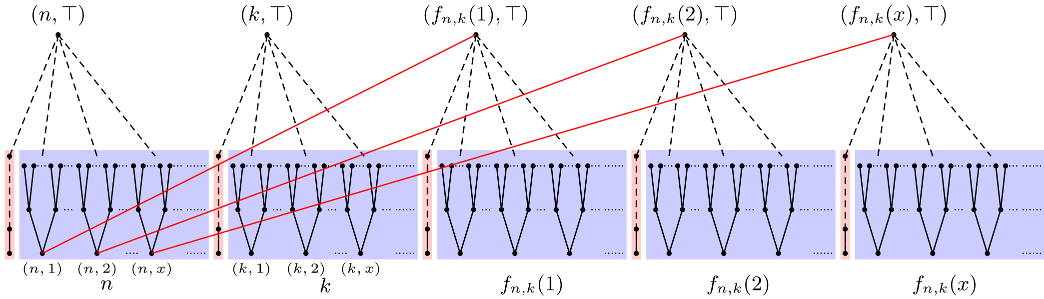

Figure 2. The strict order $<_3$.

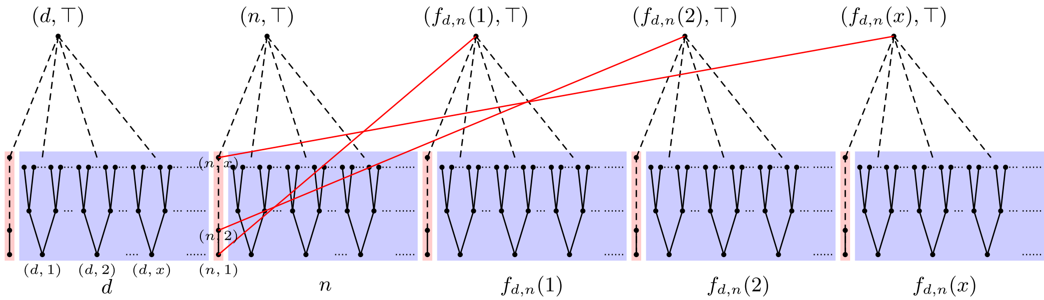

Figure 3. The strict order $<_3$.

(1) $(2, 3) <_1 (2, 3 \cdot 1)$.
(2) $(3, 12 \cdot 4) <_2 (f_{3,k}(12 \cdot 4), \top)$ for all $k > 3$.
(3) $(3, 5) <_3 (f_{d,3}(5), \top)$ for all $d < 3$. Here, the 5 in $f_{d,3}(5)$ is a member of $\mathbb{N}^{<\mathbb{N}}$ with length equaling 1.
(4) $(f_{2,3}(2 \cdot 6), 7) <_4 (f_{2,3}(2 \cdot 6 \cdot 7), \top)$.

Now let $\leq$ be the partial order on P generated by $<_1 \cup <_2 \cup <_3 \cup <_4$ (the smallest partial order relation containing all $<_i$ $(i = 1, 2, 3, 4)$). Note that the partial order $\leq$ coincides with $\leq_1$ $\cup(\leq_1 ; <_2) \cup (\leq_1 ; <_3) \cup (\leq_1 ; <_4)$, where $\leq_1$ is the reflexive closure of $<_1$ and $(R;S)$ denotes the composition of two relations R and S, defined by $x(R;S)y$ iff $\exists z : xRzSy$.

If $n = m$, then the strict order $<_1$ is depicted as in Fig. 1. For $n \neq m$, the strict orders $<_2$, $<_3$, $<_4$ of P are depicted in the following figures, respectively.

When $k > n$, the red lines in Fig. 2 illustrate three specific cases: $(n, 1) <_2 (f_{n,k}(1), \top)$, $(n, 2) <_2 (f_{n,k}(2), \top)$ and $(n, x) <_2 (f_{n,k}(x), \top)$.

The red lines in Fig. 3 illustrate the cases: $(n, 1) <_3 (f_{d,n}(1), \top)$, $(n, 2) <_3 (f_{d,n}(2), \top)$, and $(n, x) <_3 (f_{d,n}(x), \top)$.

The red lines in Fig. 4 illustrate the cases: $(f_{a,b}(1), 1) <_4 (f_{a,b}(1.1), \top)$ and $(f_{a,b}(1), x) <_4 (f_{a,b}(1.x), \top)$.

The red lines in Fig. 5 illustrate the cases: $(f_{a,b}(1), x) <_4 (f_{a,b}(1.x), \top)$ and $(f_{a,b}(1.x), y) <_4 (f_{a,b}(1.x.y), \top)$.

The red lines in Fig. 6 illustrate the cases: $(b, y) <_3 (f_{a,b}(y), \top)$; $(f_{a,b}(y), z) <_4 (f_{a,b}(y.z), \top)$; $(f_{a,b}(y.z), u) <_4 (f_{a,b}(y.z.u), \top)$.

In Fig. 7, the red lines are the same as Fig. 6 and the blue lines add the cases of $<_2$: $(a, y) <_2 (f_{a,b}(y), \top)$, $(a, y.z) <_2 (f_{a,b}(y.z), \top)$, and $(a, y.z.u) <_2 (f_{a,b}(y.z.u), \top)$.

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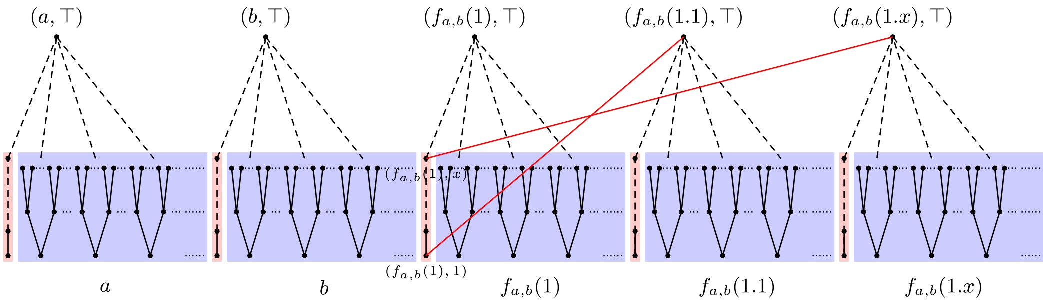

Figure 4. The strict order $<_4$.

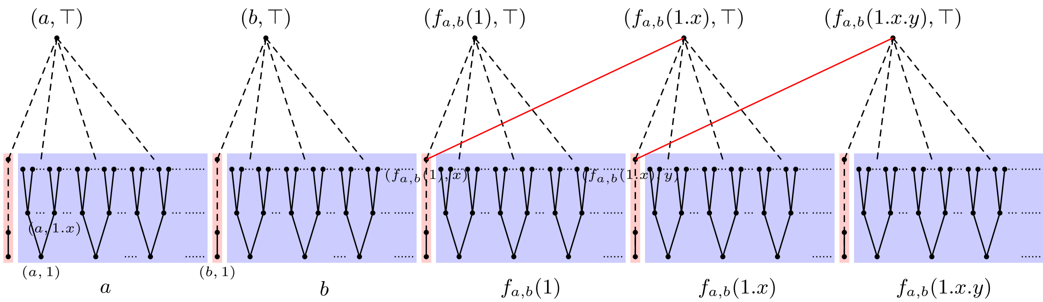

Figure 5. The strict order $<_4$.

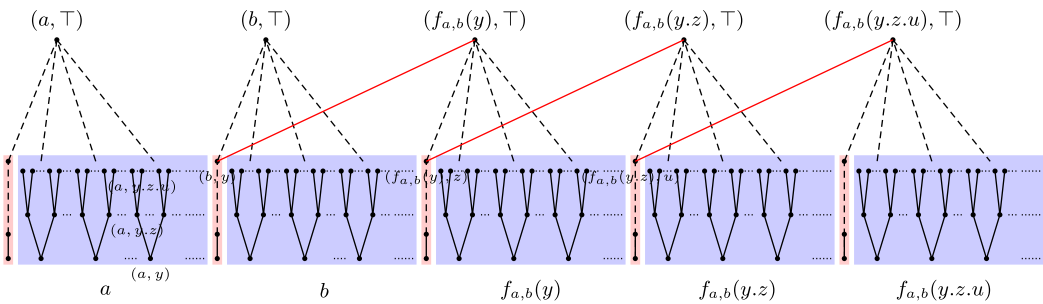

Figure 6. Assembling the strict orders $<_3$ and $<_4$.

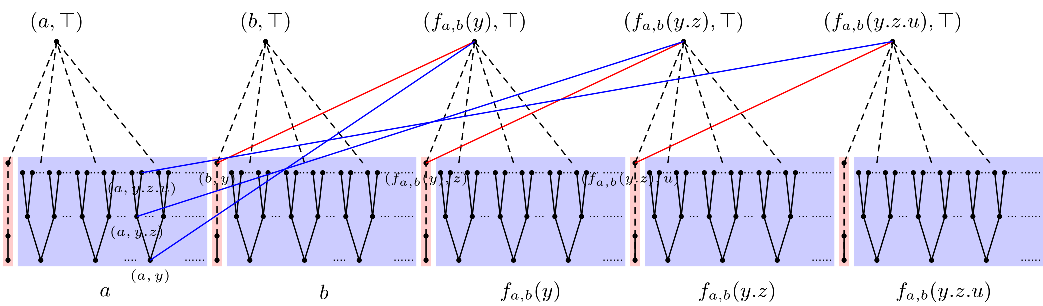

Figure 7. Assembling the strict orders $<_2$, $<_3$ and $<_4$.

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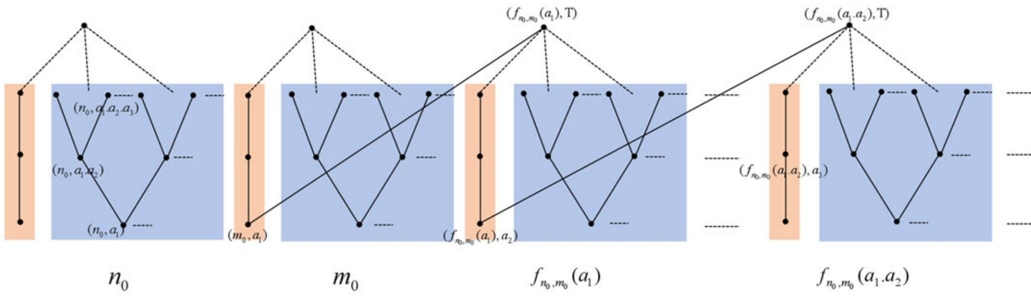

Figure 8. The proof of Lemma 3.2 ($n < m$).

Lemma 3.2. *P is an irreducible subset of ΣP.*

Proof. By the definition of irreducibility, it suffices to prove that $U \cap V \neq \emptyset$ for any two nonempty Scott open sets U, V of P.

Since U and V are nonempty Scott open sets, there exist $(n_0, \top) \in U$ and $(m_0, \top) \in V$.

If $n_0 = m_0$, then $(n_0, \top) \in U \cap V$.

Next, assume that $n_0 \neq m_0$. Without loss of generality, we just consider the case $n_0 < m_0$.

Since V is Scott open and $\bigvee\{(m_0, k) : k \in \mathbb{N}\} = (m_0, \top) \in V$, there exists $a_1 \in \mathbb{N}$ such that $(m_0, a_1) \in V$.

From the definition of $<_3$, it follows that $(m_0, a_1) <_3 (f_{n_0,m_0}(a_1), \top)$. Whence, $(f_{n_0,m_0}(a_1), \top) \in V$. By the similar reason for the existence of a_1, there is $a_2 \in \mathbb{N}$ such that $(f_{n_0,m_0}(a_1), a_2) \in V$.

By the definition of $<_4$, we have that $(f_{n_0,m_0}(a_1), a_2) <_4 (f_{n_0,m_0}(a_1.a_2), \top)$. It follows that $(f_{n_0,m_0}(a_1.a_2), \top) \in V$.

By induction on $\mathbb{N}$, for any $n \in \mathbb{N}$, there exists $(f_{n_0,m_0}(a_1.a_2.\cdots.a_n), \top) \in V$. Note that $\{(n_0, a_1.a_2.\cdots.a_k)) : k \in \mathbb{N}\}$ is an increasing sequence in P and

$$\bigvee\{(n_0, a_1.a_2.\cdots.a_k)) : k \in \mathbb{N}\} = (n_0, \top) \in U.$$

Thus, there exists $k \in \mathbb{N}$ such that $(n_0, a_1.\cdots.a_k) \in U$. By the definition of $<_2$, we have that $(n_0, a_1.\cdots.a_k) <_2 (f_{n_0,m_0}(a_1.\cdots.a_k), \top)$ (see Fig. 8 for the process of the proof).

Hence, $(f_{n_0,m_0}(a_1.\cdots.a_k), \top) \in U$, implying that $(f_{n_0,m_0}(a_1.\cdots.a_k), \top) \in U \cap V$. □

Lemma 3.3. *Let $M = \{\bigcap_{x\in E} \downarrow x : \emptyset \neq E \subseteq P\}$. Then $(M, \subseteq)$ is a bounded complete dcpo.*

Proof. If $\{A_i : i \in I\} \subseteq M$ has an upper bound in M, where $A_i = \bigcap_{x\in E_i} \downarrow x$ $(i \in I)$ with $E_i \subseteq P$, then there is $y_0 \in P$ such that $\bigcup\{A_i : i \in I\} \subseteq \downarrow y_0$. Hence, $\bigcap\{\downarrow y : \bigcup\{A_i : i \in I\} \subseteq \downarrow y\}$ is the supremum of $\{A_i : i \in I\}$ in M. It follows that M is bounded complete.

We now show that M is a dcpo. Let $B = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n < m\}$. In order to determine what the intersections of two principal ideals of P are, we first list all types of principal ideals $\downarrow x$ of P.

Type I : $\downarrow(m_0, s_0) = \{(m_0, s) : s \leq s_0\}$ for some $m_0 \in \mathbb{N}, s_0 \in \mathbb{N}^{<\mathbb{N}}$ (see Fig. 9 for Type I ideals).
Type II : $\downarrow(m_0, n_0) = \{(m_0, n) : n \leq n_0\}$ for some $m_0, n_0 \in \mathbb{N}$ (see Fig. 10 for Type II ideals).
Type III : $\downarrow(n_0, \top) = L_{n_0}$ for some $n_0 \in \mathbb{N} \setminus \bigcup_{(n,m)\in B} i(n, m)$ (see Fig. 11 for Type III ideals).
Type IV : $\downarrow(f_{m_0,n_0}(s_0), \top) = L_{f_{m_0,n_0}(s_0)} \cup \{(m_0, s_0)\} \cup \{(n_0, n) : n \leq s_0\}$ for some $(m_0, n_0) \in B$, $s_0 \in \mathbb{N}^{<\mathbb{N}}$ with $|s_0| = 1$ (see Fig. 12 for Type IV ideals).
Type V : $\downarrow(f_{m_0,n_0}(s_0), \top) = L_{f_{m_0,n_0}(s_0)} \cup \{(m_0, s) : s \leq s_0\} \cup \{(f_{m_0,n_0}(s_0^*), n) : n \leq n_0^*\}$ for some $(m_0, n_0) \in B$, $s_0 = s_0^*.n_0^* \in \mathbb{N}^{<\mathbb{N}}$ with $s_0^* \in \mathbb{N}^{<\mathbb{N}}$, $n_0^* \in \mathbb{N}$ (see Fig. 13 for Type V ideals).

The Types I, II, and III are easily understood.

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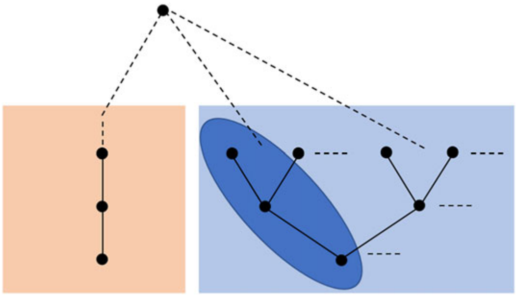

Figure 9. The Type I ideals.

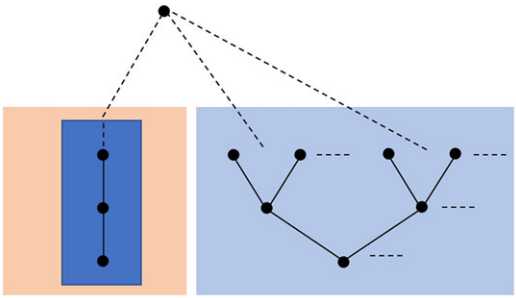

Figure 10. The Type II ideals.

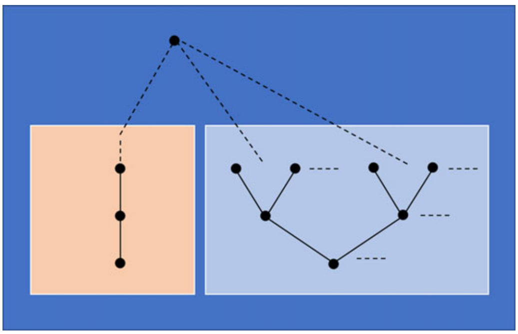

Figure 11. The Type III ideals.

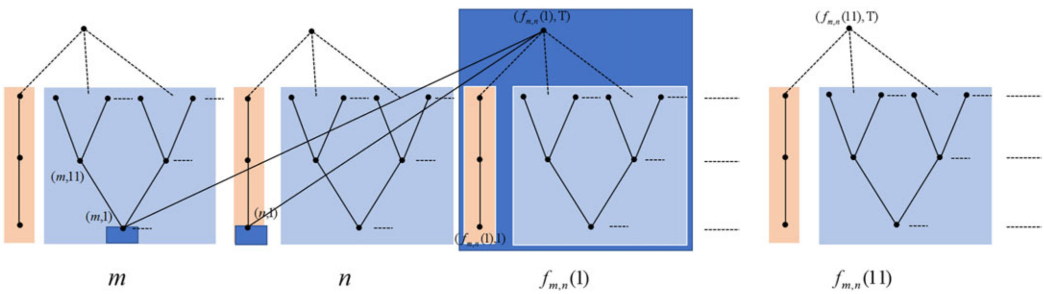

Figure 12. The Type IV ideals.

For the Type IV principle ideal $\downarrow(f_{m_0,n_0}(s_0), \top)$, clearly it contains the whole $L_{f_{m_0,n_0}(s_0)}$. Also $(m_0, s_0) <_2 (f_{m_0,n_0}(s_0), \top)$, where (m_0, s_0) is taken as an element in $\mathbb{N} \times \mathbb{N}^{<\mathbb{N}}$. Thus, $(m_0, s_0) \in \downarrow(f_{m_0,n_0}(s_0), \top)$. Note that this (m_0, s_0) is a minimal element of P. Next, $(n_0, s_0) <_3 (f_{m_0,n_0}(s_0), \top)$ where the s_0 in (n_0, s_0) is a member of $\mathbb{N}$, thus we have $\downarrow(n_0, s_0) = \{(n_0, n) : n \leq s_0\} \subseteq \downarrow(f_{m_0,n_0}(s_0), \top)$.

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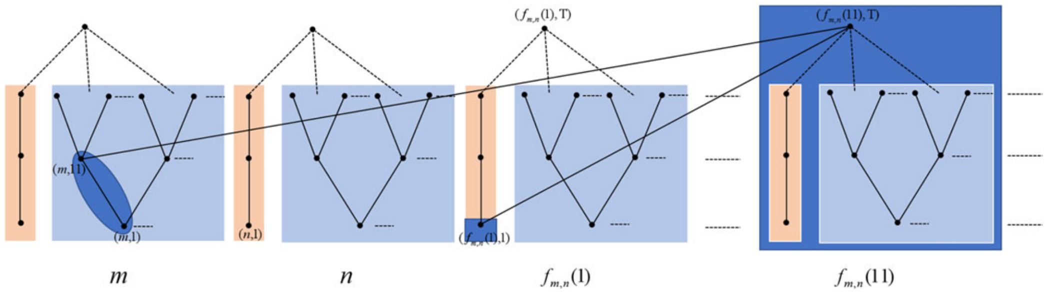

Figure 13. The Type V ideals.

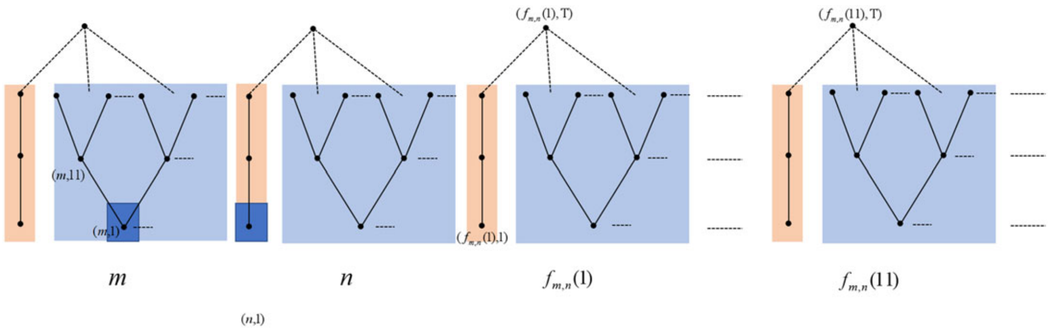

Figure 14. The Type I ∪ II[1] sets.

Now consider the Type V principle ideal $\downarrow(f_{m_0,n_0}(s_0), \top)$, where $s_0 \in \mathbb{N}^{<\mathbb{N}}$. Trivially, it contains the whole $L_{f_{m_0,n_0}(s_0)}$. Next, $(m_0, s_0) <_2 (f_{m_0,n_0}(s_0), \top)$, thus $\downarrow(m_0, s_0) = \{(m_0, s) : s \leq s_0\} \subseteq \downarrow(f_{m_0,n_0}(s_0), \top)$. Furthermore, if $s_0 = s_0^*.n_0^*$ with $n_0^* \in \mathbb{N}$, then $(f_{m_0,n_0}(s_0^*), n_0^*) <_4 (f_{m_0,n_0}(s_0^*.n_0^*), \top) = (f_{m_0,n_0}(s_0), \top)$, hence $\downarrow(f_{m_0,n_0}(s_0^*), n_0^*) = \{(f_{m_0,n_0}(s_0^*), n) : n \leq n_0^*\} \subseteq \downarrow(f_{m_0,n_0}(s_0), \top)$. And these are all the elements in $\downarrow(f_{m_0,n_0}(s_0), \top)$.

These types of principle ideas are depicted as below (as the blue regions).

We now list all the subsets of P which are the intersections of two principal ideals in the following table.

	Type I	Type II	Type III	Type IV	Type V
Type I	I/∅	∅	I/∅	I/∅	I/∅
Type II		II/∅	II/∅	II/∅	II/∅
Type III			III/∅	I/II/∅	I/∅
Type IV				I/II/IV/I∪II^1/∅	I/II/I∪II^1/I∪II^2/∅
Type V					I/II/V/I∪II^2/∅

In the aforementioned table, Type I∪II^1 sets are of the form $\{(m_0, s_0)\} \cup \{(n_0, n) : n \leq k_0\}$ for some $(m_0, n_0) \in B$, $s_0 \in \mathbb{N}^{<\mathbb{N}}$, $k_0 \in \mathbb{N}$ with $|s_0| = 1$, $k_0 \leq s_0$ (see Fig. 14 for Type I∪II^1 ideals).

Type I∪II^2 sets are of the form $\{(m_0, s) \mid s \leq s_0\} \cup \{(f_{m_0,n_0}(s_0), n) : n \leq k_0\}$ for some $(m_0, n_0) \in B$, $s_0 \in \mathbb{N}^{<\mathbb{N}}$, $k_0 \in \mathbb{N}$ (see Fig. 15 for Type I∪II^2 ideals).

The two new types of subsets are depicted in Fig. 14, as the blue regions.

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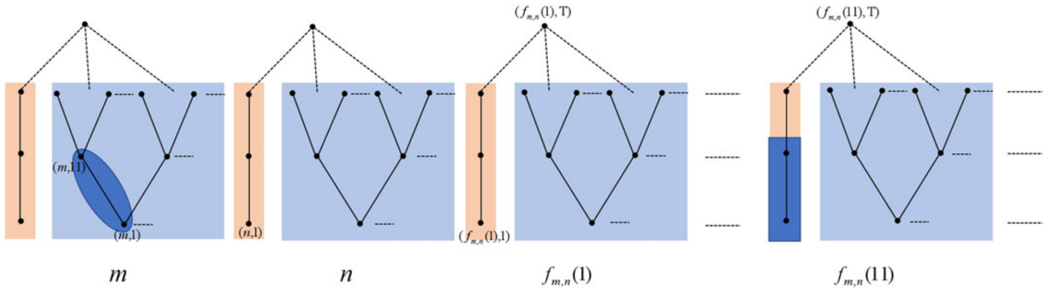

Figure 15. The Type $\mathrm{I}\cup\mathrm{II}^2$ sets.

For the intersections with Type I, Type II, and Type III principle ideals, the results are easily seen. We now explain the intersections of two Type IV principle ideas, one Type IV and one Type V principle ideals and two Type V principle ideals.

Intersections of two Type IV principle ideals

The corresponding cell for intersections of two Type IV ideals is indicated as $\mathrm{I/II/IV/I}\cup\mathrm{II}^1$, $\emptyset$, meaning that intersection can be a Type I ideal, a Type II ideal, a Type IV ideal, a Type $\mathrm{I}\cup\mathrm{II}^1$ set, or the empty set.

Let I_1 and I_2 be two Type IV principle ideals, where $I_1 = \downarrow(f_{m_1,n_1}(s_1), \top) = L_{f_{m_1,n_1}(s_1)} \cup \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq s_1\}$ for some $(m_1, n_1) \in B, s_1 \in \mathbb{N}^{<\mathbb{N}}$ with $|s_1| = 1$, and $I_2 = \downarrow(f_{m_2,n_2}(s_2), \top) = L_{f_{m_2,n_2}(s_2)} \cup \{(m_2, s_2)\} \cup \{(n_2, n) : n \leq s_2\}$ for some $(m_2, n_2) \in B, s_2 \in \mathbb{N}^{<\mathbb{N}}$ with $|s_2| = 1$.

We prove this by considering the following different cases for $f_{m_2,n_2}(s_2)$.

(1) $f_{m_2,n_2}(s_2) < m_1$.
In this case, as $m_2 < n_2 < f_{m_2,n_2}(s_2) < m_1 < n_1 < f_{m_1,n_1}(s_1)$, it follows that $I_1 \cap I_2 = \emptyset$.

(2) $f_{m_2,n_2}(s_2) = m_1$.
Then $m_2 < n_2 < f_{m_2,n_2}(s_2) = m_1 < n_1 < f_{m_1,n_1}$, so $\{(m_1, s_1)\} \subseteq I_2$ and $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(3) $m_1 < f_{m_2,n_2}(s_2) < n_1$.
Let $m_2 \neq m_1$. Then $I_1 \cap I_2 = \emptyset$.
Let $m_2 = m_1$.
If $s_1 \neq s_2$, then $I_1 \cap I_2 = \emptyset$. Otherwise, $I_1 \cap I_2 = \{(m_1, s_1)\} = \{(m_2, s_2)\}$, which is a Type I ideal.

(4) $f_{m_2,n_2}(s_2) = n_1$.
Then, as $m_2 < n_2 < f_{m_2,n_2}(s_2) = n_1$, we have $m_2 < n_2 < n_1 < f_{m_1,n_1}(s_1)$.
If $m_2 \neq m_1$, then $I_1 \cap I_2 = \{(n_1, n) : n \leq s_1\}$, which is a Type II ideal.
If $m_2 = m_1$ and $s_1 = s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq s_1\}$, which is a Type $\mathrm{I}\cup\mathrm{II}^1$ set.
If $m_2 = m_1$ and $s_1 \neq s_2$, then $I_1 \cap I_2 = \{(n_1, n) : n \leq s_1\}$, which is a Type II ideal.

(5) $n_1 < f_{m_2,n_2}(s_2) < f_{m_1,n_1}(s_1)$.
If $m_1 \neq m_2, n_1 \neq n_2$, then $I_1 \cap I_2 = \emptyset$.
If $m_1 = m_2, n_1 \neq n_2$ and $s_1 \neq s_2$.
Then $I_1 \cap I_2 = \emptyset$.
If $m_1 = m_2, n_1 \neq n_2$ and $s_1 = s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.
Now consider the case $m_1 = m_2, n_1 = n_2$.

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By $f_{m_2,n_2}(s_2) < f_{m_1,n_1}(s_1)$, we have $s_1 \neq s_2$. Hence, $I_1 \cap I_2 = \{(n_1, n) : n \leq min\{s_1, s_2\}\}$.
For the case $m_1 \neq m_2$, $n_1 = n_2$, we have $I_1 \cap I_2 = \{(n_1, n) : n \leq \min\{s_1, s_2\}\}$, which is a Type II ideal.

(6) $f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1)$
Then $(m_1, n_1) = (m_2, n_2)$, $s_1 = s_2$ by the property of i and f_{m_1,n_1}, thus $I_1 \cap I_2 = I_1$, which is a type IV ideal.

(7) $f_{m_1,n_1}(s_1) < f_{m_2,n_2}(s_2)$
By interchanging m_1 and m_2, n_1 and m_2 in the cases (1)–(5), we deduce again that $I_1 \cap I_2$ is of Type I/II/IV/$\mathrm{I} \cup \mathrm{II}^1$/∅.

Intersections of one Type IV and one Type V principle ideals

Let I_1 be a Type IV ideal and I_2 a Type V ideal.

Specifically, $I_1 = \downarrow(f_{m_1,n_1}(s_1), \top) = L_{f_{m_1,n_1}(s_1)} \cup \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq s_1\}$ for some $(m_1, n_1) \in B$, $s_1 \in \mathbb{N}^{<\mathbb{N}}$ with $|s_1| = 1$, and $I_2 = \downarrow(f_{m_2,n_2}(s_2), \top) = L_{f_{m_2,n_2}(s_2)} \cup \{(m_2, s) : s \leq s_2\} \cup \{(f_{m_2,n_2}(s_2^*), n) : n \leq n_2^*\}$ for some $(m_2, n_2) \in B$, $s_2 = s_2^*.n_2^* \in \mathbb{N}^{<\mathbb{N}}$ with $s_2^* \in \mathbb{N}^{<\mathbb{N}}$, $n_2^* \in \mathbb{N}$.

Note that in this case, $s_1 \neq s_2$.

We prove this by considering the following cases for $f_{m_2,n_2}(s_2)$.

(1) $f_{m_2,n_2}(s_2) < m_1$.
Then $m_2 < n_2 < f_{m_2,n_2}(s_2) < m_1 < n_1 < f_{m_1,n_1}(s_1)$ and
$f_{m_2,n_2}(s_2^*) < f_{m_2,n_2}(s_2)$.
Hence, $I_1 \cap I_2 = \emptyset$.

(2) $f_{m_2,n_2}(s_2) = m_1$.
Then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(3) $m_1 < f_{m_2,n_2}(s_2) < n_1$
- (3.1) $m_2 \neq m_1$
  $I_1 \cap I_2 = \emptyset$.
- (3.2) $m_2 = m_1$
  If $s_1 \not\leq s_2$, then $I_1 \cap I_2 = \emptyset$.
  If $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(4) $f_{m_2,n_2}(s_2) = n_1$
- (4.1) $m_2 \neq m_1$
  $I_1 \cap I_2 = \{(n_1, n) : n \leq s_1\}$, which is a Type II ideal.
- (4.2) $m_2 = m_1$
  If $s_1 \not\leq s_2$, then we have the same result as in case $m_2 \neq m_1$.
  If $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\} \cup \{(n_1, s) : s \leq s_1\}$, which is a Type $\mathrm{I}\cup\mathrm{II}^1$ set.

(5) $n_1 < f_{m_2,n_2}(s_2) < f_{m_1,n_1}(s_1)$
- (5.1) $m_1 \neq m_2$, $n_1 \neq f_{m_2,n_2}(s_2^*)$
  $I_1 \cap I_2 = \emptyset$.
- (5.2) $m_1 = m_2$, $n_1 \neq f_{m_2,n_2}(s_2^*)$
  If $s_1 \not\leq s_2$, $I_1 \cap I_2 = \emptyset$.
  If $s_1 \leq s_2$, $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.
- (5.3) $m_1 = m_2$, $n_1 = f_{m_2,n_2}(s_2^*)$
  If $s_1 \not\leq s_2$, then $I_1 \cap I_2 = \{(n_1, n) : n \leq \min\{s_1, n_2^*\}\}$, which is a Type II ideal. Otherwise
  If $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq \min\{s_1, n_2^*\}\}$, which is a Type $\mathrm{I} \cup \mathrm{II}^1$ set.
- (5.4) $m_1 \neq m_2$, $n_1 = f_{m_2,n_2}(s_2^*)$
  $I_1 \cap I_2 = \{(n_1, n) : n \leq \min\{s_1, n_2^*\}\}$, which is a Type II ideal.

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(6) $f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1)$
Then $(m_1, n_1) = (m_2, n_2), s_1 = s_2$ by the property of i and f_{m_1,n_1}, which contradicts the assumption that $s_1 \neq s_2$. Thus, this case does not exist.

(7) $f_{m_2,n_2}(s_2) > f_{m_1,n_1}(s_1)$

(7.1) $f_{m_1,n_1}(s_1) < m_2$
Then $I_1 \cap I_2 = \emptyset$.

(7.2) $f_{m_1,n_1}(s_1) = m_2$
Then $I_1 \cap I_2 = \{(m_2, s) : s \leq s_2\}$, which is a Type I ideal.

(7.3) $m_2 < f_{m_1,n_1}(s_1) < f_{m_2,n_2}(s_2^*)$
If $m_2 \neq m_1$, $I_1 \cap I_2 = \emptyset$.
If $m_2 = m_1$ and $s_1 \not\leq s_2$, then $I_1 \cap I_2 = \emptyset$.
If $m_2 = m_1$ and $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(7.4) $f_{m_1,n_1}(s_1) = f_{m_2,n_2}(s_2^*)$
Then $(m_1, n_1) = (m_2, n_2), s_2^* = s_1$ by the property of functions i and f_{m_1,n_1}. This implies that $I_1 \cap I_2 = \{(m_1, s_1)\} \cup \{(f_{m_1,n_1}(s_1), n) : n \leq n_2^*\}$ which is a Type $\mathrm{I} \cup \mathrm{II}^2$ set.

(7.5) $f_{m_2,n_2}(s_2^*) < f_{m_1,n_1}(s_1) < f_{m_2,n_2}(s_2)$

(7.5.1) $m_1 \neq m_2, n_1 \neq f_{m_2,n_2}(s_2^*)$, then $I_1 \cap I_2 = \emptyset$.

(7.5.2) $m_1 = m_2, n_1 \neq f_{m_2,n_2}(s_2^*)$
If $s_1 \not\leq s_2$, then $I_1 \cap I_2 = \emptyset$.
If $s_1 \leq s_2$, $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(7.5.3) $m_1 = m_2, n_1 = f_{m_2,n_2}(s_2^*)$.
If $s_1 \not\leq s_2$, then $I_1 \cap I_2 = \{(n_1, n) : n \leq \min\{s_1, n_2^*\}\}$, which is a Type II ideal.
If $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_2, s_1)\} \cup \{(f_{m_2,n_2}(s_2^*), n) : n \leq \min\{s_1, n_2^*\}\}$, which is a Type $\mathrm{I} \cup \mathrm{II}^1$ set.

(7.5.4) $m_1 \neq m_2, n_1 = f_{m_2,n_2}(s_2^*)$
$I_1 \cap I_2 = \{(n_1, n) : n \leq \min\{s_1, n_2^*\}\}$, which is a Type II ideal.

These cover all possible cases and we are done.

Intersections of two Type V principle ideals

Next, we show that the intersection of two Type V ideals has the form I/II/V/I∪II^2/∅, that is, either a Type I ideal, a Type II ideal, a Type V ideal, and a Type $\mathrm{I} \cup \mathrm{II}^2$ set or the empty set.

Let I_1, I_2 be two Type V ideals, where
$I_1 = \downarrow(f_{m_1,n_1}(s_1), \top) = L_{f_{m_1,n_1}(s_1)} \cup \{(m_1, s) : s \leq s_1\} \cup \{(f_{m_1,n_1}(s_1^*), n) : n \leq n_1^*\}$ for some $(m_1, n_1) \in B$, $s_1 = s_1^*.n_1^* \in \mathbb{N}^{<\mathbb{N}}$ with $s_1^* \in \mathbb{N}^{<\mathbb{N}}$, $n_1^* \in \mathbb{N}$, and
$I_2 = \downarrow(f_{m_2,n_2}(s_2), \top) = L_{f_{m_2,n_2}(s_2)} \cup \{(m_2, s) : s \leq s_2\} \cup \{(f_{m_2,n_2}(s_2^*), n) : n \leq n_2^*\}$ for some $(m_2, n_2) \in B$, $s_2 = s_2^*.n_2^* \in \mathbb{N}^{<\mathbb{N}}$ with $s_2^* \in \mathbb{N}^{<\mathbb{N}}$, $n_2^* \in \mathbb{N}$.

We prove for the case $f_{m_2,n_2}(s_2) \leq f_{m_1,n_1}(s_1)$. The proof for the case $f_{m_1,n_1}(s_1) \leq f_{m_2,n_2}(s_2)$ is similar.

(1) $f_{m_2,n_2}(s_2) < m_1$
Then $I_1 \cap I_2 = \emptyset$.

(2) $f_{m_2,n_2}(s_2) = m_1$
Then $I_1 \cap I_2 = \{(m_1, s) : s \leq s_1\}$, which is a Type I ideal.

(3) $m_1 < f_{m_2,n_2}(s_2) < f_{m_1,n_1}(s_1^*)$

(3.1) $m_2 \neq m_1$
Then $I_1 \cap I_2 = \emptyset$.

(3.2) $m_2 = m_1$
If $\downarrow s_1 \cap \downarrow s_2 = \emptyset$, then $I_1 \cap I_2 = \emptyset$.
If $\downarrow s_1 \cap \downarrow s_2 = \emptyset$, then $I_1 \cap I_2 = \{(m_1, s) : s \leq \inf\{s_1, s_2\}$, which is a Type I ideal.

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(4) $f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1^*)$
Then $(m_1, n_1) = (m_2, n_2)$, $s_1^* = s_2$ by the property of i and f_{m_1,n_1}.
Note that $s_2 < s_1$ in this case. We have

$$I_1 \cap I_2 = \{(m_2, s) : s \leq s_2)\} \cup \{(f_{m_2,n_2}(s_2), n) : n \leq n_1^*\}$$

which is a Type I ∪ II^2 set.

(5) $f_{m_1,n_1}(s_1^*) < f_{m_2,n_2}(s_2) < f_{m_1,n_1}(s_1)$

(5.1) $m_1 \neq m_2, f_{m_1,n_1}(s_1^*) \neq f_{m_2,n_2}(s_2^*)$
Then $I_1 \cap I_2 = \emptyset$.

(5.2) $m_1 = m_2, f_{m_1,n_1}(s_1^*) \neq f_{m_2,n_2}(s_2^*)$
If $\downarrow s_1 \cap \downarrow s_2 = \emptyset$, then $I_1 \cap I_2 = \emptyset$.
If $\downarrow s_1 \cap \downarrow s_2 \neq \emptyset$, then $I_1 \cap I_2 = \{(m_1, s) : s \leq \inf\{s_1, s_2\}\}$, which is a Type I ideal.

(5.3) $m_1 = m_2, f_{m_1,n_1}(s_1^*) = f_{m_2,n_2}(s_2^*)$
Then $(m_1, n_1) = (m_2, n_2)$, $s_1^* = s_2^*$ by the property of i and f_{m_1,n_1}. Thus, $I_1 \cap I_2 = \{(m_1, s) : s \leq s_1^*)\} \cup \{(f_{m_1,n_1}(s_1^*), n) : n \leq \min\{n_1^*, n_2^*\}\}$, which is a Type I ∪ II^2 set.

(5.4) $m_1 \neq m_2, f_{m_1,n_1}(s_1^*) = f_{m_2,n_2}(s_2^*)$
The second equality implies $m_1 = m_2$, contradicting the first inequality $m_1 \neq m_2$. Thus, this case does not exist.

(6) $f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1)$
Then $(m_1, n_1) = (m_2, n_2)$, $s_1 = s_2$ by the property of i and f_{m_1,n_1}. This reveals that $I_1 \cap I_2 = I_1$, which is a type V ideal.

This covers all possible cases, and we have confirmed that the intersections of two Type V ideals can be a Type I ideal, a Type II ideal, a Type V ideal, Type I ∪ II^2 set, or the empty set.

Next, we consider the intersections of Type I∪II^1 (Type I∪II^2, resp.) sets with Type I∪II^1, I∪II^2 sets, Type I, Type II, Type III, Type IV, and Type V ideals.

The results are shown in the following table.

	Type I	Type II	Type III	Type IV	Type V	Type I∪II^1	Type I∪II^2
Type I∪II^1	I/∅	II/∅	I/II/∅	I/II/I∪II^1/∅	I/II/I∪II^1/∅	I/II/I∪II^1/∅	I/II/I∪II^1/∅
Type I∪II^2	I/∅	II/∅	I/II/∅	I/II/I∪II^1/∅	I/II/I∪II^2/∅		I/II/I∪II^2/∅

We now explain the aforementioned table.

We only consider the following nontrivial cases.

Intersections of a Type IV ideal and a Type I∪II^1 set

Let $I_1 = \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq k_1\}$ for some $(m_1, n_1) \in B, s_1 \in \mathbb{N}^{<\mathbb{N}}, k_1 \in \mathbb{N}$ with $|s_1| = 1$, $k_1 \leq s_1$, and
$I_2 = \downarrow(f_{m_2,n_2}(s_2), \top) = L_{f_{m_2,n_2}(s_2)} \cup \{(m_2, s_2)\} \cup \{(n_2, n) : n \leq s_2\}$ for some $(m_2, n_2) \in B, s_2 \in \mathbb{N}^{<\mathbb{N}}$ with $|s_2| = 1$.

Then I_1 is a Type I∪II^1 set and I_2 is a type IV ideal.
We prove this by considering the following cases for $f_{m_2,n_2}(s_2)$.

(1) $f_{m_2,n_2}(s_2) < m_1$
Then $I_1 \cap I_2 = \emptyset$.

(2) $f_{m_2,n_2}(s_2) = m_1$
Then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

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(3) $m_1 < f_{m_2,n_2}(s_2) < n_1$
  (3.1) $m_2 \neq m_1$
    Then $I_1 \cap I_2 = \emptyset$.
  (3.2) $m_2 = m_1$
    If $s_2 \neq s_1$, then $I_1 \cap I_2 = \emptyset$.
    If $s_2 = s_1$, then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(4) $f_{m_2,n_2}(s_2) = n_1$
  (4.1) $m_2 \neq m_1$
    Then $I_1 \cap I_2 = \{(n_1, n) : n \leq k_1\}$, which is a Type II ideal.
  (4.2) $m_2 = m_1$
    If $s_2 \neq s_1$, then we have the same result as in case $m_2 \neq m_1$.
    If $s_2 = s_1$, then $I_1 \cap I_2 = I_1$, which is a Type I∪II^1 set.

(5) $f_{m_2,n_2}(s_2) > n_1$
  If $m_2 \neq m_1$, $n_2 \neq n_1$, then $I_1 \cap I_2 = \emptyset$.
  If $m_2 \neq m_1$, $n_2 = n_1$, then $I_1 \cap I_2 = \{(n_2, n) : n \leq \min\{k_1, s_2\}\}$, which is a Type II ideal.
  If $m_2 = m_1$, $n_2 \neq n_1$, and $s_2 \neq s_1$, then $I_1 \cap I_2 = \emptyset$.
  If $m_2 = m_1$, $n_2 \neq n_1$, and $s_2 = s_1$, then $I_1 \cap I_2 = \{(m_2, s_2)\}$, which is a Type I ideal.
  If $m_2 = m_1$, $n_2 = n_1$, and $s_2 \neq s_1$, then $I_1 \cap I_2 = \{(n_2, n) : n \leq \min\{k_1, s_2\}\}$, which is a Type II ideal.
  If $m_2 = m_1$, $n_2 = n_1$, and $s_2 = s_1$, then $I_1 \cap I_2 = \{(m_2, s_2)\} \cup \{(n_2, n) : n \leq \min\{k_1, s_2\}\}$, which is a Type I∪II^1 set.

These cover all possible cases and we are done.

Intersections of a Type V ideal and a Type I∪II^1 set

Let $I_1 = \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq k_1\}$ for some $(m_1, n_1) \in B, s_1 \in \mathbb{N}^{<\mathbb{N}}, k_1 \in \mathbb{N}$ with $|s_1| = 1$, $k_1 \leq s_1$, and
$I_2 = \downarrow(f_{m_2,n_2}(s_2), \top) = L_{f_{m_2,n_2}(s_2)} \cup \{(m_2, s) : s \leq s_2\} \cup \{(f_{m_2,n_2}(s_2^*), n) : n \leq n_2^*\}$ for some $(m_2, n_2) \in B, s_2 = s_2^*.n_2^* \in \mathbb{N}^{<\mathbb{N}}$ with $s_2^* \in \mathbb{N}^{<\mathbb{N}}, n_2^* \in \mathbb{N}$.

Then I_1 is a Type I∪II^1 set and I_2 is a type V ideal.
Note that in this case, $s_1 \neq s_2$ because $|s_1| = 1$ and $|s_2| > 1$.
We prove this by considering the following cases for $f_{m_2,n_2}(s_2)$.

(1) $f_{m_2,n_2}(s_2) < m_1$
  Then $I_1 \cap I_2 = \emptyset$.

(2) $f_{m_2,n_2}(s_2) = m_1$
  Then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(3) $m_1 < f_{m_2,n_2}(s_2) < n_1$
  (3.1) $m_2 \neq m_1$
    Then $I_1 \cap I_2 = \emptyset$.
  (3.2) $m_2 = m_1$
    If $s_1 \nleq s_2$, then $I_1 \cap I_2 = \emptyset$.
    If $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

(4) $f_{m_2,n_2}(s_2) = n_1$
  (4.1) $m_2 \neq m_1$
    Then $I_1 \cap I_2 = \{(n_1, n) : n \leq k_1\}$, which is a Type II ideal.
  (4.2) $m_2 = m_1$
    If $s_1 \nleq s_2$, then we have the same result as in case $m_2 \neq m_1$.
    If $s_1 \leq s_2$, then $I_1 \cap I_2 = I_1$, which is a Type I∪II^1 set.

(5) $f_{m_2,n_2}(s_2) > n_1$

(5.1) $m_2 \neq m_1$

If $f_{m_2,n_2}(s_2^*) \neq n_1$, then $I_1 \cap I_2 = \emptyset$.

If $f_{m_2,n_2}(s_2^*) = n_1$, then $I_1 \cap I_2 = \{(f_{m_2,n_2}(s_2^*), n) : n \leq \min\{k_1, n_2^*\}\}$, which is a Type II ideal.

(5.2) $m_2 = m_1$

If $f_{m_2,n_2}(s_2^*) \neq n_1$ and $s_1 \nleq s_2$, then $I_1 \cap I_2 = \emptyset$.

If $f_{m_2,n_2}(s_2^*) \neq n_1$ and $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

$f_{m_2,n_2}(s_2^*) = n_1$ and $s_1 \nleq s_2$, then $I_1 \cap I_2 = \{(f_{m_2,n_2}(s_2^*), n) : n \leq \min\{k_1, n_2^*\}\}$, which is a Type II ideal.

If $f_{m_2,n_2}(s_2^*) = n_1$ and $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\} \cup \{(f_{m_2,n_2}(s_2^*), n) : n \leq \min\{k_1, n_2^*\}\}$, which is a Type I∪II^1 set.

These cover all possible cases and we are done.

Intersections of two Type I∪II^1 sets

Let $I_1 = \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq k_1\}$ for some $(m_1, n_1) \in B, s_1 \in \mathbb{N}^{<\mathbb{N}}, k_1 \in \mathbb{N}$ with $|s_1| = 1$, $k_1 \leq s_1$, and
$I_2 = \{(m_2, s_2)\} \cup \{(n_2, n) : n \leq k_2\}$ for some $(m_2, n_2) \in B, s_2 \in \mathbb{N}^{<\mathbb{N}}, k_2 \in \mathbb{N}$ with $|s_2| = 1, k_2 \leq s_2$.

Then I_1, I_2 are both Type I∪II^1 sets.
We prove this by considering the following cases.

(1) $m_2 \neq m_1, n_2 \neq n_1$
Then $I_1 \cap I_2 = \emptyset$.

(2) $m_2 \neq m_1, n_2 = n_1$
Then $I_1 \cap I_2 = \{(n_2, n) : n \leq \min\{k_1, k_2\}\}$, which is a Type II ideal.

(3) $m_2 = m_1$, $n_2 \neq n_1$, and $s_2 \neq s_1$
Then $I_1 \cap I_2 = \emptyset$.

(4) $m_2 = m_1$, $n_2 \neq n_1$, and $s_2 = s_1$
Then $I_1 \cap I_2 = \{(m_2, s_2)\}$, which is a Type I ideal.

(5) $m_2 = m_1$, $n_2 = n_1$, and $s_2 \neq s_1$
then $I_1 \cap I_2 = \{(n_2, n) : n \leq \min\{k_1, k_2\}\}$, which is a Type II ideal.

(6) $m_2 = m_1$, $n_2 = n_1$, and $s_2 = s_1$
then $I_1 \cap I_2 = \{(m_2, s_2)\} \cup \{(n_2, n) : n \leq \min\{k_1, k_2\}\}$, which is a Type I∪II^1 set.
These cover all possible cases and we are done.

Intersections of a Type I∪II^1 set and a I∪II^2 set

Let $I_1 = \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq k_1\}$ for some $(m_1, n_1) \in B, s_1 \in \mathbb{N}^{<\mathbb{N}}, k_1 \in \mathbb{N}$ with $|s_1| = 1$, $k_1 \leq s_1$, and
$I_2 = \{(m_2, s) : s \leq s_2\} \cup \{(f_{m_2,n_2}(s_2), n) : n \leq k_2\}$ for some $(m_2, n_2) \in B, s_2 \in \mathbb{N}^{<\mathbb{N}}, k_2 \in \mathbb{N}$.

Then I_1 is a Type I∪II^1 set and I_2 is a Type I∪II^2 set.
We prove this by considering the following cases.

(1) $m_2 \neq m_1$
If $f_{m_2,n_2}(s_2) \neq n_1$, then $I_1 \cap I_2 = \emptyset$.
If $f_{m_2,n_2}(s_2) = n_1$, then $I_1 \cap I_2 = \{(f_{m_2,n_2}(s_2), n) : n \leq \min\{k_1, k_2\}\}$, which is a Type II ideal.

(2) $m_2 = m_1$

(2.1) $f_{m_2,n_2}(s_2) \neq n_1$
If $s_1 \nleq s_2$, then $I_1 \cap I_2 = \emptyset$.
If $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\}$, which is a Type I ideal.

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(2.2) $f_{m_2,n_2}(s_2) = n_1$
If $s_1 \not\leq s_2$, then $I_1 \cap I_2 = \{(f_{m_2,n_2}(s_2), n) : n \leq \min\{k_1, k_2\}\}$, which is a Type II ideal.
If $s_1 \leq s_2$, then $I_1 \cap I_2 = \{(m_1, s_1)\} \cup \{(n_1, n) : n \leq \min\{k_1, k_2\}\}$, which is a Type I∪II^1 set.

These cover all possible cases and we are done.

Intersections of a Type IV ideal and a Type I∪II^2 set

Let $I_1 = \{(m_1, s) : s \leq s_1)\} \cup \{(f_{m_1,n_1}(s_1), n) : n \leq k_1\}$ for some $(m_1, n_1) \in B$, $s_1 \in \mathbb{N}^{<\mathbb{N}}$ with $k_1 \in \mathbb{N}$, and
$I_2 = \downarrow(f_{m_2,n_2}(s_2), \top) = L_{f_{m_2,n_2}(s_2)} \cup \{(m_2, s_2)\} \cup \{(n_2, n) : n \leq s_2\}$ for some $(m_2, n_2) \in B$, $s_2 \in \mathbb{N}^{<\mathbb{N}}$ with $|s_2| = 1$.

Then I_1 is a Type I∪II^2 set and I_2 is a Type IV ideal.
We prove this by considering the following cases for $f_{m_2,n_2}(s_2)$.

(1) $f_{m_2,n_2}(s_2) < m_1$
Then $I_1 \cap I_2 = \emptyset$.

(2) $f_{m_2,n_2}(s_2) = m_1$
Then $I_1 \cap I_2 = \{(m_1, s) : s \leq s_1\}$, which is a Type I ideal.

(3) $m_1 < f_{m_2,n_2}(s_2) < f_{m_1,n_1}(s_1)$

(3.1) $m_2 \neq m_1$
Then $I_1 \cap I_2 = \emptyset$.

(3.2) $m_2 = m_1$
If $s_2 \not\leq s_1$, then $I_1 \cap I_2 = \emptyset$.
If $s_2 \leq s_1$, then $I_1 \cap I_2 = \{(m_2, s_2)\}$, which is a Type I ideal.

(4) $f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1)$
Then $(m_2, n_2) = (m_1, n_1)$, $s_1 = s_2$ by the property of i and f_{m_1,n_1}.
So $I_1 \cap I_2 = I_1$, which is a Type I∪II^2 set.

(5) $f_{m_2,n_2}(s_2) > f_{m_1,n_1}(s_1)$

(5.1) $m_2 \neq m_1$
If $n_2 \neq f_{m_1,n_1}(s_1)$, then $I_1 \cap I_2 = \emptyset$.
If $n_2 = f_{m_1,n_1}(s_1)$, then $I_1 \cap I_2 = \{(n_2, n) : n \leq \min\{k_1, s_2\}\}$, which is a Type II ideal.

(5.2) $m_2 = m_1$
If $n_2 \neq f_{m_1,n_1}(s_1)$ and $s_2 \not\leq s_1$, then $I_1 \cap I_2 = \emptyset$.
If $n_2 \neq f_{m_1,n_1}(s_1)$ and $s_2 \leq s_1$, then $I_1 \cap I_2 = \{(m_2, s_2)\}$, which is a Type I ideal.
If $n_2 = f_{m_1,n_1}(s_1)$ and $s_2 \not\leq s_1$, then $I_1 \cap I_2 = \{(n_2, n) : n \leq \min\{k_1, s_2\}\}$, which is a Type II ideal.
If $n_2 = f_{m_1,n_1}(s_1)$, and $s_2 \leq s_1$, then $I_1 \cap I_2 = \{(m_2, s_2)\} \cup \{(n_2, n) : n \leq \min\{k_1, s_2\}\}$, which is a Type I∪II^1 set.

These cover all possible cases and we are done.

Intersections of a Type V ideal and a Type I∪II^2 set

Let $I_1 = \{(m_1, s) : s \leq s_1)\} \cup \{(f_{m_1,n_1}(s_1), n) : n \leq k_1\}$ for some $(m_1, n_1) \in B$, $s_1 \in \mathbb{N}^{<\mathbb{N}}$ with $k_1 \in \mathbb{N}$, and
$I_2 = \downarrow(f_{m_2,n_2}(s_2), \top) = L_{f_{m_2,n_2}(s_2)} \cup \{(m_2, s) : s \leq s_2\} \cup \{(f_{m_2,n_2}(s_2^*), n) : n \leq n_2^*\}$ for some $(m_2, n_2) \in B$, $s_2 = s_2^*.n_2^* \in \mathbb{N}^{<\mathbb{N}}$ with $s_2^* \in \mathbb{N}^{<\mathbb{N}}$, $n_2^* \in \mathbb{N}$.

Then I_1 is a Type I∪II^2 set and I_2 is a Type V ideal.
We prove this by considering the following cases for $f_{m_2,n_2}(s_2)$.

(1) $f_{m_2,n_2}(s_2) < m_1$
Then $I_1 \cap I_2 = \emptyset$.

(2) $f_{m_2,n_2}(s_2) = m_1$
Then $I_1 \cap I_2 = \{(m_1, s) : s \leq s_1\}$, which is a Type I ideal.
(3) $m_1 < f_{m_2,n_2}(s_2) < f_{m_1,n_1}(s_1)$
(3.1) $m_2 \neq m_1$
Then $I_1 \cap I_2 = \emptyset$.
(3.2) $m_2 = m_1$
If $\downarrow s_1 \cap \downarrow s_2 = \emptyset$, then $I_1 \cap I_2 = \emptyset$.
If $\downarrow s_1 \cap \downarrow s_2 \neq \emptyset$, then $I_1 \cap I_2 = \{(m_1, s) : s \leq \inf\{s_1, s_2\}\}$, which is a Type I ideal.
(4) $f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1)$
Then $(m_2, n_2) = (m_1, n_1)$, $s_1 = s_2$ by the property of i and f_{m_1,n_1}. Thus $I_1 \cap I_2 = I_1$, which is a Type I∪II^2 set.
(5) $f_{m_2,n_2}(s_2) > f_{m_1,n_1}(s_1)$
(5.1) $m_2 \neq m_1$
If $f_{m_2,n_2}(s_2^*) \neq f_{m_1,n_1}(s_1)$, then $I_1 \cap I_2 = \emptyset$.
If $f_{m_2,n_2}(s_2^*) = f_{m_1,n_1}(s_1)$, then $(m_2, n_2) = (m_1, n_1)$, $s_2^* = s_1$ by the property of i and f_{m_1,n_1}, which contradicts $m_2 \neq m_1$. Thus, this case does not occur.
(5.2) $m_2 = m_1$
If $f_{m_2,n_2}(s_2^*) \neq f_{m_1,n_1}(s_1)$ and $\downarrow s_1 \cap \downarrow s_2 = \emptyset$, then $I_1 \cap I_2 = \emptyset$.
If $f_{m_2,n_2}(s_2^*) \neq f_{m_1,n_1}(s_1)$ and $\downarrow s_1 \cap \downarrow s_2 \neq \emptyset$, then $I_1 \cap I_2 = \{(m_1, s) : s \leq \inf\{s_1, s_2\}\}$, which is a Type I ideal.
If $f_{m_2,n_2}(s_2^*) = f_{m_1,n_1}(s_1)$, then $(m_2, n_2) = (m_1, n_1)$, $s_2^* = s_1$.
It follows that $I_1 \cap I_2 = \{(m_1, s) : s \leq s_2^*\} \cup \{(f_{m_2,n_2}(s_2^*), n) : n \leq \min\{k_1, n_2^*\}\}$, which is a Type I∪II^2 set.
These cover all possible cases and we are done.

Intersections of two Type I∪II^2 sets

Let $I_1 = \{(m_1, s) : s \leq s_1\} \cup \{(f_{m_1,n_1}(s_1), n) : n \leq k_1\}$ for some $(m_1, n_1) \in B$, $s_1 \in \mathbb{N}^{<\mathbb{N}}$ with $k_1 \in \mathbb{N}$, and
$I_2 = \{(m_2, s) : s \leq s_2\} \cup \{(f_{m_2,n_2}(s_2), n) : n \leq k_2\}$ for some $(m_2, n_2) \in B$, $s_2 \in \mathbb{N}^{<\mathbb{N}}$ with $k_2 \in \mathbb{N}$.

Then I_1, I_2 are both Type I∪II^2 sets. We prove this by considering the following cases.

(1) $m_2 \neq m_1, f_{m_2,n_2}(s_2) \neq f_{m_1,n_1}(s_1)$
Then $I_1 \cap I_2 = \emptyset$.
(2) $m_2 \neq m_1, f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1)$
Then $(m_2, n_2) = (m_1, n_1)$, $s_2 = s_1$, which contradicts the assumption that $m_2 \neq m_1$. Hence, this case does not occur.
(3) $m_2 = m_1, f_{m_2,n_2}(s_2) \neq f_{m_1,n_1}(s_1)$, and $\downarrow s_2 \cap \downarrow s_1 = \emptyset$
Then $I_1 \cap I_2 = \emptyset$.
(4) $m_2 = m_1, f_{m_2,n_2}(s_2) \neq f_{m_1,n_1}(s_1)$, and $\downarrow s_2 \cap \downarrow s_1 \neq \emptyset$
Then $I_1 \cap I_2 = \{(m_2, s) : s \leq \min\{s_1, s_2\}\}$, which is a Type I ideal.
(5) $m_2 = m_1, f_{m_2,n_2}(s_2) = f_{m_1,n_1}(s_1)$, then $(m_2, n_2) = (m_1, n_1)$, $s_2 = s_1$.
It follows that $I_1 \cap I_2 = \{(m_2, s) : s \leq s_2\} \cup \{(f_{m_2,n_2}(s_2), n) : n \leq \min\{k_1, k_2\}\}$, which is a Type I∪II^2 set.

These cover all possible cases and we are done.

It follows that the intersection of finite number of principle ideals of P is either a principle ideal, or a Type I∪II^1 set, or a Type I∪II^2 set, or the empty set.

Let

$$\Psi = \{\text{all principle ideals of } P\} \cup \{\text{all Type I} \cup \text{II}^1 \text{ sets of } P\} \cup \{\text{all Type I} \cup \text{II}^2 \text{ sets of } P\} \cup \{\emptyset\}.$$

By the above arguments, we deduce that Ψ is closed under finite non-nullary intersections. In addition, since all elements of Ψ are principal ideals or finite subsets of P, and P has no infinite decreasing chains, we have that Ψ does not contain an infinite decreasing chain.

For any nonempty $E \subseteq P$,

$$\bigcap\{\downarrow x : x \in E\} = \bigcap\{\bigcap\{\downarrow x : x \in F\} : F \text{ is a finite subset of } E\}.$$

Each $\bigcap\{\downarrow x : x \in F\}$ is a member of Ψ, so $\{\bigcap\{\downarrow x : x \in F\} : F \text{ is a finite subset of } E\}$ is a filter of members of Ψ. Since Ψ does not have infinite decreasing chains, this family must have a smallest member, which equals $\bigcap\{\downarrow x : x \in E\}$ and is in Ψ. Therefore, $M = \Psi$.

We now show that every directed subset of M has a supremum in M.

Let $\mathcal{D} = \{I_i : i \in I\}$ be a directed subset of M. If $\mathcal{D}$ has a largest member, then its supremum exists and equals the largest member. Now we assume that $\mathcal{D}$ does not have a largest member. Then $\mathcal{D}$ contains an infinite chain. Since Type III ideals, Type IV ideals and Type V ideals are maximal elements of M, we deduce that none of the I_i is a Type III ideal, Type IV ideal, and Type V ideal.

We consider the following two remaining cases.

Case 1: All members of $\{I_i : i \in I\}$ are Type I ideals or Type II ideals.

Then, as a Type I ideal and a Type II ideal are not comparable, it follows that either all I_i are Type I ideals or they are all Type II ideals. Thus, $\sup_{i\in I} I_i$ exists because P is a dcpo.

Case 2: There exists $i_0 \in I$ such that I_{i_0} is a Type $\mathrm{I} \cup \mathrm{II}^1$ set or a Type $\mathrm{I} \cup \mathrm{II}^2$ set.

Then all members of $\{I_i : I_i \geq I_{i_0}\}$ are either Type $\mathrm{I} \cup \mathrm{II}^1$ sets or Type $\mathrm{I} \cup \mathrm{II}^2$ sets. Note that there exist no infinite increasing chains consisting of Type $\mathrm{I} \cup \mathrm{II}^1$ sets. Hence, there exists i_1 such that $\mathcal{D}_1 = \{I_i : I_i \geq I_{i_1}\}$ consists of only Type $\mathrm{I} \cup \mathrm{II}^2$ sets. In order to prove $\sup \mathcal{D}$ exists, it is enough to prove $\sup \mathcal{D}_1$ exists.

Let $\mathcal{D}_1 = \{I_j : j \in J\}$, where J is a subset of I.

For each $j \in J$, let $I_j = \{(m_j, s) : s \leq s_j\} \cup \{(f_{m_j,n_j}(s_j), n) : n \leq k_j\}$ for some $(m_j, n_j) \in B$, $s_j \in \mathbb{N}^{<\mathbb{N}}$, $k_j \in \mathbb{N}$.

Then $f_{m_i,n_i}(s_i) = f_{m_j,n_j}(s_j)$ for all $i, j \in J$, which implies that $(m_i, n_i) = (m_j, n_j)$ and $s_i = s_j$. Let $m_j = m_0$, $n_j = n_0$ and $s_j = s_0$ $(j \in J)$. Then, $I_j = \{(m_0, s) : s \leq s_0\} \cup \{(f_{m_0,n_0}(s_0), n) : n \leq k_j\}$. Since $\mathcal{D}$ does not have a largest member, $\mathcal{D}_1$ has no a largest member either. It follows that the set $\{k_j : j \in J\}$ does not have a largest element.

We claim that $\sup \mathcal{D}_1 = \downarrow(f_{m_0,n_0}(s_0), \top)$.

Clearly, $\downarrow(f_{m_0,n_0}(s_0), \top)$ is an upper bound of $\mathcal{D}_1$. For any upper bound I of $\mathcal{D}_1$ in M, we have that $\bigcup\{\{(f_{m_0,n_0}(s_0), n) : n \leq k_j\} : j \in J\} \subseteq I$. Note that I is a Scott closed set of P, and $\bigcup\{\{(f_{m_0,n_0}(s_0), n) : n \leq k_j\} : j \in J\}$ is a directed subset of P. Therefore, $\sup \bigcup\{\{(f_{m_0,n_0}(s_0), n) : n \leq k_j\} : j \in J\} = (f_{m_0,n_0}(s_0), \top) \in I$.

All the above together show that $\sup \mathcal{D} = \sup \mathcal{D}_1 = \downarrow(f_{m_0,n_0}(s_0), \top)$. Thus, M is a dcpo. □

Remark 3.4. From the structure of M, it is easy to see that the dcpo M does not contain an infinite decreasing chain. Thus, in the dual poset M^{op} of M, every element is compact.

Theorem 3.5. *M is a countable bounded complete dcpo whose Scott space ΣM is not sober.*

Proof. Because every element of M is a finite intersection of principal downsets of P, and P is countable, M is countable. By Lemma 3.3, it remains to verify that ΣM is not sober.

Define the mapping $g : P \to M$ by $g(x) = \downarrow x$ for all $x \in P$. By the last part of the proof of Lemma 3.3, we see easily that g is Scott continuous. By Lemma 3.2, P is irreducible; thus, $g(P)$ is irreducible. Clearly, the closure of $g(P)$ equals M, so M is irreducible (with respect to the Scott topology). As M does not have a top element, it is not the closure of a singleton set. It follows that ΣM is not sober. □

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Let $\hat{M} = M \cup \{\top\}$ be the poset obtained by adding a top element $\top$ to M. Then $\hat{M}$ is a countable non-sober complete lattice.

Corollary 3.6. *There exists a countable non-sober complete lattice.*

Note that the non-sober complete lattice $\hat{M}$ constructed above is not distributive. Thus, it remains to know whether there is a distributive countable non-sober complete lattice.

By applying the Lemma 2.3, we give a negative answer to the above problem.

Theorem 3.7. *Let $\mathcal{F} = \{\downarrow F : F \subseteq_{fin} \hat{M}\}$. Then, $(\mathcal{F}, \subseteq)$ is a countable non-sober distributive complete lattice.*

Proof. First, every element in the dual $\hat{M}^{op}$ of the complete lattice $\hat{M}$ is compact. Now apply Lemma 2.3 to $\hat{M}^{op}$, we deduce that $\mathcal{F} = \{\downarrow F : F \subseteq_{fin} \hat{M}\}$ is a distributive complete lattice. It is countable because $\hat{M}$ is countable.

Define $g : \hat{M} \to \mathcal{F}$ by $g(x) = \downarrow x$ and $f : \mathcal{F} \to \hat{M}$ by $f(A) = \sup A$ for any $A \in \mathcal{F}$. Clearly, both f and g are monotone. In addition, for any $x \in \hat{M}$ and $\downarrow F \in \mathcal{F}$, we have

$$f(\downarrow F) \le x \text{ iff } \bigvee F \le x \text{ iff } \downarrow F \subset \downarrow x = g(x).$$

Hence, f is the left adjoint of g. Then f preserves all suprema, in particular the suprema of directed subsets. Hence, it is Scott continuous. Also by the structure of $\hat{M}$ and $\mathcal{F}$, we easily see that g also preserves the suprema of directed subsets; hence, it is also Scott continuous. Note that $f \circ g = id_{\hat{M}}$. It thus follows that $\Sigma\hat{M}$ is a retraction of $\Sigma\mathcal{F}$. Since $\Sigma\hat{M}$ is non-sober, thus $\Sigma\mathcal{F}$ is non-sober. □

4. A Sufficient Condition for Complete Lattices to be Sober

In this section, we prove some positive results on the sobriety of Scott spaces of dcpos. One immediate corollary is that every complete lattice L with $Id(L)$ countable has a sober Scott space.

From the above section, we know that a countable complete lattice may not be sober in the Scott topology. In this section, we show that the Scott space of a complete lattice with $Id(P)$ countable is sober.

The following lemma is critical for our later discussions.

Lemma 4.1. *Let P, Q be two posets. If $|Id(P)|, |Id(Q)|$ are both countable, then $\Sigma(P \times Q) = \Sigma P \times \Sigma Q$.*

Proof. Obviously, $\sigma(P) \times \sigma(Q) \subseteq \sigma(P \times Q)$. It remains to prove that $\sigma(P \times Q) \subseteq \sigma(P) \times \sigma(Q)$. Let U be a nonempty Scott open set and $(a_1, b_1) \in U$. We denote $Id(P)$ and $Id(Q)$ by $\{I_n^P \mid n \in \mathbb{N}\}$ and $\{I_n^Q \mid n \in \mathbb{N}\}$, respectively.

For $n = 1$, $A_1 = \{a_1\}, B_1 = \{b_1\}$.

For $n = 2$, we define A_2 and B_2 below:

If $\sup I_1^P \in \uparrow A_1$, then $(\sup I_1^P, b_1) \in U$. It follows that there exists $d_1^P \in I_1^P$ such that $(d_1^P, b_1) \in U$ by the Scott openness of U. Let $A_2 = \{d_1^P\}$ in this case, and $A_2 = \emptyset$ otherwise. Note that $(A_1 \cup A_2) \times B_1 \subseteq U$.

If $\sup I_1^Q \in \uparrow B_1$, we have $(A_1 \cup A_2) \times \{\sup I_1^Q\} \subseteq U$. For each $a \in A_1 \cup A_2$, we can choose a d_a form I_1^Q satisfying $(a, d_a) \in U$ by the Scott openness of U. Since $A_1 \cup A_2$ is finite and I_1^Q is directed, there exists $d_1^Q \in I_1^Q$ such that $(A_1 \cup A_2) \times \{d_1^Q\} \subseteq U$. Let $B_2 = \{d_1^Q\}$ in this case, and $B_2 = \emptyset$ otherwise. We conclude that $(A_1 \cup A_2) \times (B_1 \cup B_2) \subseteq U$.

For $n = 3$, we first consider the two index sets:

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$$E_1 = \left\{ i \in \{1\} \mid \sup I_i^P \notin \uparrow A_1 \text{ and } \sup I_i^P \in \uparrow A_2 \right\} \cup \left\{ i \in \{2\} \mid \sup I_i^P \in \uparrow (A_1 \cup A_2) \right\} \text{ and}$$
$$F_1 = \left\{ i \in \{1\} \mid \sup I_i^Q \notin \uparrow B_1 \text{ and } \sup I_i^Q \in \uparrow B_2 \right\} \cup \left\{ i \in \{2\} \mid \sup I_i^Q \in \uparrow (B_1 \cup B_2) \right\}.$$

However, if $\sup I_1^P \notin \uparrow A_1$ and $\sup I_1^Q \notin \uparrow B_1$, then $A_2 = \emptyset$ and $B_2 = \emptyset$ from the above step. In this way, $\left\{ i \in \{1\} \mid \sup I_i^P \notin \uparrow A_1 \text{ and } \sup I_i^P \in \uparrow A_2 \right\}$ and $\left\{ i \in \{1\} \mid \sup I_i^Q \notin \uparrow B_1 \text{ and } \sup I_i^Q \in \uparrow B_2 \right\}$ must be empty.

Next, we define A_3 and B_3 in the similar way as before.

If $E_1 \neq \emptyset$, then $E_1 = \{2\}$. We have $\{\sup I_2^P\} \times (B_1 \cup B_2) \subseteq U$. Through the similar discussion process, we can deduce that there exists $d_2^P \in I_2^P$ such that $\{d_2^P\} \times (B_1 \cup B_2) \subseteq U$ because $B_1 \cup B_2$ is finite and I_2^P is directed. Let $A_3 = \{d_2^P\}$ in this case, and $A_3 = \emptyset$ otherwise. Note that $(A_1 \cup A_2 \cup A_3) \times (B_1 \cup B_2) \subseteq U$.

If $F_1 \neq \emptyset$, then $F_1 = \{2\}$. Thus, $(A_1 \cup A_2 \cup A_3) \times \{\sup I_2^Q\} \subseteq U$. Note that $A_1 \cup A_2 \cup A_3$ is a finite set. It follows that there exists $d_2^Q \in I_2^Q$ such that $(A_1 \cup A_2 \cup A_3) \times \{d_2^Q\} \subseteq U$. Let $B_3 = \{d_2^Q\}$ in this case, and $B_3 = \emptyset$ otherwise. We conclude that $(A_1 \cup A_2 \cup A_3) \times (B_1 \cup B_2 \cup B_3) \subseteq U$.

For $n = 4$, we also consider the two index sets:

$$E_2 = \left\{ i \in \{1,2\} \mid \sup I_i^P \notin \bigcup_{k=1}^{i} \uparrow A_k \text{ and } \sup I_i^P \in \bigcup_{k=i+1}^{3} \uparrow A_k \right\} \cup \left\{ i \in \{3\} \mid \sup I_i^P \in \bigcup_{k=1}^{3} \uparrow A_k \right\},$$
$$F_2 = \left\{ i \in \{1,2\} \mid \sup I_i^Q \notin \bigcup_{k=1}^{i} \uparrow B_k \text{ and } \sup I_i^Q \in \bigcup_{k=i+1}^{3} \uparrow B_k \right\} \cup \left\{ i \in \{3\} \mid \sup I_i^Q \in \bigcup_{k=1}^{3} \uparrow B_k \right\}.$$

Next, we define A_4 and B_4 in the following:

If $E_2 \neq \emptyset$, then $i \in \{1, 2\}$ implies $\sup I_i^P \in \bigcup_{k=i+1}^3 \uparrow A_k \subseteq \bigcup_{k=1}^3 \uparrow A_k$, and $i = 3$ implies $\sup I_i^P \in \bigcup_{k=1}^3 \uparrow A_k$. Thus, $\sup I_i^P \in \bigcup_{k=1}^3 \uparrow A_k$ for all $i \in E_2$. So for each $i \in E_2$, $\{\sup I_i^P\} \times (B_1 \cup B_2 \cup B_3) \subseteq U$ implies that there exists $d_i^P \in I_i^P$ such that $\{d_i^P\} \times (B_1 \cup B_2 \cup B_3) \subseteq U$ because $B_1 \cup B_2 \cup B_3$ is finite and I_i^P is directed. Let $A_4 = \{d_i^P \mid i \in E_2\}$ in this case, and $A_4 = \emptyset$ otherwise. Note that

$$\left(\bigcup_{k=1}^{4} \uparrow A_k \right) \times \left(\bigcup_{k=1}^{3} \uparrow B_k \right) \subseteq U.$$

If $F_2 \neq \emptyset$, then $i \in \{1, 2\}$ implies $\sup I_i^Q \in \bigcup_{k=i+1}^3 \uparrow B_k \subseteq \bigcup_{k=1}^3 \uparrow B_k$, and $i = 3$ implies $\sup I_i^Q \in \bigcup_{k=1}^3 \uparrow B_k$. Thus, $\sup I_i^Q \in \bigcup_{k=1}^3 \uparrow B_k$ for all $i \in F_2$. So for each $i \in F_2$, $(\bigcup_{k=1}^4 A_k) \times \{\sup I_i^Q\} \subseteq U$ implies that there exists $d_i^Q \in I_i^Q$ such that $(\bigcup_{k=1}^4 A_k) \times \{d_i^Q\} \subseteq U$ since $\bigcup_{k=1}^4 A_k$ is a finite set and I_i^Q is directed. Let $B_4 = \{d_i^Q \mid i \in F_2\}$ in this case, and $B_4 = \emptyset$ otherwise. We conclude that

$$\left(\bigcup_{k=1}^{4} A_k \right) \times \left(\bigcup_{k=1}^{4} B_k \right) \subseteq U.$$

For $n \geq 4$, we assume that

$$\left(\bigcup_{k=1}^{n-1} A_k \right) \times \left(\bigcup_{k=1}^{n-1} B_k \right) \subseteq U.$$

Then, we define A_n and B_n inductively.

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We first consider the following two index sets:

$$E_{n-2} = \left\{ i \in \{1, \ldots, n-2\} \mid \sup I_i^P \notin \bigcup_{k=1}^{i} \uparrow A_k \text{ and } \sup I_i^P \in \bigcup_{k=i+1}^{n-1} \uparrow A_k \right\}$$
$$\cup \left\{ i \in \{n-1\} \mid \sup I_i^P \in \bigcup_{k=1}^{n-1} \uparrow A_k \right\},$$
$$F_{n-2} = \left\{ i \in \{1, \ldots, n-2\} \mid \sup I_i^Q \notin \bigcup_{k=1}^{i} \uparrow B_k \text{ and } \sup I_i^Q \in \bigcup_{k=i+1}^{n-1} \uparrow B_k \right\}$$
$$\cup \left\{ i \in \{n-1\} \mid \sup I_i^Q \in \bigcup_{k=1}^{n-1} \uparrow B_k \right\}.$$

Note that $\left\{ i \in \{1, \ldots, n-2\} \mid \sup I_i^P \notin \bigcup_{k=1}^{i} \uparrow A_k \text{ and } \sup I_i^P \in \bigcup_{k=i+1}^{n-1} \uparrow A_k \right\}$ and $\left\{ i \in \{1, \ldots, n-2\} \mid \sup I_i^Q \notin \bigcup_{k=1}^{i} \uparrow B_k \text{ and } \sup I_i^Q \in \bigcup_{k=i+1}^{n-1} \uparrow B_k \right\}$ may not be empty.

If $E_{n-2} \neq \emptyset$, similarly, we can deduce $\{\sup I_i^P\} \times \left(\bigcup_{k=1}^{n-1} B_k \right) \subseteq U$ for any $i \in E_{n-2}$. Note that $\bigcup_{k=1}^{n-1} B_k$ is a finite set and each I_i^P is directed. Thus, there exists $d_i^P \in I_i^P$ such that $\{d_i^P\} \times \left(\bigcup_{k=1}^{n-1} B_k \right) \subseteq U$ for any $i \in E_{n-2}$. Let $A_n = \{d_i^P | i \in E_{n-2}\}$ in this case, and $A_n = \emptyset$ otherwise. It follows that

$$\left(\bigcup_{k=1}^{n} A_k \right) \times \left(\bigcup_{k=1}^{n-1} B_k \right) \subseteq U.$$

If $F_{n-2} \neq \emptyset$, then $\left(\bigcup_{k=1}^{n} A_k \right) \times \{\sup I_i^Q\} \subseteq U$ for any $i \in F_{n-2}$. Note that $\bigcup_{k=1}^{n} A_k$ is a finite set. This means that there exists $d_i^Q \in I_i^Q$ such that $\left(\bigcup_{k=1}^{n} A_k \right) \times \{d_i^Q\} \subseteq U$ for any $i \in F_{n-2}$. Let $B_n = \{d_i^Q \mid i \in F_{n-2}\}$ in this case, and $B_n = \emptyset$ otherwise. We conclude that

$$\left(\bigcup_{k=1}^{n} A_k \right) \times \left(\bigcup_{k=1}^{n} B_k \right) \subseteq U.$$

Let $A = \bigcup_{n \in \mathbb{N}} A_n$ and $B = \bigcup_{n \in \mathbb{N}} B_n$. It is easy to see that $(a_1, b_1) \in A_1 \times B_1 \subseteq \uparrow A \times \uparrow B \subseteq U$. It suffices to prove that $\uparrow A$, $\uparrow B$ are both Scott open.

Let D be a directed subset of P with $\sup D \in \uparrow A$. If $\sup D \in D$, then $D \cap \uparrow A \neq \emptyset$. If $\sup D \notin D$, i.e., D contains no maximal element, then $\downarrow D \in Id(P)$. Thus, there exists $n_0 \in \mathbb{N}$ such that $\downarrow D = I_{n_0}^P$.

Therefore, $\sup D \in \uparrow A$ can imply that $\sup I_{n_0}^P \in \uparrow A$. Let $n_1 = \inf\{n \in \mathbb{N} \mid \sup I_{n_0}^P \in \uparrow A_n\}$. Then $\sup I_{n_0}^P \in \uparrow A_{n_1}$. Now we need to distinguish between the following two cases for n_0, n_1.

Case 1, $n_0 < n_1$. If $n_0 = 1, n_1 = 2$, then $\sup I_1^P \notin \uparrow A_1$ implies $A_2 = \emptyset$, which contradicts the condition $\sup I_1^P \in \uparrow A_2$. So $n_1 \geqslant 3$. The fact that $\sup I_{n_0}^P \notin \bigcup_{k=1}^{n_0} \uparrow A_k$ and $\sup I_{n_0}^P \in \bigcup_{k=n_0+1}^{n_1} \uparrow A_k$ can imply $n_0 \in E_{n_1-1}$. This means that $I_{n_0}^P \cap A_{n_1+1} \neq \emptyset$. Hence, $D \cap \uparrow A \neq \emptyset$.

Case 2, $n_0 \geq n_1$. If $n_0 = n_1 = 1$, then $\sup I_1^P \in \uparrow A_1$ implies $I_1^P \cap A_2 \neq \emptyset$. If $n_0 \geqslant 2$, then $\sup I_{n_0}^P \in \uparrow A_{n_1} \subseteq \bigcup_{k=1}^{n_0} \uparrow A_k$, which implies $n_0 \in E_{n_0-1}$. It follows that $I_{n_0}^P \cap A_{n_0+1} \neq \emptyset$. Therefore, $D \cap \uparrow A \neq \emptyset$.

Hence, $\uparrow A$ is Scott open, and $\uparrow B$ is Scott open by the similar proof. □

Theorem 4.2. *Let L be a dcpo with $Id(L)$ countable. If ΣL is coherent and well filtered, then ΣL is sober.*

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Proof. Let A be an irreducible closed subset of ΣL. It suffices to prove that A is directed, which means that $\uparrow x \cap \uparrow y \cap A \neq \emptyset$ for any $x, y \in A$.

Write $B = \{(a, b) \in L \times L \mid \uparrow a \cap \uparrow b \subseteq L \backslash A\}$. We claim that B is Scott open in $L \times L$. Obviously, B is an upper set. Let $(x_i, y_i)_{i \in I}$ be a directed subset of $L \times L$ with $\sup_{i \in I}(x_i, y_i) \in B$. Then $(\sup_{i \in I} x_i, \sup_{i \in I} y_i) \in B$, which is equivalent to saying that $\uparrow \sup_{i \in I} x_i \cap \uparrow \sup_{i \in I} y_i \subseteq L \backslash A$. It follows that $\bigcap_{i \in I}(\uparrow x_i \cap \uparrow y_i) \subseteq L \backslash A$. Since ΣL is coherent and well filtered, we can find some index $i \in I$ such that $\uparrow x_i \cap \uparrow y_i \subseteq L \backslash A$. This implies that $(x_i, y_i) \in B$. Thus, B is Scott open.

It is worth noting that $Id(L)$ is countable. $\Sigma(L \times L) = \Sigma L \times \Sigma L$ from Lemma 4.1. For the sake of contradiction, we assume that there are $x, y \in A$ such that $\uparrow x \cap \uparrow y \cap A = \emptyset$. The fact that $(x, y) \in B \subseteq \sigma(L \times L)$ implies that we can find $U_x, U_y \in \sigma(L)$ such that $(x, y) \in U_x \times U_y \subseteq B$. Note that $x \in U_x \cap A$ and $y \in U_y \cap A$. By the irreducibility of A, we have $A \cap U_x \cap U_y \neq \emptyset$. Pick $a \in A \cap U_x \cap U_y$. Then $(a, a) \in U_x \times U_y \subseteq B$, that is, $a \in \uparrow a \cap \uparrow a \subseteq L \backslash A$. It contradicts the assumption that $a \in A$. Hence, A is directed and $\sup A \in A$. So $A = \downarrow \sup A$. □

Corollary 4.3. *Let L be a complete lattice. If Id(L) is countable, then ΣL is sober.*

Proof. From Jia et al. (2016) and Xi and Lawson (2017), we deduce that ΣL is well filtered and coherent. The result is evident by Theorem 4.2. □

5. Conclusions

In this paper, we constructed a countable complete lattice whose Scott space is non-sober, thus answering a problem posed by Achim Jung. Based on this complete lattice, we further obtained a countable distributive complete lattice whose Scott space is not sober.

The countable complete lattice constructed here is not a frame (or complete Heyting algebra). Thus, the following problem is still open.

Problem. Is there a countable frame whose Scott space is non-sober?

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Author contribution. Xiaoyong Xi gives the main idea of the counterexample. Hualin Miao and Qingguo Li make up the details of the counterexample and give a sufficient condition for the open problem posed by Achim Jung. Dongsheng Zhao writes the main manuscript text, and all authors reviewed the manuscript.

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Appendix

Remark 6.1. There exists a monotone bijection $f : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$.

In fact, consider the set of prime numbers $\mathbf{P} = \{p_1, p_2, \cdots p_n, \cdots\}$, where $p_i < p_{i+1}$ for all $i \geq 1$. For any element $a = n_1.n_2. \cdots .n_k \in \mathbb{N}^{<\mathbb{N}}$, we set $f(a) = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$. Then f is monotone and injective. Notice that image of f is orderly isomorphic to $\mathbb{N}$.

The above explanation was suggested by one of the referees.

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